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Correspondences and groupoids

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Abstract

Our definition of correspondence between groupoids (which generalizes the notion of homomorphism) is obtained by weakening the conditions in the definition of equivalence of groupoids in [5]. We prove that such a correspondence induces another one between the associated C*-algebras, and in some cases besides a Kasparov element. We wish to apply the results obtained in the particular case of K-oriented maps of leaf spaces in the sense of [3].

Key words: Groupoid, C-algebra, correspondence, KK-group.*
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1 Introduction

A. Connes introduced in [1] the notion of correspondence in the theory of Von Neumann algebras. It is a concept of morphism, which gives the well known notion of correspondence of C*-algebras. In [5], P.S. Mulhy, J.N. Renault and D. Williams defined the notion of *equivalence* of groupoids and showed that if two groupoids are equivalent, then the associated C*-algebras are Morita-equivalent. But, this notion is too strong: here we give a definition of *correspondence of groupoids*, by weakening the conditions of equivalence in [5]. We prove that such a correspondence induces another one (in the sense of A. Connes) between the associated reduced C*-algebras. We show that a groupoid homomorphism satisfying some additional conditions can be thought as a correspondence between them, and induces a Kasparov element between the associated C*-algebras. Our fundamental interest is to apply these results to the study of foliated spaces where, in many cases, homomorphisms between holonomy groupoids appear (for example, in [2]).

2 Correspondences and groupoids

Let G_i $(i = 1, 2)$ be a second countable locally compact Hausdorff groupoid. Let $G_i^{(0)}$ be the unit space and $s_i, r_i : G_i \longrightarrow G_i^{(0)}$ the source and range map, respectively (we do not assume that r_i, s_i are open). Let $G_{i,x} = s_i^{-1}(x)$, for $x \in G_i^{(0)}$. Let Z be a second countable locally compact Hausdorff space, $\rho : Z \longrightarrow G_1^{(0)}$ a continuous and surjective map, and the space $G_1 * Z = \{(\gamma_1, z) \in G_1 \times Z : s_1(\gamma_1) = \rho(z)\}$.

Definition 1 A *left action* of G_1 on Z is a continuous and surjective map $\Phi : G_1 * Z \longrightarrow Z$, noted $\Phi(\gamma_1, z) = \gamma_1.z$, with the following properties:

1) $\rho(\gamma_1.z) = r_1(\gamma_1)$, for $(\gamma_1, z) \in G_1 * Z$,
2) $\gamma_1'.(\gamma_1.z) = (\gamma_1'\gamma_1).z$, when both sides of the equality are defined,
3) $\rho(z).z = z$, for each $z \in Z$,

and we say thus that Z is a *left* G_1*-space*. Z is called *proper*, when the map $\Phi_1 : G_1 * Z \longrightarrow Z \times Z$ defined by $\Phi_1(\gamma_1, z) = (\gamma_1.z, z)$, is proper.

In the same manner, we define a *right* G_2*-action* on Z, using a continuous and surjective map $\sigma : Z \longrightarrow G_2^{(0)}$ and $Z * G_2 = \{(z, \gamma_2) \in Z \times G_2 : r_2(\gamma_2) = \sigma(z)\}$.

Definition 2 Let G_1 and G_2 be second countable locally compact Hausdorff groupoids and Z a second countable locally compact Hausdorff space. The space Z is a *correspondence* from G_1 to G_2, when:

1) there is a left proper G_1-action on Z and a right proper G_2-action on Z, and they commute,
2) $\rho : Z \longrightarrow G_1^{(0)}$ is open and induces a bijection of Z/G_2 onto $G_1^{(0)}$.

The difference with the definition of equivalence in [5], is that here we do not suppose that the actions are free, we do not assume that σ is open and above all, we do not suppose that σ induces a bijection of $G_1 \backslash Z$ onto $G_2^{(0)}$.

If $V \subset Z$, $Sat_2(V) = \{z.\gamma_2 \in Z : z \in V \text{ and } (z, \gamma_2) \in Z * G_2\}$ is its saturation with respect to the G_2-action and (2) of definition 2 is true, then $Sat_2(V) = \rho^{-1}\rho(V)$ and the quotient map $Z \to Z/G_2$ is open. Moreover, if the G_2-action is proper, then Z/G_2 is a locally compact Hausdorff space.

Definition 3 Let A and B be C*-algebras. The couple (E, ϕ) is a *correspondence from* A *to* B, if it satisfies the following properties:

1) E is a right Hilbert B-module,

2) ϕ is a $*$-homomorphism from A into $\mathcal{L}_B(E)$, the set of bounded adjointable operators on E.

If $\phi(A) \subset \mathcal{K}_B(E)$ (the closure of the linear span of $\{\theta_{\xi,\eta}\}_{\xi,\eta\in E}$, where $\theta_{\xi,\eta} \in \mathcal{L}_B(E)$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle$, for $\zeta \in E$), then $(E, \phi, 0)$ is a Kasparov module for trivially graded C*-algebras (A, B) and gives an element $[E]$ of $KK(A, B)$. Note that each $*$-homomorphism between C*-algebras induces a correspondence between them.

