Mathematical and numerical aspects in fluid dynamics and turbulent flows. Fluid Equations under stochastic forces.

C. Gorria[1]

[1]Department of Applied Mathematics, Statistics and O.R. (UPV/EHU)

Master MORE, Bilbao, 2020

Index

Context

Physics:

- Astrophysics.
- Dynamical systems with noise.
- Fluid dynamics in the presence of random forces and turbulence.

Mathematics and finance:

- Probability theory.
- Prediction in finance models and time series.
- Traffic models and non-deterministic interaction between particles.

Applications

Applications:

- Stability analysis of the solutions of equations under stochastic interactions.
- Exploration of the boundary values in predictive models for different scenarios.
- Resistance conditions of structures under stochastic forces.

Motivation

Motivation by an example:

- Lets consider a particle leaving initially from the origin, $x(0) = 0$ and moving over the time, $x(t)$.
- Each time interval Δt the particle moves to the left $x_i = 0$ or to the right $x_i = 1$, both with probability $1/2$.
- The step length is Δx and the movements x_i are independent.

The question is to estimate the probability of being at time t at each possible position.

Random variable:

The random dynamics explained above can be expressed by a random variable $x_i \in \{0, 1\}$ with probabilities $p(0) = p(1) = 1/2$, mean $\langle x_i \rangle = 1/2$ and variance $\mathrm{var}(x_i) = (1/2)^2$.

Motivation

Random walk:

The position of the particle at any time $t = t_n = n\Delta t$ can be expressed by the sum of individual steps $S_n = \sum_{i=1}^{n} x_i$:

$$x(t) = S_n \Delta x - (n - S_n)\Delta x = (2S_n - n)\Delta x.$$

The itinerary of x along the time is a random walk.
Due to the independence between x_i, the mean and variance are

$$\mathrm{E}(x(t)) = \mathrm{E}\left(\sum_{i=1}^{n}(2x_i - 1)\Delta x\right) = \sum_{i=1}^{n}(2\frac{1}{2} - 1)\Delta x = 0,$$

$$\mathrm{Var}(x(t)) = \mathrm{E}\left(\sum_{i=1}^{n}(2x_i - 1)^2\Delta x^2\right) = n\Delta x^2$$

Motivation

Density function for the position of the particle:

Assuming $\frac{\Delta x^2}{\Delta t} = C$ constant, then $\mathrm{Var}(x(t)) = Ct$ and $\Delta x = \sqrt{\frac{Ct}{n}}$.
The random variable associated to the position $x(t)$ reaches

$$x(t) = (2S_n - n)\Delta x = \frac{(2S_n - n)\sqrt{Ct}}{\sqrt{n}} = \frac{(S_n - n/2)}{\sqrt{n/4}}\sqrt{Ct}$$

Taking into account that $\langle S_n \rangle = n/2$ and $\sigma_x = \sqrt{n/4}$, the probability function for $x(t)$ is calculated by the Central Limit Th.

$$P(a \leq x(t) \leq b) = \frac{1}{\sqrt{2\pi}} \int_{a/\sqrt{Ct}}^{b/\sqrt{Ct}} e^{-\frac{x^2}{2}} \mathrm{d}x = \frac{1}{\sqrt{2\pi Ct}} \int_a^b e^{-\frac{x^2}{2Ct}} \mathrm{d}x$$

Motivation

Central Limit Theorem:

Let $\{X_1, \ldots, X_n\}$ be a random sample of size n of identical and independent random variables with mean μ and variance σ^2, then the limit for $n \to \infty$ of arithmetic mean of the sample, $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, converges in distribution to a standard normal distribution in the following way:

$$\lim_{n\to\infty} \frac{S_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1).$$

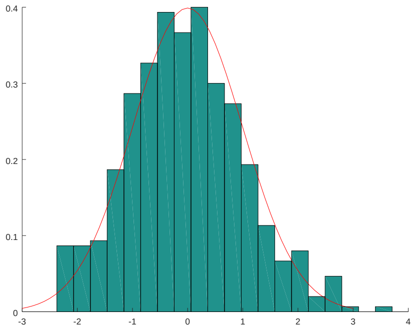
Observation:

For the validity of this theorem only the independence between the events governed by identically distributed random variables is crucial, no matter the random variables (uniformly distributed, exponential, Gamma, binomial, Poisson, chi square,...).

