A thermal paradox: which gets warmer?

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Abstract
In this paper we address a common misconception concerning the thermal behaviour of matter, namely that the front surface of a very thin plate, uniformly illuminated by a constant light beam, reaches a higher temperature than the front surface of a very thick slab made out of the same material. We present analytical solutions for the temperature rise above the ambient of thin and thick samples, after a heating lamp is switched on. It is shown that the temperature rise at the illuminated surface of the thick sample is twice that of the thin plate. However, the temperature rise of the thin plate is much faster (minutes) than the heating of the thick one (hours). This explains why our intuition, which learns from what happens at the very beginning of the process, leads us to the feeling that the front surface of a thin plate will get warmer than the illuminated surface of a thick slab, i.e. the above-mentioned misconception.

1. Introduction

Let us start with the following question: If a light beam of constant intensity illuminates uniformly the surface of a thin plate or of a thick slab of an opaque material, which one will get warmer? Most physics students (and even physics teachers before thinking twice) would answer that the surface of a thin plate would reach a higher temperature. This is what intuition tells us. However, what really happens is the opposite; the front surface of a thick slab will become warmer. Actually, the temperature rise of a very thick slab is twice that of a very thin plate. The reason for this is that thermal equilibrium is reached between the inwards light flux and the outwards heat losses by conduction and convection to the surrounding air, as well as by radiation. As the thinner plate has two surfaces from which heat can be lost, the final temperature rise will be smaller. Then, what is the origin of the above misconception? The reason is that reaching thermal equilibrium takes a long time (hours) while our intuition draws
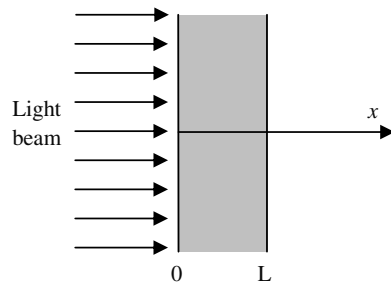
Figure 1. Diagram of an opaque plate uniformly illuminated by a light beam.

its conclusions from what happens during the very first seconds, at most minutes, rather than hours.

The intended reader of this paper is an undergraduate student in physics who has already passed through first courses of classical thermodynamics and who is familiar with the use of the Laplace transform to solve differential equations.

The aim of this work is to show, using undergraduate physics and mathematics, that the temperature of the illuminated surface of a thin plate increases much faster than that of a very thick slab, although the final temperature is higher for the latter. We present analytical solutions of the temperature evolution of both a very thick slab and a very thin plate, after switching on a light lamp. In addition, we show the results of experiments performed using the equipment available in our educational laboratories, confirming the theoretical predictions. This paper is intended to serve the pedagogical purpose of using a puzzle question to attract the attention of physics students to heat propagation. Moreover, the experimental verification can easily be implemented as a clarifying laboratory demonstration for university students.

2. Theoretical model

Figure 1 shows the diagram for an opaque plate of thickness L uniformly illuminated by a light beam. The intensity of this light beam is zero for $t < 0$ and is I_o for $t > 0$ (i.e. a Heaviside function). To obtain the time evolution of this sample we need to solve the one-dimensional diffusion equation. In the absence of heat sources the equation is written as [1]

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{D}\frac{\partial T}{\partial t} = 0, \tag{1}$$

where $D = K/(\rho c)$ is the thermal diffusivity, K is the thermal conductivity, ρ is the density and c is the specific heat. Throughout this work T represents the temperature rise above the ambient. In transient problems it is useful to work in the Laplace space [2]. In this way, the Laplace transform of the diffusion equation is

$$\frac{d^2 \bar{T}}{dx^2} - q^2 \bar{T} = 0, \tag{2}$$

where $q^2 = s/D$, s being the Laplace variable, and $\bar{T}$ is the Laplace transform of the temperature

$$\bar{T}(x, s) = \int_0^\infty e^{-st} T(x, t)\, dt. \tag{3}$$

Equation (2) is a very well known differential equation in physics, whose general solution is written as

$$\bar{T}(x, s) = A\, e^{qx} + B\, e^{-qx}, \tag{4}$$

where A and B are constants to be obtained from the boundary conditions, which in our problem are the heat fluxes at the front and rear surfaces:

$$\Phi(x = 0) = -K \left. \frac{dT}{dx} \right|_{x=0} = -h_f T(x = 0) + \begin{cases} 0 & \text{for} \quad t < 0 \\ I_o & \text{for} \quad t > 0 \end{cases} \tag{5a}$$

$$\Phi(x = L) = -K \left. \frac{dT}{dx} \right|_{x=L} = h_r T(x = L), \tag{5b}$$

where hT accounts, in a phenomenological way, for heat losses by convection and radiation, h_f and h_r being the heat transfer coefficients at the front and rear surfaces, respectively. Heat losses by conduction to the surrounding gas are negligible due to the very low thermal conductivity of the air and do not need to be included in the boundary conditions. The Laplace transforms of equations (5) are