For $i \in \{1, 2\}$, let λ^i be a right Haar system of G_i (this condition implies that r_i and s_i are open). Let $C_c(G_i)$ be the $*$-algebra of compactly supported continuous functions, where the product and the involution are defined by:

$$(ab)(\gamma_i) = \int_{G_i} a(\gamma_i{\gamma_i'}^{-1})b(\gamma_i')d\lambda^i_{s_i(\gamma_i)}(\gamma_i') \quad \text{and} \quad a^*(\gamma_i) = \overline{a(\gamma_i^{-1})},$$

for $a, b \in C_c(G_i)$ and $\gamma_i \in G_i$. For $x \in G_i^{(0)}$, we define a representation $\pi_{i,x}$ of $C_c(G_i)$ on $L^2(G_{i,x}, \lambda^i_x)$ by:

$$(\pi_{i,x}(a)\zeta)(\gamma_i) = \int_{G_i} a(\gamma_i{\gamma_i'}^{-1})\zeta(\gamma_i')d\lambda^i_x(\gamma_i')$$

for $a \in C_c(G_i)$, $\zeta \in L^2(G_{i,x}, \lambda^i_x)$ and $\gamma_i \in G_{i,x}$. We define the reduced norm by

$$\|a\| = \sup_{x\in G_i^{(0)}} \|\pi_{i,x}(a)\|,$$

and the reduced groupoid C*-algebra $C_r^*(G_i)$ is the completion of $C_c(G_i)$ by the reduced norm.

Theorem 4 (see [4]) *Let (G_i, λ^i) ($i \in \{1, 2\}$) be a second countable locally compact Hausdorff groupoid with a right Haar system λ^i and Z a correspondence from G_1 to G_2. There exists a correspondence from $C_r^*(G_2)$ to $C_r^*(G_1)$.*

3 Homomorphisms of groupoids

Let G_1 and G_2 be as in the previous section and let f be a continuous homomorphism of G_1 onto G_2. We denote by $f^{(0)}$ the restriction of f to $G_1^{(0)}$, which is a map onto $G_2^{(0)}$. The kernel of f, $H = \{\gamma_1 \in G_1 : f(\gamma_1) \in G_2^{(0)}\}$, is a closed subgroupoid of G_1 and we have $H^{(0)} = G_1^{(0)}$. There is a natural right action of H on G_1, which is proper since H is closed. We define the map $(r,s)_H : H \longrightarrow H^{(0)} \times H^{(0)}$ by $(r,s)_H(\gamma) = (r_H(\gamma), s_H(\gamma))$, for $\gamma \in H$, where r_H and s_H are the range and source map of H, respectively. Then:

Theorem 5 (see [4]) *Let G_1 and G_2 be second countable locally compact Hausdorff groupoids, let f be a continuous homomorfism of G_1 onto G_2 and let H be the kernel of f. Suppose that the following properties are satisfied:*

(C1) *the quotient map $q_H : G_1 \longrightarrow G_1/H$ is open,*

(C2) $r_1 : G_1 \longrightarrow G_1^{(0)}$ *is open,*

(C3) $(r,s)_H : H \longrightarrow H^{(0)} \times H^{(0)}$ *is proper,*

(C4) *for each $x \in G_1^{(0)}$, $f(G_{1,x}) = G_{2,f(x)}$,*

(C5) $f : G_1 \longrightarrow G_2$ *is open, and*

(C6) $f^0 : G_1^{(0)} \longrightarrow G_2^{(0)}$ *is locally one-to-one.*

Then, G_1/H is a correspondence from G_1 to G_2.

An homomorphism of groupoids does not induce, in general, an homomorphism between the associated C*-algebras, but the following result holds:

Theorem 6 (see [4]) *Let (G_i, λ_i) be a second countable locally compact Hausdorff groupoid with a right Haar system λ^i for $i = 1, 2$, and let f be a continuous homomorphism of G_1 onto G_2. Suppose that the conditions* (C1) *to* (C6)*, and the following condition are satisfied:*

(C7) $f^0 : G_1^{(0)} \longrightarrow G_2^{(0)}$ *is proper.*

Then, there is a correspondence (E, ϕ) from $C_r^*(G_2)$ to $C_r^*(G_1)$, such that $\phi(C_r^*(G_2)) \subset \mathcal{K}_{C_r^*(G_1)}(E)$. Thus, $(E, \phi, 0)$ is a Kasparov module for the couple $(C_r^*(G_2), C_r^*(G_1))$, and we obtain an element of $KK(C_r^*(G_2), C_r^*(G_1))$.