Example 1

Example 1. Central limit theorem:

The phenomenon explained by the central limit theorem can by observed for some statistical distribution like the χ^2



Exercise 1

Exercise 1. Central limit theorem:

Verify experimentally the assertion given by the central limit theorem for some of the statistical distributions given by octave/matlab in the pdf (probability density function) mode:
`https://octave.org/doc/v4.2.0/Distributions.html`

Octave/matlab functions used:

- Standard normal distribution: `stdnormal_pdf(x)`
- Graphics combination: `hold on ... hold off`
- Plotting histogram: `hist(vector,interv,norm);`

Example 2

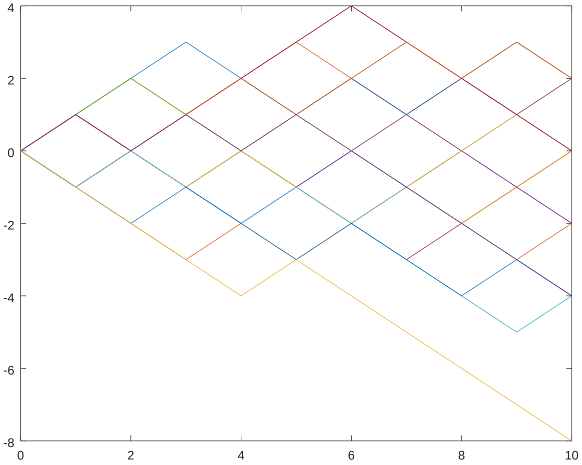
Example 2. Random walk:

Let's be a random walk that for each time instant from 0 to 10, $t_i = i$, $i = 0, \ldots, 10$it is given a step of length $\Delta x = 1$ either right or left with probability $p = 0.5$ each. We can implement a program in any language as octave/matlab in order to get a series of itineraries obtained for different simulations.

Octave/matlab functions used:

- Vectors: `zeros(nrow,ncol)`
- Statistics: `binornd(n,p)`, `hist(endpoint, 10, 1.0)`
- Numerical: `min(v)`, `max(v)`
- Graphics and evaluation: `plot()`, `subplot()`, `eval()`

Example 2



Exercise 2

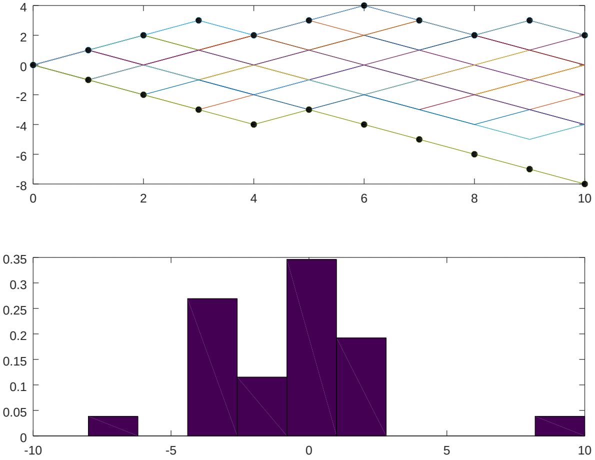
Exercise 2. Random walk:

Let's be a random walk that in each time instant $t_i = i$, $i = 0, \ldots, 10$ goes a step Δx either left or right with probability $p = 0.5$.

a) Build a code for calculating 100 different simulations of the random walk and store in a matrix by columns
b) Keep in a matrix with 3 columns the boundaries of the random walks: 1st column for the instant $t = t_0, t_1, \ldots$, 2nd column for the minimum value of every random walk and 3rd column for the maximum value of every random walk.
c) Draw a histogram of the final positions of the random walks (the frequencies corresponding to each position

In the following slide is a graphic with a sample of the results that have to be obtained.

Exercise 2



Brownian motion

Definition:

A Brownian motion or Wiener process is a random variable $W(t)$ defined for each time t characterized by

$$\begin{cases} \langle W(t) \rangle = 0, \\ \mathrm{Var}(W(t)) = t, \end{cases}$$

and the following conditions have to be reached:

- $W(0) = 0$,
- $W(t) - W(s) \in \mathcal{N}(0, t-s)$, $t \geq s \geq 0$,
- The following random variables are independent for any choose of $0 \leq t_1 \leq t_2 \ldots \leq t_n$:

$$\{W(t_2) - W(t_1), \quad W(t_3) - W(t_2), \ldots, W(t_n) - W(t_{n-1})\}$$

Brownian motion

Several positions at different times:

When a particle dynamics follow Brownian motion, the probability of being at different times in certain positions is calculated by a multiple integral of conditional probability density functions,

$$P(a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, \ldots, a_n \leq x_n \leq b_n)$$
$$= \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} g(x_1, t_1|0) \cdot g(x_2, t_2 - t_1|x_1) \cdots$$
$$g(x_n, t_n - t_{n-1}|x_{n-1}) \mathrm{d}x_n \cdots \mathrm{d}x_1.$$

Correlation:

It can be proven that the correlation between two measurements of the Brownian taken at different times is

$$\langle W(t)W(s)\rangle = t \wedge s = \min\{s, t\}.$$

White noise

Concept of white noise:

The white noise can be defined as an occasional impulse which has influence over the behavior of some magnitude. It emerges in different contests. The motion dynamics of particles in the presence of white noise are expressed by stochastic differential equations.

Definition (heuristic):

Although the differentiation for a Wiener process is not formally defined, this heuristic definition of the white noise can be accepted,

$$\dot{W}(t) = \frac{\mathrm{d}W(t)}{\mathrm{d}t} = \xi(t), \quad \text{where} \quad \langle \xi(t)\xi(s) \rangle = \delta_0(s-t).$$

White noise

Autocorrelation function:

$X : [0, \infty] \times \mathbb{R} \to \mathbb{R}^n$ stochastic process with $\langle X^2(t) \rangle < \infty$ then $r(s,t) = \langle X(t)X(s) \rangle$ is the autocorrelation function. When $r(s,t) = c(s-t)$ the stochastic process is called stationary. For example the white noise $\xi(t)$ is stationary with $c(\cdot) = \delta_0$.