$$\bar{\Phi}(x = 0) = -K \left. \frac{d\bar{T}}{dx} \right|_{x=0} = -h_f \bar{T}(x = 0) + \frac{I_o}{s} \tag{6a}$$

$$\bar{\Phi}(x = L) = -K \left. \frac{d\bar{T}}{dx} \right|_{x=L} = h_r \bar{T}(x = L), \tag{6b}$$

where I_o/s is the Laplace transform of the illumination (a Heaviside function) [3]. By substituting equation (4) into equations (6) the Laplace transform of the temperature at any point of the plate is obtained:

$$\bar{T}(x, s) = \frac{I_o}{s} \frac{1}{Kq} \frac{(1 + H_r)\, e^{qL}\, e^{-qx} + (1 - H_r)\, e^{-qL}\, e^{qx}}{(1 + H_f)(1 + H_r)\, e^{qL} - (1 - H_f)(1 - H_r)\, e^{-qL}}, \tag{7}$$

where $H_f = \frac{h_f}{Kq}$ and $H_r = \frac{h_r}{Kq}$. By applying the inverse Laplace transform to this last equation the time evolution of the slab temperature can be obtained. Unfortunately there is no analytical solution for it, so a numerical inversion must be performed, using, for instance, the Stehfest algorithm [4, 5]. According to the pedagogical aim of this paper we look for analytical solutions corresponding to the two extreme cases of practical interest for the problem we are dealing with: the surface temperature of (a) a very thick slab and (b) a very thin plate.

2.1. A very thick sample

If the sample is thick ($e^{-qL} \approx 0$), the Laplace transform of the temperature at both surfaces, where the temperature is usually sensed, is obtained from equation (7) and reduces to

$$\bar{T}(x = 0, s) \approx \frac{I_o}{s} \frac{1}{Kq} \frac{1}{(1 + H_f)} = \frac{I_o}{e} \frac{1}{s\left(\sqrt{s} + \frac{h_f}{e}\right)}, \tag{8a}$$

$$\bar{T}(x = L, s) \approx 0, \tag{8b}$$

where $e = \sqrt{\rho c K} = K/\sqrt{D}$ is the thermal effusivity [6, 7]. Their inverse Laplace transform gives[1]

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[1] This inverse Laplace transform does not appear in usual tables, but it can be obtained using Mathematica.

$$T(x = 0, t) \approx \frac{I_o}{h_f} \left[1 - e^{(h_f/e)^2 t} \, \text{Erfc}\left(\frac{h_f \sqrt{t}}{e} \right) \right], \tag{9a}$$

$$T(x = L, t) \approx 0, \tag{9b}$$

where Erfc is the complementary error function. Equation (9a) indicates that the asymptotic value of the front surface temperature, I_o/h_f, depends on the intensity of the light and on the coefficient of heat losses by convection and radiation. On the other hand, the time evolution of the surface temperature is governed by the ratio h_f/e. Regarding the rear surface temperature, it remains close to zero.

2.2. A very thin sample

If the sample is thin ($e^{\pm qL} \approx 1 \pm qL$), the Laplace transform of temperature at both surfaces is obtained from equation (7) and is written as

$$\bar{T}(x = 0) \approx \bar{T}(x = L) \approx \frac{I_o}{s} \frac{1}{Kq} \frac{1}{H_f + H_r + qL} = \frac{I_o}{\rho cL} \frac{1}{s\left(\sqrt{s} + \frac{h_f + h_r}{\rho cL}\right)}. \tag{10}$$

Their inverse Laplace transform gives[1]

$$T(x = 0, t) \approx T(x = L, t) \approx \frac{I_o}{h_f + h_r} \left(1 - e^{-\frac{h_f + h_r}{\rho cL} t} \right), \tag{11}$$

indicating that temperature is the same at both surfaces. In particular, their asymptotic value, $I_o/(h_f + h_r)$, depends on the intensity of the light and on the coefficient of heat losses by convection and radiation at the front and at the rear surfaces. On the other hand, the time evolution of the surface temperature is governed by the ratio $(h_f + h_r)/(\rho cL)$.

