Eur. Phys. J. Special Topics **153**, 383–386 (2008)

DOI: 10.1140/epjst/e2008-00467-7

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Propagation of thermal waves in multilayered cylinders using the thermal quadrupole method

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Abstract. In this work the surface temperature oscillation of multilayered cylindrical samples which are uniformly heated by a modulated light beam is calculated using the thermal quadrupole method. This temperature is compared with that calculated in cylinders with continuously varying in-depth thermal conductivity. Following this theoretical approach, photothermal techniques can be used for the quantitative thermophysical characterization of hardened steel wires, tubes and nails.

1 Introduction

The quadrupole method is a unified exact method of representing linear systems. It has been applied in the framework of conductive transfer to calculate the surface temperature of flat multilayered samples [1]. Here we exploit this method to express the surface temperature of multilayered cylindrical samples in a compact manner. For the sake of simplicity we focus in this work on multilayered cylinders which are uniformly illuminated. We compare these results with those calculated in cylinders with in-depth varying thermal conductivity. This work opens the way to develop inverse procedures for the thermophysical characterization of hardened steel samples as wires, tubes and nails.

2 Theory

Let us first consider an infinite and opaque multilayered cylinder whose outer surface is uniformly illuminated by a light beam of intensity I_o modulated at a frequency f. It is made of N layers of different thicknesses and materials (see Fig. 1). The thermophysical properties of layer i are labeled by subindex i and its outer and inner radii by a_i and a_{i+1} respectively. According to the quadrupole method, in the absence of heat losses the temperature at the outer and inner surfaces of the cylinder are given by [2]:

$$T(a_1) = \frac{I_o}{2}\frac{A}{C} \qquad \text{and} \qquad T(a_{N+1}) = \frac{I_o}{2}\frac{1}{C}, \tag{1}$$

where the frequency dependent coefficients A and C are obtained from the following matrix product:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \prod_{i=1}^{N} \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}, \tag{2}$$

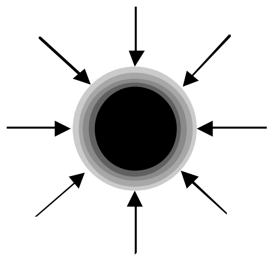

Fig. 1. Cross-section of the cylinder.

with

$$A_i = \left[H'_o(q_i a_{i+1}) J_o(q_i a_i) - J'_o(q_i a_{i+1}) H_o(q_i a_i)\right] / E_i,$$
$$B_i = \left[J_o(q_i a_{i+1}) H_o(q_i a_i) - H_o(q_i a_{i+1}) J_o(q_i a_i)\right] / E_i K_i q_i,$$
$$C_i = K_i q_i \left[H'_o(q_i a_{i+1}) J'_o(q_i a_i) - J'_o(q_i a_{i+1}) H'_o(q_i a_i)\right] / E_i,$$
$$D_i = \left[J_o(q_i a_{i+1}) H'_o(q_i a_i) - H_o(q_i a_{i+1}) J'_o(q_i a_i)\right] / E_i,$$

and

$$E_i = H'_o(q_i a_{i+1}) J_o(q_i a_{i+1}) - H_o(q_i a_{i+1}) J'_o(q_i a_{i+1}).$$

Here $q = \sqrt{i\omega/d}$ is the thermal wave vector, K is the thermal conductivity, d is the thermal diffusivity and J_o and H_o are the zeroth order of the Bessel and Hankel functions respectively. It is worth noting that Eqs. (1) and (2) are the same as those obtained for a homogeneous and semi-infinite slab whose front surface is periodically illuminated by a uniform light beam [1], but the coefficients A to D depend now on Bessel functions instead of on hyperbolic ones.

Let us now consider a solid cylinder of radius a_1 whose thermal conductivity varies radially as a polynomial: $K(r) = K_o + K_1 r + K_2 r^2 + \cdots + K_p r^p$, while the heat capacity ρc remains constant. As before, its surface is uniformly illuminated by a radial light beam of intensity I_o modulated at a frequency f. Temperature at any point of the cylinder is given by [3]:

$$T(r) = \frac{I_o}{2K(a_1)} \frac{\theta(r)}{\theta'(a_1)}, \tag{3}$$

where $\theta(r) = \sum_{n=0}^{\infty} c_n r^n$ is the ingoing thermal wave, and $\theta'(a_1)$ is its derivative evaluated at $r = a_1$. Coefficients c_n are given by: $c_o = 1$, $c_1 = 0$ and the remainder coefficients satisfy the following recurrence relation: $c_{n+2}(n+2)^2 K_o + c_{n+1}(n+2)(n+1)K_1 + c_n(n+2)nK_2 + c_{n-1}(n+2)(n-1)K_3 + \ldots + c_{n-p+2}(n+2)(n-p+2)K_p + i\omega\rho c c_n = 0$. In this last equation coefficients with negative subindex are null. For instance $c_2 = -\frac{i\omega\rho c}{4K_o}$ and $c_3 = \frac{i\omega\rho c K_1}{6K_o^2}$.