Spectral density:

The Fourier transform of the autocorrelation function is known as the "spectral density". It gives a measure of the periodicity in time with which the stochastic process repeats a pattern.

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(t) e^{-i\lambda t} \mathrm{d}t.$$

White noise

Brownian motion by random Fourier series:

Let be $\{\varphi_n(t)\} \subset L^2(0,1)$ orthonormal function space and

$$\begin{cases} \xi(t) = \sum_{n=1}^{\infty} A_n \varphi_n(t), \\ A_n = \int_0^1 \xi(t)\varphi_n(t)\mathrm{d}t, \quad \text{random variables,} \end{cases}$$

then $\langle A_n, A_m \rangle = \delta_{nm}$ and the Brownian motion can be defined as

$$W(t) = \int_0^t \xi(s)\mathrm{d}s = \sum_{n=1}^{\infty} A_n \int_0^t \varphi_n(s)\mathrm{d}s.$$

Stochastic integrals

Stochastic differential Equation:

A stochastic differential Equation is composed by an deterministic component $\vec{b}(\vec{x},t)$ and a stochastic component $\vec{B}(\vec{x},t)$

$$\begin{cases} d\vec{X} = \vec{b}(\vec{X},t)dt + \vec{B}(\vec{X},t)d\vec{W}, \\ \vec{X}(0) = \vec{X}_0, \end{cases}$$

The evolution of $\vec{X}$ magnitude is governed by the integral formula

$$\vec{X}(t) = \vec{X}_0 + \int_0^t \vec{b}(\vec{X},s)ds + \int_0^t \vec{B}(\vec{X},s)d\vec{W}.$$

Questions about stochastic integrals:

How could we define the integrals $\int \vec{G}d\vec{W}$ and $\int \vec{W}d\vec{W}$ when $d\vec{W}$ is a vector Wiener process?

Stochastic integrals

Stochastic integral for scalar function:

For $g : [0,1] \longrightarrow \mathbb{R}, \quad g \in C[0,1]$ function it is defined

$$\int_0^1 g(t) \mathrm{d}W = -\int_0^1 g'(t) W(t) \mathrm{d}t$$

as random integral reaching the following conditions:

$$E\left(\int_0^1 g(t)\mathrm{d}W\right) = 0 \quad \text{and} \quad E\left(\left(\int_0^1 g(t)\mathrm{d}W\right)^2\right) = \int_0^1 g^2 \mathrm{d}t.$$

It can be proved that exists a function succession $\{g_n\} \subset C^1[0,1]$ that $\lim_{n\to\infty} \int_0^1 (g_n(t) - g(t))^2 \mathrm{d}W = 0$.

Stochastic integrals

Riemann sum for Wiener process:

Let's be $P^n = \{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = T\}$ a partition of $[0, T]$, where $|P^n| = \max_k |t_{k+1}^n - t_k^n|$ is the size of the net and $\tau_k = (1-\lambda)t_k^n + \lambda t_{k+1}^n$, $0 \leq \lambda \leq 1$ an interior point, then the Riemann sum corresponding to $\int_0^T W(t)\mathrm{d}W$ is defined by

$$R_n = R_n(P, \lambda) = \sum_{k=0}^{m_n - 1} W(\tau_k) \overbrace{(W(t_{k+1}) - W(t_k))}^{\approx \mathrm{d}W(\tau_k)}$$

Stochastic integrals

Lemma (Stochastic integrals):

The L^2 limit (by quadratic mean) of R_n when $(n \to \infty)$ is

$$\lim_{n\to\infty} R_n = \frac{W(T)^2}{2} + \left(\lambda - \frac{1}{2}\right) T$$

Ito's and Stratonovich's representations:

The chose of $0 \leq \lambda \leq 1$ determines the integration formula:

Ito ($\lambda = 0$): $$\int_0^T W(t)\mathrm{d}t = \frac{W(T)^2}{2} - \frac{T}{2},$$

Stratonovich ($\lambda = 1/2$): $$\int_0^T W(t) \circ \mathrm{d}t = \frac{W(T)^2}{2}.$$

Stochastic integrals

Lemma (Quadratic variation):

Let's be $P^n = \{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = T\}$ a partition of $[0, T]$ and $\lim_{n\to\infty} |P^n| = 0$, then the limit in L^2 of the sum of squares of the differentials of W is

$$\lim_{n\to\infty} \sum_{k=0}^{m_n-1} (W(t_{k+1}) - W(t_k))^2 = (b-a) = \int_a^b \mathrm{d}t.$$

Observation: This result is true for arbitrary interval, $[t, t+\Delta t]$. Here $\Delta W = \sqrt{\Delta t}$ and the rule of the stochastic term in the numerical integration of a SDE is much heavier than the deterministic term, for example $\Delta t = 0.01 \Rightarrow \Delta W = 0.1$.