2.3. Discussion

By comparing equations (9a) and (11) it is clear that the asymptotic temperature at the front surface of a thick sample is higher than the corresponding value of a thin plate of the same material. In fact, for a sample surrounded by air $h_f \approx h_r$ and therefore the final front surface temperature of a very thick sample is actually twice the temperature of the thin sample: $T_{\text{thick}}(x = 0) \approx 2T_{\text{thin}}(x = 0)$. On the other hand, using the large argument expansion of the complementary error function $\text{Erfc}(z) \approx e^{-z^2}/(z\sqrt{\pi})$ (see equation (7.1.23) in [3]) analytical expressions of the temperature of a thick sample and of a thin sample in the long time limit are obtained, respectively:

$$T_{\text{thick}}(x = 0, t \to \infty) \approx \frac{I_o}{h_f} \left[1 - \frac{e}{h_f \sqrt{\pi t}} \right] \tag{12a}$$

$$T_{\text{thin}}(x = 0, t \to \infty) \approx T_{\text{thin}}(x = L, t \to \infty) \approx \frac{I_o}{h_f + h_r}, \tag{12b}$$

indicating that the thin sample has already reached the asymptotic final value while the surface temperature of the thick one is still rising.

In figure 2 we show the front surface temperature rise as a function of time for two samples made of the same bad thermal conductor (a typical polymer) $D = 0.1$ mm^2 s^{-1}, $K = 0.3$ W m^{-1} K^{-1}, $h_f = h_r = 10$ W m^{-2} K^{-1}, illuminated with the same light intensity $I_o = 100$ W m^{-2}, but having different thicknesses: $L = 1$ mm (plot of equation (9a)) and $L = 50$ cm (plot of equation (11)). As expected, (a) the asymptotic temperature for the thick
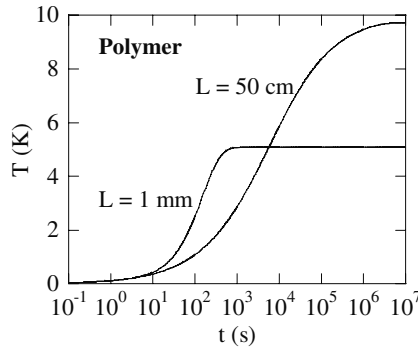
Figure 2. Calculated temperature rise of the illuminated surface as a function of time for two samples made of the same polymer, $D = 0.1$ mm^2 s^{-1}, $K = 0.3$ W m^{-1} K^{-1}, $h_f = h_r = 10$ W m^{-2} K^{-1}, illuminated with the same light intensity $I_o = 100$ W m^{-2}, but having different thicknesses: $L = 1$ mm and $L = 50$ cm.

sample is twice the corresponding value of the thin sample, but (b) the temperature rise of the thin plate is much faster than that of the thicker one. In fact, during the first minutes the temperature of the thin plate remains higher than that of the thicker one. It takes about an hour for the thick slab to beat the thin sheet and even many days to reach the asymptotic value. This result can explain the origin of our misconception, since who in the real world (i.e. outside the lab) waits so long to get the right conclusion?

It is worth mentioning the origin of the value of the heat transfer coefficient, $h = 10$ W m^{-2} K^{-1}, that we have used. For a small temperature rise above the ambient ($T_{amb} \approx 300$ K) the radiative heat transfer is $h_{rad} \approx 4\varepsilon\sigma T_{amb}^3$, where ε is the surface emissivity and σ is the Stefan–Boltzmann constant [8]. Therefore, for a black sample with an emissivity close to 1, $h_{rad} \approx 6$ W m^{-2} K^{-1}. On the other hand, for a vertical plate surrounded by air at room temperature and at room pressure, the convective transfer coefficient is $h_{conv} \approx 1.5(\Delta T/b)^{0.25}$, where ΔT is the temperature difference between the sample surface and the surrounding air and b is the plate height [9]. For instance, by taking $\Delta T = 10$ K and $b = 20$ cm, then $h_{conv} \approx 4$ W m^{-2} K^{-1}. Accordingly, a realistic value of the combined heat transfer coefficient is $h = h_{conv} + h_{rad} \approx 10$ W m^{-2} K^{-1}.

Finally, the validity of equations (9) and (11) deserves some comments. From numerical calculations, by comparing the general solution given by the Laplace transform of equation (7) and the approximation for thin samples given by equation (11), we have found the following condition for the sample thickness to verify the very thin approximation:

$$L_{thin} \leqslant 0.02 \frac{\lambda K}{h}, \tag{13}$$

where λ is the error (%) in the asymptotic temperature and $h = h_f = h_r$. Note that this expression is independent of D. In contrast, we have not found a simple expression for the validity of the very thick approximation given by equation (9). However, as an approximate rule of thumb, we can say that equation (9) is valid if the thickness of the sample is at least 500 times the value given by equation (12): $L_{thick} \geqslant 500 L_{thin}$. For instance, for the polymer used in figure 2, assuming $h = 10$ W m^{-2} K^{-1} and accepting an error $\lambda = 2\%$, the thickness of a thin sample should be smaller than 1 mm and, for a thick sample, should be larger than 50 cm. In the case of a very good thermal conductor like copper ($K = 400$ W m^{-1} K^{-1}), the limits would be $L_{thin} \leqslant 1.5$ m and $L_{thick} \geqslant 750$ m respectively. This indicates that for good
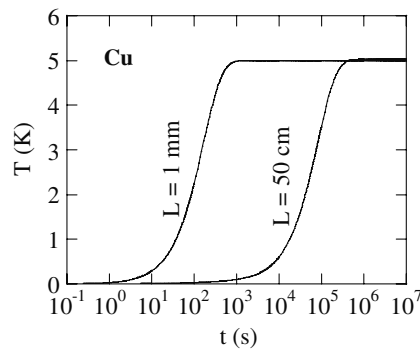
Figure 3. Calculated temperature rise of the front surface as a function of time for two copper samples, $D = 120 \text{ mm}^2 \text{ s}^{-1}$, $K = 400 \text{ W m}^{-1} \text{ K}^{-1}$, $h_f = h_r = 10 \text{ W m}^{-2} \text{ K}^{-1}$, illuminated with the same light intensity $I_o = 100 \text{ W m}^{-2}$, but having different thicknesses: $L = 1$ mm and $L = 50$ cm.

thermal conductors (metals, alloys, some ceramics, etc) the thick condition is very difficult to fulfil. In figure 3 we show the front surface temperature rise for two copper samples ($D = 120 \text{ mm}^2 \text{ K}^{-1}$, $K = 400 \text{ W m}^{-1} \text{ K}^{-1}$, $h_f = h_r = 10 \text{ W m}^{-2} \text{ K}^{-1}$) of the same thicknesses as that of the polymer shown in figure 2, illuminated with the same light intensity $I_o = 100 \text{ W m}^{-2}$. For this material even the sample of $L = 50$ cm is thin, and therefore the final temperature at the front surface is the same in both cases according to equation (11). However, the temperature rise of the thinner sample is very much faster than that of the thicker one. As can be seen, the 1 mm thick plate reaches the final temperature when the temperature of the 50 cm slab starts rising.

Anyway, it is worth mentioning that the question we raised in the introduction only refers to the front surface temperature, which is the one directly accessible. If we considered the temperature of the whole sample the answer would be different. A thin sample reaches a uniform temperature, while the temperature of a thick slab varies in depth: it reaches a higher temperature than the thin plate at the front surface, decreases exponentially as we go deeper inside the material and is zero (ambient) at the rear surface. Accordingly, the average temperature of a thick sample is lower than the temperature of a thin plate.

3. Experimental results and conclusion

To validate the results of the previous section we have prepared two black plasticine samples of about 10 cm × 10 cm. Their thicknesses are $L \approx 1$ mm and $L \approx 10$ cm. The thicker one has been surrounded by a thick thermally isolating layer in order to reduce lateral heat losses as much as possible. We used a Xe lamp of 150 W as a light source. Its beam was defocused by means of a spherical lens of 10 cm focal length to illuminate the whole sample surface. We used an infrared video camera to measure the sample surface temperature as a function of time. In figure 4 we show the temperature rise of the two plasticine samples as a function of time after the Xe lamp is switched on. As theoretically predicted the temperature of the thin plasticine plate rises faster than that of the thick one, and remains warmer for almost 1 h. After some hours the temperature of the thick sample clearly rises above the thin plate's temperature, although the asymptotic value is not reached even after 9 h. As predicted theoretically, some days would be needed to achieve thermal equilibrium in a thick sample.
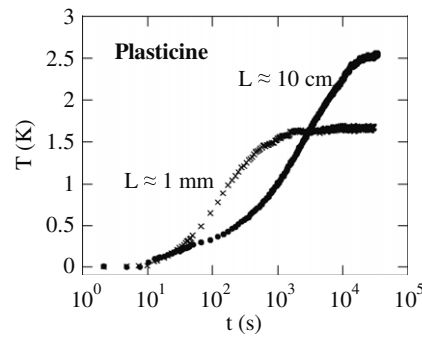
Figure 4. Experimental temperature history of the front surface of two plasticine samples of very different thicknesses illuminated by a Xe lamp.

Note that in figure 4 the asymptotic value of the 1 mm thick plasticine plate is not completely flat. This is due to the fact that the lamp driver heats the laboratory, and therefore this small slope in the asymptotic value of the thin plate accounts for the rise of the ambient temperature.

We have demonstrated theoretically and verified experimentally that the front surface of a thick sample reaches a higher equilibrium temperature than a thin one when they are illuminated by a lamp of constant intensity. Our intuition leads us to draw the opposite and wrong conclusion because the temperature rise of the thin sheet is very much faster than that of the thick one, and only after some hours the real asymptotic value is reached. The experiments we have performed can easily be implemented as automated student demonstrations, which might clarify the concept of heat losses. We expect this paper to be useful for physics lecturers to attract the attention of physics students on heat propagation.

Acknowledgment

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