4 Some examples

Let G_i $(i = 1, 2)$, f and H be as in Theorem 5. Suppose that they satisfy the conditions (C1) to (C6). Set $Z = G_1/H$. Denote by λ^i a right Haar system of G_i. It follows from Theorems 4 and 5 that we have a correspondence from $C_r^*(G_2)$ to $C_r^*(G_1)$. Denote by (E, ϕ) the correspondence constructed in the proof of Theorem 4. If the condition (C7) is satisfied, then $(E, \phi, 0)$ is a

Kasparov module and gives an element of $KK(C_r^*(G_2), C_r^*(G_1))$ by Theorem 6. In this section, we study some examples where groupoids are topological spaces, topological groups and transformation groups, respectively.

4.1 Topological Spaces

Let X_i be a topological space and suppose that G_i is the trivial groupoid X_i, $i \in \{1,2\}$. Then, $f : X_1 \longrightarrow X_2$ is continuous and surjective and $C_r^*(G_i)$ is the commutative C*-algebra $C_0(X_i)$ of continuous functions vanishing at infinity. Remark that $f^{(0)} = f$, $H = X_1$ and $X_1/H = X_1$. We have $E = C_0(X_1)$ and thus, ϕ is the $*$-homomorphism $\phi : C_0(X_2) \longrightarrow \mathcal{M}(C_0(X_1))$ ($\mathcal{M}(C_0(X_1))$ is the multiplier algebra of $C_0(X_1)$), defined by $\phi(b) = b(f(x_1))$, for $b \in C_0(X_2)$ and $x_1 \in X_1$. If (C7) is satisfied, then f is proper and $\phi(C_0(X_2)) \subset C_0(X_1)$.

4.2 Topological Groups

Let Γ_i be a topological group and suppose that $G_i = \Gamma_i$. Then, $f : \Gamma_1 \longrightarrow \Gamma_2$ is an epimorphism and $H = Ker(f)$. By (C5), f is open. Therefore, Γ_1/H and Γ_2 are isomorphic topological groups, and thus f can be thought as the quotient map $f : \Gamma_1 \longrightarrow \Gamma_1/H$. Since $G_i^{(0)} = \{e_i\}$ (e_i is the unit of Γ_i), $f^{(0)}$ is trivial and (C7) is always satisfied. Moreover, H is a compact group by (C3). We define λ^i as a right Haar measure on Γ_i.

4.3 Transformation groups

Let Γ_i be a topological group, X_i a right Γ_i-space and $G_i = X_i \times \Gamma_i$. The groupoid structure of G_i is defined by $r_i(x_i, g_i) = x_i$, $s_i(x_i, g_i) = x_i g_i$ and $(x_i, g_i)(x_i g_i, g_i') = (x_i, g_i g_i')$, where we identify $G_i^{(0)}$ with X_i. Moreover, we suppose that there is a surjective map $f^{(0)} : X_1 \longrightarrow X_2$ and an epimorphism $\varphi : \Gamma_1 \longrightarrow \Gamma_2$, such that $f(x, g) = (f^{(0)}(x), \varphi(g))$ and $f^{(0)}(xg) = f^{(0)}(x)\varphi(g)$. By (C5), $f^{(0)}$ and φ are open maps. If $\Xi = Ker(\varphi)$, we identify Γ_1/Ξ with Γ_2. Then, φ is the quotient map. We have $H = X_1 \times \Xi$ and $Z = X_1 \times \Gamma_2$. The condition (C3) is satisfied if and only if the Ξ-action is proper. We define $\rho : Z \longrightarrow X_1$ and $\sigma : Z \longrightarrow X_2$ by $\rho(x_1, g_2) = x_1$ and $\sigma(x_1, g_2) = f^{(0)}(x_1)g_2$. The G_1-action and the G_2-action on Z are defined respectively by $(x_1 g_1^{-1}, g_1).(x_1, g_2) = (x_1 g_1^{-1}, g_1.g_2)$ and $(x_1, g_2).(f^{(0)}(x_1)g_2, g_3) = (x_1, g_2 g_3)$, for $(x_1, g_2) \in Z$, $(x_1 g_1^{-1}, g_1) \in G_1$ and $(f^{(0)}(x_1)g_2, g_3) \in G_2$.

5 Further research

If $(M_i, \mathcal{F}_i)$, $i \in \{1,2\}$, are foliated manifolds and $f : M_1/\mathcal{F}_1 \longrightarrow M_2/\mathcal{F}_2$ is a K-oriented morphism of leaf spaces (i.e., a correspondence between its holonomy groupoids), we look for an element of the Kasparov group $KK(C_r^*(G_1), C_r^*(G_2))$, where G_i is the holonomy groupoid of $(M_i, \mathcal{F}_i)$. At present, we study the case of transversely affine foliations, since their holonomy groupoids are Hausdorff. But holonomy groupoids are non Hausdorff in many cases of interesting foliated spaces. Thus, we intend to extend our results to non Hausdorff groupoids.

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