Stochastic integrals

Proof:

$$Q_n = \sum_{k=0}^{m_n-1} \left(W(t_{k+1}^n) - W(t_k^n)\right)^2 \Rightarrow E\left((Q_n - (b-a))^2\right) =$$

$$\sum_{k=0}^{m_n-1} \sum_{j=0}^{m_n-1} E\left[\left(W(t_{k+1}^n) - W(t_k^n)\right)^2 - (t_{k+1}^n - t_k^n)\right]$$

$$\left[\left(W(t_{j+1}^n) - W(t_j^n)\right)^2 - (t_{j+1}^n - t_j^n)\right] = (*)$$

By the independence between cross uncorrelated terms and $E\left(\left(W(t_{k+1}^n) - W(t_k^n)\right)^2\right) = \text{var}\left(W(t_{k+1}^n) - W(t_k^n)\right) = t_{k+1}^n - t_k^n$, taking $y_k = \frac{W(t_{k+1}^n) - W(t_k^n)}{\sqrt{t_{k+1}^n - t_k^n}} \sim \mathcal{N}(0,1)$, now

$$(*) = \sum_{k=0}^{m_n-1} E\left((y_k^2 - 1)^2 (t_{k+1}^n - t_k^n)^2\right) \le c|P^n|(b-a) \to 0$$

Stochastic integrals

Definitions of continuity in a probability space:

(i) With probability 1 (w.p.1):

$$P\left(\{w \in \Omega : \lim_{s \to t} |X(s,w) - X(t,w)| = 0\}\right) = 1.$$

(ii) In mean square: $\lim_{s \to t} E\left(|X(s,w) - X(t,w)|\right)^2 = 0.$

(iii) In probability:

$$\lim_{s \to t} P\left(\{w \in \Omega : |X(s,w) - X(t,w)| \geq \epsilon\}\right) = 0, \ \forall \epsilon > 0.$$

(iv) In distribution: $\lim_{s \to t} F_s(x) = F_t(x), \ \forall x, \ F_t(x)$ continuous.

Implications: (i)$\Rightarrow$ (iii), (ii)$\Rightarrow$ (iii), (iii)$\Rightarrow$ (iv),

Itô's integral

Theorem: Properties of Itô's integral

$\forall a, b \in \mathbb{R}, \ \forall G, H \in L^2(0,T)$

1. $\int_0^T (aG + bH)\mathrm{d}W = a \int_0^T G\mathrm{d}W + b \int_0^T H\mathrm{d}W,$
2. $E\left(\int_0^T G\mathrm{d}W\right) = 0,$
3. $E\left(\left(\int_0^T G\mathrm{d}W\right)^2\right) = E\left(\int_0^T G^2\mathrm{d}t\right),$
4. $E\left(\int_0^T G\mathrm{d}W \int_0^T H\mathrm{d}W\right) = \int_0^T GH\mathrm{d}W,$

Itô's formula

Definition: Itô's formula for differentiation:

Let's be $X(\cdot) \in \mathbb{R}$ stochastic process that for $F \in L^1(0,T)$ and $G \in L^2(0,T)$ it reaches $X(r) = X(s) + \int_s^r F\mathrm{d}t + \int_s^r G\mathrm{d}W$, then the stochastic differential of X is $\mathrm{d}X = F\mathrm{d}t + G\mathrm{d}W$.

Theorem: Itô's formula for integration

For $Y(t) = u(X(t),t)$, where $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ and $\partial u/\partial t$, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ exist and are continuous and $\mathrm{d}X = F\mathrm{d}t + G\mathrm{d}W$, the following equalities are reached:

$$\mathrm{d}Y = \frac{\partial u}{\partial t}\mathrm{d}t + \frac{\partial u}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 u}{\partial X^2}G^2\mathrm{d}t =$$
$$\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial X}F + \frac{1}{2}\frac{\partial^2 u}{\partial X^2}G^2\right)\mathrm{d}t + \frac{\partial u}{\partial X}G\mathrm{d}W,$$
$$Y(r) = Y(s) + \int_s^r \left[\frac{\partial u}{\partial t} + \frac{\partial u}{\partial X}F + \frac{1}{2}\frac{\partial^2 u}{\partial X^2}G^2\right]\mathrm{d}t + \int_s^r \frac{\partial u}{\partial X}G\mathrm{d}W.$$

Itô's formula

Properties of Itô's formula:

(i) $\mathrm{d}(W^2) = 2W\mathrm{d}W + \mathrm{d}t$,

(ii) $\mathrm{d}(tW) = W\mathrm{d}t + t\mathrm{d}W$,

(iii) $\forall F_i \in L^1(0,T),\ G_j \in L^2(0,T)$ and $\begin{cases} \mathrm{d}X_1 = F_1\mathrm{d}t + G_1\mathrm{d}W, \\ \mathrm{d}X_2 = F_2\mathrm{d}t + G_2\mathrm{d}W, \end{cases}$

$$\mathrm{d}(X_1X_2) = X_2\mathrm{d}X_1 + X_1\mathrm{d}X_2 + G_1G_2\mathrm{d}t.$$