3 Numerical simulations and discussion

In Fig. 2 we show the frequency dependence of the amplitude and phase of the surface temperature of a cylinder of radius 1 mm, whose thermal conductivity varies linearly from $K = 10\,\mathrm{Wm^{-1}K^{-1}}$ at the center to $K = 5\,\mathrm{Wm^{-1}K^{-1}}$ at the surface ($K = 10 - 5 \times 10^3\, r$), while the heat capacity $\rho c = 2.5 \times 10^6$ S.I. remains constant. In all the simulations the temperature is normalized to a homogeneous cylinder of the same radius and heat capacity but with a constant thermal conductivity $K = 10\,\mathrm{Wm^{-1}K^{-1}}$. The continuous line is the exact calculation performed using Eq. (2) and is compared with calculations performed in a multilayered cylinder with layers of uniform thickness: Dashed and dotted lines represent a 6-layer and a 10-layer cylinder respectively. As can be seen results differ at high frequencies and a high number of layers would be required to fit the linear behavior.

In order to reduce the number of layers needed to simulate the linear behavior we use layers of different thickness, thinner outside and thicker inside the cylinder. In Fig. 3 we show the

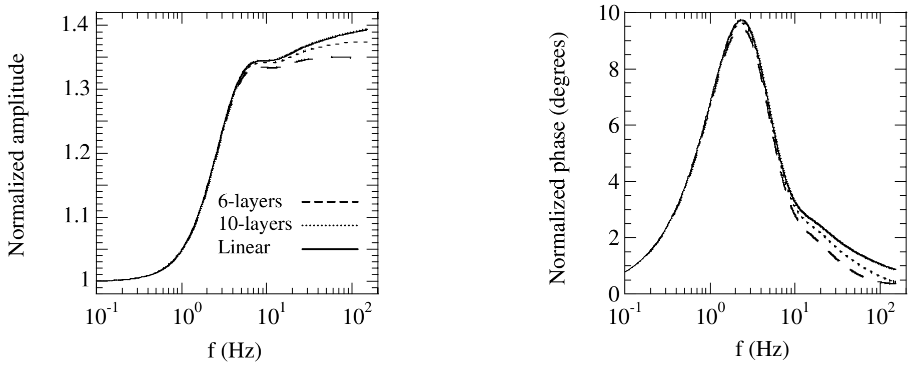

Fig. 2. Amplitude and phase of the surface temperature for a linearly varying in-depth thermal conductivity. Layers of the same thickness are used.

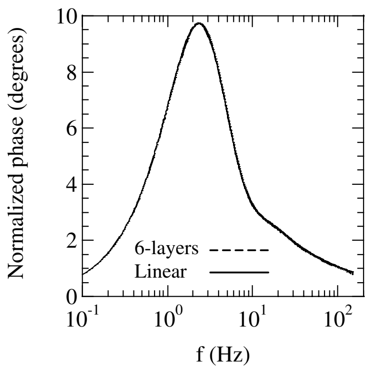

Fig. 3. The same as Fig. 2 but using layers of different thicknesses.

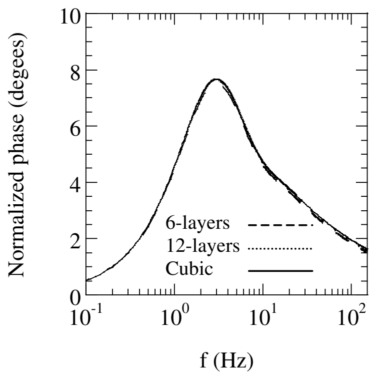

Fig. 4. Phase of the surface temperature for a cubically varying in-depth thermal conductivity. Layers of different thicknesses are used.

phase of the surface temperature for the same cylinder as in Fig. 2 but using layers whose thicknesses are multiple of the outer one. In this case with only 6 layers the linear behavior is perfectly reproduced.

Finally in Fig. 4 we show the phase of the surface temperature as a function of the modulation frequency for a cylinder of the same size and heat capacity as before but whose thermal conductivity varies as $K = 10 - 5 \times 10^9 r^3$. This means that at the center and at the surface of the cylinder the thermal conductivities are the same as before, $K = 10\,\mathrm{Wm^{-1}K^{-1}}$ and $K = 5\,\mathrm{Wm^{-1}K^{-1}}$ respectively, but K varies slowly at the inner part of the cylinder and abruptly at the outer part. Unlike in the case of linear thermal conductivity now 12 layers of increasing thickness are needed to reproduce the cubic behavior. This fact is related to the sharp change of thermal conductivity at the outer part of the cylinder. On the other hand, when comparing the phases for the linear and the cubic conductivities, this last one produces a smaller phase contrast at low frequencies ($\sim$8° against $\sim$10° at the peak) but a higher contrast at high frequencies.

The quadrupole method is useful to calculate the surface temperature of a multilayered cylinder. Using this method it has been demonstrated that, with a few number of layers of different thicknesses, the surface temperature of a cylindrical sample with continuously varying in-depth thermal conductivity can be calculated with high accuracy. Consequently the quadrupole method allows the development of inverse procedures, with a low number of unknowns, to characterize the thermal properties of hardened steel wires, tubes and nails, in which the thermal conductivity varies in-depth depending on the type of the applied thermal treatment.

This work has been supported by the Ministerio de Educación y Ciencia through research grant No. MAT2005-02999.

References

1. D. Maillet, S. André, J.C. Batsale, A. Degiovanni, C. Moyne, *Thermal Quadrupoles* (John Wiley & Sons, 2000)
2. A. Salazar, R. Celorrio, J. Appl. Phys. **100**, 113535 (2006)
3. A. Salazar, R. Celorrio, J. Appl. Phys. (to be published)