Itô's formula

Corollary: Generalized Itô's formula

$$\begin{cases} dX^{(i)} = F^{(i)}dt + G^{(i)}dW, \\ F^{(i)} \in L^1(0,T),\ G^{(i)} \in L^2(0,T), \\ u : \mathbb{R}^n \times [0,T] \to \mathbb{R},\ \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{continuous,} \end{cases}$$

Then

$$d\big(u(X^{(1)}, \dots, X^{(n)}, t)\big) =$$

$$\frac{\partial u}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} dX^{(i)} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} G^{(i)} G^{(j)} dt$$

Stratonovich Integral

Definition: Stratonovich Integral

For $P_n = \{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = T\}$ partition of $[0, T]$ interval with $|P^n| = \max_{k=1,\ldots,m_n} |t_k^n - t_{k-1}^n|$, the integral of the Brownian motion W in Stratonovich sense is

$$\int_0^T W \circ \mathrm{d}W =$$

$$\lim_{|P^n|\to 0} \sum_{k=0}^{m_n-1} W\left(\frac{t_{k+1}^n + t_k^n}{2}\right)\left[W(t_{k+1}^n) - W(t_k^n)\right] = \frac{W^2(T)}{2}.$$

Stratonovich Integral

Definition: solution of the Stratonovich Integral

Let $\vec{X}(\cdot)$ be the solution of the Stratonovich integral,

$$\vec{X}(t) = \vec{X}(t) + \int_0^t b(\vec{X}, s)\mathrm{d}s + \int_o^t \vec{B}(\vec{X}, s) \circ \mathrm{d}\vec{W},$$

then $\mathrm{d}\vec{X} = \vec{b}(\vec{X}, t)\mathrm{d}t + \vec{B}(\vec{X}, t) \circ \mathrm{d}\vec{W}$.

Theorem: chain rule derivative

For $\vec{X} \in \mathbb{R}^n$, $Y(t) = u(\vec{X}(t), t) \in \mathbb{R}$ and $\mathrm{d}\vec{X}$ as previous def,

$$\mathrm{d}Y = \frac{\partial u}{\partial t}\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \circ \mathrm{d}\vec{X}^i =$$

$$\left(\frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} b^i\right) \mathrm{d}t + \sum_{i=1}^{n}\sum_{k=1}^{m} \frac{\partial u}{\partial x_i} B^{ik} \circ \mathrm{d}W^k.$$

Conversion Itô ↔ Stratonovich

Conversion Itô ↔ Stratonovich. Integral form.

$$\left[\int_0^T \vec{B}(\vec{W},t)\circ \mathrm{d}\vec{W}\right]^i =$$

$$\left[\int_0^T \vec{B}(\vec{W},t)\mathrm{d}\vec{W}\right]^i + \frac{1}{2}\int_0^T \frac{\partial B^{ij}}{\partial x_j}(\vec{W},t)\mathrm{d}t.$$

Conversion Itô ↔ Stratonovich

Conversion Itô ↔ Stratonovich, differential form

Itô

$$\begin{cases} \mathrm{d}\vec{X} = \vec{b}(\vec{X},t)\mathrm{d}t + \vec{B}(\vec{X},t)\mathrm{d}\vec{W}, \\ \vec{X}(0) = \vec{X}_0. \end{cases}$$

Stratonovich

$$\begin{cases} \mathrm{d}\vec{X} = \left[\vec{b}(\vec{X},t) - \frac{1}{2}\vec{c}(\vec{X},t)\right]\mathrm{d}t + \vec{B}(\vec{X},t)\circ\mathrm{d}\vec{W}, \\ \vec{X}(0) = \vec{X}_0, \\ c^i(x,t) = \sum_{k=1}^{m}\sum_{j=1}^{n}\frac{\partial B^{ik}}{\partial x_j}(x,t)B^{jk}(x,t). \end{cases}$$

Numerical schemes for solving SDE

Initial value problem ODE

Given an initial value problem in general case $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$:

$$\begin{cases} \mathbf{x}' = f(t, \mathbf{x}(t)), \\ \mathbf{x}(t_0) = v \end{cases}$$

Given a discretization of the time interval $0 = t_0 < \cdots < t_n = T$, numerical methods try to approximate the behavior of the solution over the time $\mathbf{x}_i \approx \mathbf{x}(t_i)$ leaving from $\mathbf{x}_0$ and going ahead through the direction of the derivative $f(t, \mathbf{x}(t))$.

Explicit Euler method for ODE

The explicit Euler method is defined by the recursion formula:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \Delta t f(t_i, \mathbf{x}_i), \qquad \mathbf{x}_0 = v$$

Numerical schemes for solving SDE

Explicit Euler method for ODE

A little more sophisticated technique is the implicit Euler method. It estimates the progressing direction by using a weighted mean of the gradient, including the end-point of each iteration. The recursion formula is:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{\Delta t}{2}\left(f(t_i, \mathbf{x}_i) + f(t_{i+1}, \mathbf{x}_{i+1})\right), \qquad \mathbf{x}_0 = \nu.$$

The toll paid by implementing this improvement is the necessity of solving an equation (in general nonlinear) in each iteration. When the function $f(t, \mathbf{x})$ is smooth enough it is easily proved that for a Δt small enough, a fix-point scheme is convergent for finding the solution of the equivalent equation $\mathbf{x}_{i+1} = G(t_{i+1}, \mathbf{x}_{i+1})$.

Numerical schemes for solving SDE

Implicit Euler method

- Scalar case:

$$Y_{n+1} = Y_n + [\alpha a(\tau_{n-1}, Y_{n-1}) + (1-\alpha) a(\tau_n, Y_n)]\Delta +$$

$$b(\tau_n, Y_n)\Delta W, \quad \Delta = \tau_{n-1} - \tau_n, \quad \alpha \in [0,1].$$

- Vector case for the k'th component:

$$Y^k_{n+1} = Y^k_n + [\alpha A^k(\tau_{n-1}, Y_{n-1}) + (1-\alpha) A^k(\tau_n, Y_n)]\Delta +$$

$$\sum_{j=1}^{m} B^{kj}(\tau_n, Y_n)\Delta W^j, \quad A = (A^k),\ B = (B^{kj}).$$

Example 3

Example 3. Numerical schemes for solving ODE:

Let it be a second order differential equation with oscillatory type solutions:

second order equation — first order equation system

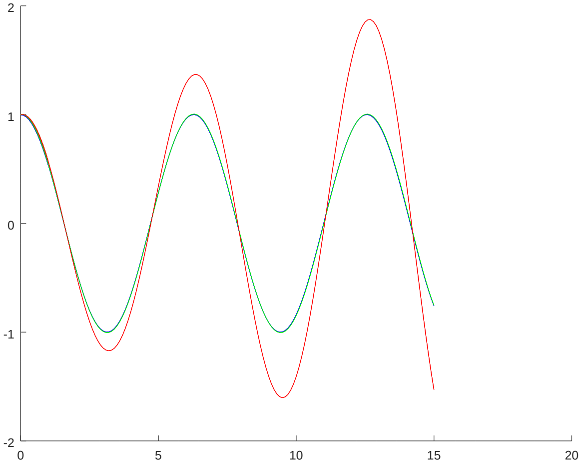
$$\left\{\begin{array}{l} x''(t)+x(t)=0, \\ x(0)=1,\ x'(0)=0. \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} x'(t)=y(t), \\ y'(t)=-x(t), \\ x(0)=1,\ y(0)=0. \end{array}\right.$$

Explicit and implicit Euler methods:

$$x_{n+1}=x_n+hf(t_n,x_n) \quad \text{Explicit Euler method}$$

$$x_{n+1}=x_n+\frac{h}{2}(f(t_n,x_n)+f(t_{n+1},x_{n+1})) \quad \text{Implicit Euler method}$$

Example 3



Exercise 3

Exercise 3: Explicit and implicit Euler methods

Solve numerically by explicit and implicit Euler methods the following ODE system corresponding to Lotka-Volterra equations:

$$\begin{cases} x'(t) = a_1 x(t) - a_2 x(t) y(t), \\ y'(t) = -b_1 y(t) + b_2 x(t) y(t), \end{cases}$$

$$a_1 = 0.4,\ a_2 = 0.018,\ b_1 = 0.8,\ b_2 = 0.023$$

$$x(0) = 30,\ y(0) = 4,\ tmin = 0,\ tmax = 20,\ h = 0.1$$

Draw the solutions in the plane (t, x) and in the phase plane (x, y).

Example 4

Brownian motion equations:

Let $X = X(t)$ symbolizes certain magnitude or the position of a particle with respect to time. In the following examples we will analyze some equations of Brownian motion of the following type:

$$\mathrm{d}X = b(t, X)\mathrm{d}t + \sigma(t, X)\mathrm{d}W$$

We will find analytically the solutions $X = f(t, W)$ for certain function f by comparison with the Itô's formula:

$$\mathrm{d}X = \left(\frac{\partial f(t, W)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t, W)}{\partial x^2}\right)\mathrm{d}t + \frac{\partial f(t, W)}{\partial x}\mathrm{d}W$$

In addition, we will also provide the numerical solutions given by the Euler implicit method (trapezoidal rule).

Example 4

Example 4: Brownian geometric motion

Let it be the stochastic equation:

$$\begin{cases} \mathrm{d}X = \mu X \mathrm{d}t + \sigma X \mathrm{d}W, \\ X(0) = x_0 > 0,\ \mu, \sigma > 0. \end{cases}$$

Assuming a solution of the type $X = f(t, W)$ and comparing its corresponding Itô's formula with the previous equation, we obtain:

$$\begin{cases} \mu f(t,W) = f_t(t,W) + \frac{1}{2} f_{xx}(t,W) \\ \sigma f(t,W) = f_x(t,W) \end{cases} \Rightarrow$$

$$X(t,W) = x_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}.$$

Exercise 4

Exercise 4: Ornstein-Uhlenbeck process

Let it be the stochastic equation proposed by Ornstein and Uhlenbeck for modelling the variation of the velocity of a particle under diffusion in short time interval:

$$\begin{cases} \mathrm{d}X = -\alpha X \mathrm{d}t + \sigma \mathrm{d}W, \\ X(0) = x_0 > 0, \ \mu, \sigma > 0. \end{cases}$$

Assuming a solution of the type $X = a(t)\left[x_0 + \int_0^t b(s)\mathrm{d}W(s)\right]$ and repeating the process of previous example the analytical solution is $X(t) = x_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)}\mathrm{d}W(s)$

Calculate the numerical solution of the above equation in the interval $t \in [0,2]$ by the implicit Euler method for: $\alpha = 1$, $\sigma = 1$, $x_0 = 3$ and $dt = 0.01$ and compare with the exact solution.

Exercise 4

Exercise 5: Brownian bridge

Let it be the stochastic equation:

$$\begin{cases} \mathrm{d}X = -\frac{X}{1-t}\mathrm{d}t + \mathrm{d}W, \\ X_0 = 0, \quad t \in [0,1). \end{cases}$$

Assuming a solution of the type $X = a(t)\left[x_0 + \int_0^t b(s)\mathrm{d}W(s)\right]$ and repeating the process of previous example the analytical solution is $X(t) = (1-t)\int_0^t \frac{1}{1-s}\mathrm{d}W(s)$

Calculate the numerical solution of the above equation in the interval $t \in [0, 0.99]$ by the implicit Euler method for $dt = 0.01$ and compare with the exact solution.

Equations of Fluid Dynamics

Navier-Stokes Eq. for incompressible fluids

$$\begin{cases} \dfrac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) = -\nabla p(\mathbf{x},t) + \nu\Delta\mathbf{u}(\mathbf{x},t), \\ \nabla\cdot\mathbf{u}(\mathbf{x},t) = 0. \end{cases}$$

Here ν is the viscosity and the pressure p is defined by the equation

$$\Delta p(\mathbf{x},t) = -\nabla\cdot\mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t).$$

Burgers equation

In absence of pressure, $p = 0$, and under the action of an external force $F(x,t)$ the expression can be reduced to the Burgers eq.

$$\frac{\partial}{\partial t}u(x,t) + \frac{1}{2}\frac{\partial}{\partial x}\left(u(x,t)^2\right) = \nu\frac{\partial^2}{\partial x^2}u(x,t) + F(x,t).$$

Equations of Fluid Dynamics

Hopf-Cole Transformation

In presence of viscosity, by the Hopf-Cole transformation

$$\begin{cases} \varphi(x,t) = e^{-(1/2v)\int u\,\mathrm{d}x} \Leftrightarrow u(x,t) = -2v\dfrac{\varphi_x}{\varphi}, \\ u(x,t) = -\dfrac{1}{2v}\dfrac{\partial}{\partial x}(\ln\varphi(x,t)), \end{cases}$$

the Burgers equation can be transformed in a parabolic equation

$$\frac{\partial}{\partial t}\varphi(x,t) = v\frac{\partial^2}{\partial x^2}\varphi(x,t).$$

This last equation has an exact solution for each initial condition $\varphi(x,t_0) = \varphi_0(x)$ than can be evaluated by Fourier series expansion or by an integral equation in terms of the heat-kernel.

Equations of Fluid Dynamics

Inviscid Burgers equation

In absence of viscosity and external forces, the analysis of the Burgers equation can be made by the characteristic method,

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) + \dfrac{1}{2}\dfrac{\partial}{\partial x}\left(u(x,t)^2\right) = 0, \quad x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{cases}$$

Now we find the trajectories along which the solutions are constant, $x(t)$ where $u(x(t),t) = u_0(x(0))$. In case of being constant over the time, the total derivatives with respect to time vanishes:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}u(x(t),t) = \tfrac{\partial}{\partial t}u(x(t),t) + x'(t)\tfrac{\partial}{\partial x}u(x(t),t) = 0 =$$

$$\tfrac{\partial}{\partial t}u(x(t),t) + u(x(t),t)\tfrac{\partial}{\partial x}u(x(t),t)$$

Equations of Fluid Dynamics

Characteristics

The conclusion is that $x'(t) = u(x(t), t)$ seams to be implicitly defined. But $u(x(t), t) =$ constant $\Rightarrow x'(t) = u(x(t), t) = u_0(x)$. These trajectories are named "*characteristic lines*". Thus, for any $x \in \mathbb{R}$ then $u(x, t) = u_0(x_0)$ for the value $x = x_0 + u_0(x_0)t$.

Shocks

Unfortunately the previous solutions can be multivalued, this means that shock can occur. It will happen for the first time when for two different $x_1 \neq x_2$ the following is reached

$$x(t) = x_1 + u_0(x_1)t = x_2 + u_0(x_2)t \Leftrightarrow t = -\frac{1}{\frac{u_0(x_2)-u_0(x_1)}{x_2-x_1}} > 0.$$

This shock will happen when $\exists x \in R : u_0'(x) < 0$.

Example 5

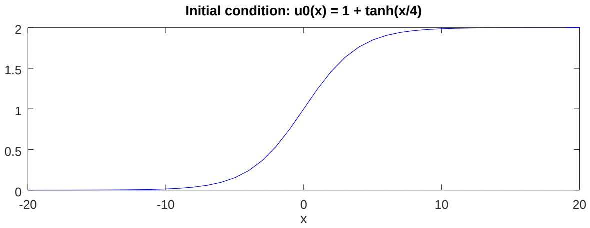
Numerical solution of the inviscid Burgers Equation by the method of the characteristic:

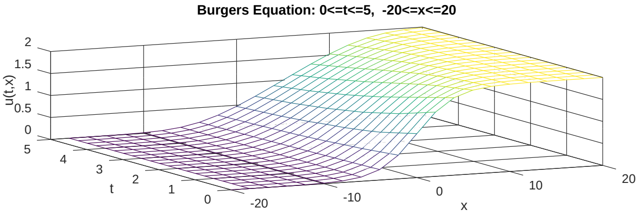
Let $u(x,t) = u_0(x_0)$ be the solution of the inviscid Burgers Equation for the value $x = x_0 + u_0(x_0)t$ without shocks at any (x,t). We will represent the solution for the initial value $u_0(x) = 1 + \tanh x/4$ in the spatial domain $-20 \leq x \leq 20$ over the time interval $0 \leq t \leq 5$.

Exercise 5. Shock in the inviscid Burgers Equation:

Let's repeat the numerical simulation of the inviscid Burgers equation for the initial condition $u_0(x) = e^{-\left(\frac{x}{5}\right)^2}$ in the spatial domain $-20 \leq x \leq 20$ over the time interval $0 \leq t \leq 10$.

Example 5


Initial condition: u0(x) = 1 + tanh(x/4)
2
1.5
1
0.5
0
-20
-10
0
10
20
x


Burgers Equation: 0<=t<=5, -20<=x<=20
u(t,x)
2
1.5
1
0.5
0
5
4
3
2
1
0
t
-20
-10
0
10
20
x

Example 5

Numerical solution of the inviscid Burgers Equation by the method of the characteristic:

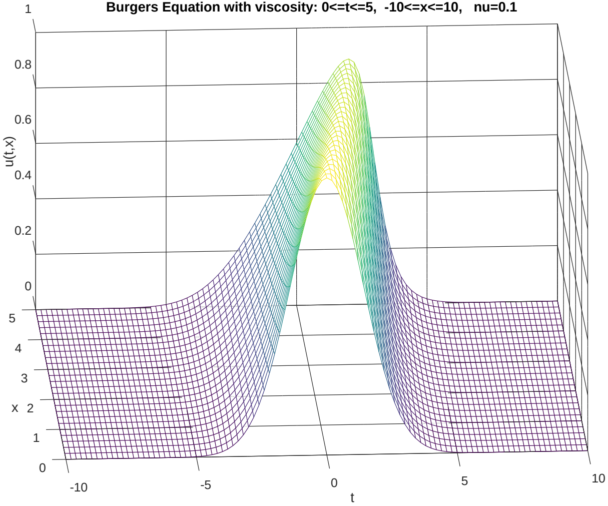
Let $u(x,t)$ be the solution of the Burgers Equation with viscosity,

$$\frac{\partial}{\partial t}u(x,t) + \frac{1}{2}\frac{\partial}{\partial x}\left(u(x,t)^2\right) = \nu\Delta u(x,t).$$

A technique to get an approximation $u_{n,k} \approx u(t_n, x_k)$ of the solution of the equation in the interval $[t_{min}, t_{max}] \times [x_{min}, x_{max}]$ over a discrete mesh (t_n, x_k) where $t_n = t_{min} + n\mathrm{d}t$ and $x_k = x_{min} + n\mathrm{d}x$, is to use finite differences to approximate derivatives in space and implicit Euler method to advance in time.

$$\frac{\partial u(t_n,x_k)}{\partial x} \approx \frac{u_{n,k+1} - u_{n,k-1}}{2\mathrm{d}x}, \quad \frac{\partial^2 u(t_n,x_k)}{\partial x^2} \approx \frac{u_{n,k+1} - 2u_{n,k} - u_{n,k-1}}{\mathrm{d}x^2}$$

Numerical solution of the Burgers equation with viscosity



Stochastic Burgers' equation

Stochastic Burgers' equation

The simplest way of considering a stochastic turbulent phenomenon in the dynamics modeled by the viscous Burgers' equation is to introduce a white noise η with intensity σ. Equivalently an stochastic force in terms of a related is a Brownian motion sheet W can be considered,

$$\frac{\partial}{\partial t}u(x,t) + \frac{1}{2}\frac{\partial}{\partial x}\left(u(x,t)^2\right) = \nu\frac{\partial^2}{\partial x^2}u(x,t) + \sigma(t,x,u)\frac{\partial^2 W}{\partial t \partial x}.$$

where $(t,x) \in [0,T] \times \mathbb{R}$ and u_0 is a nonrandom initial condition.