Singular Solutions for the Uehling Uhlenbeck Equation

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Abstract. In this paper we prove the existence of solutions of the Uehling-Uhlenbeck equation that behave like $k^{-7/6}$ as $k \to 0$. From the physical point of view, such solutions can be thought as particle distributions in the space of momentum having a sink (or a source) of particles with zero momentum. Our construction is based on the precise estimates of the semigroup for the linearized equation around the singular function $k^{-7/6}$ that we obtained in [5].

Key words. Kinetic equation, singular solution.

1 Introduction

We consider the initial value problem asociated to the Uehling Uhlenbeck (U-U) equation:

$$\frac{\partial f}{\partial t}(t,k) = Q(f)(t,k) \qquad (1.1)$$

$$f(0,k) = f_0(k) \qquad (1.2)$$

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where

$$Q(f)(k_1) = \int_{D(k_1)} W(k_1,k_2,k_3,k_4)\, q(f) dk_3 dk_4 \tag{1.3}$$

$$q(f) = f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \tag{1.4}$$

$$D(k_1) \equiv \{(k_3,k_4) : k_3 + k_4 \geq k_1\} \tag{1.5}$$

$$W(k_1,k_2,k_3,k_4) = \frac{\min\left(\sqrt{k_1},\sqrt{k_2},\sqrt{k_3},\sqrt{k_4}\right)}{\sqrt{k_1}} \tag{1.6}$$

$$k_2 = k_3 + k_4 - k_1. \tag{1.7}$$

We are interested in solutions which are singular at the origin, and more particularly behaving like $k^{-7/6}$ as $k \to 0$. The choice of this specific asymptotics is due to the fact that, as it is proved by A. M. Balk, V. E. Zakharov in [3], $A\, k^{-7/6}$ is a stationary solution of the equation

$$\widetilde{Q}(f)(k_1) = 0 \tag{1.8}$$

for all $A > 0$, where

$$\widetilde{Q}(f)(k_1) = \int_{D(k_1)} W(k_1,k_2,k_3,k_4)\, \tilde{q}(f) dk_3 dk_4 \tag{1.9}$$

$$\tilde{q}(f) = f_3 f_4 (f_1 + f_2) - f_1 f_2 (f_3 + f_4). \tag{1.10}$$

Notice that $\tilde{q}(f)$ contains the largest terms of $q(f)$ for large values of f. We therefore consider initial data f_0 which also behave in that way at the origin.

1.1 Physical Motivation

Let us define

$$\rho_0 := \int_0^\infty \frac{\sqrt{k}\, dk}{e^k - 1} = \frac{\sqrt{\pi}}{2}\zeta(3/2) \tag{1.11}$$

where ζ is the classical Riemann's zeta function.

The U-U equation describes a dilute gas of Bose particles. It has a one parameter family of steady states $\mathcal{B}_\rho$ characterized by their total density $\rho > 0$ as follows:

- If $0 < \rho \leq \rho_0$ then

$$\mathcal{B}_\rho(k) \equiv F_\mu(k) := \frac{1}{e^{k+\mu} - 1} \quad \text{where} \quad \rho = \int_0^\infty \frac{\sqrt{k}\, dk}{e^{\mu+k} - 1},\ \mu \geq 0. \tag{1.12}$$

- If $\rho > \rho_0$

$$\mathcal{B}_\rho(k) \equiv \frac{1}{e^k - 1} + (\rho - \rho_0)\frac{\delta_0}{\sqrt{k}} \tag{1.13}$$

Notice that in both cases $\int_0^\infty \mathcal{B}_\rho(k)\sqrt{k}\,dk = \rho$. The solutions $\mathcal{B}_\rho(k)$ in (1.12) are the classical Bose-Einstein equilibrium distributions if $\mu > 0$ and the Planck distribution if $\mu = 0$. On the other hand, the solutions (1.13) are the classical distributions that describe the thermal equilibrium of a family of bosons with Bose-Einstein condensate of particles having zero momentum.

In this paper we construct solutions of (1.1)-(1.7) that behave like $k^{-7/6}$ near the origin. The physical meaning of such asymptotics is that these particle distributions have a nonzero flux of particles towards the origin (cf. [3, 7, 10]). More precisely, the asymptotics

$$f(t,k) \sim a(t)\,k^{-7/6}, \qquad \text{as } k \to 0 \tag{1.14}$$

means that the rate gain of particles towards the particles with zero momentum is

$$\lim_{K\to 0} \frac{d}{dt}\left(\int_{|k_1|\le K} \sqrt{k_1}\, f(k_1,t)\,dk_1\right) = -\frac{(a(t))^3}{3} U'(7/6) \tag{1.15}$$

where

$$U(\nu) := \int_{D(1)} a(\xi_2,\xi_3,\xi_4)\,d\xi_3\,d\xi_4$$

and

$$a(\xi_2,\xi_3,\xi_4) := \left[W(\xi_1,\xi_2,\xi_3,\xi_4)\,q(\xi^{-\nu})\right]_{|\xi_1=1}$$

There are several different ways of deriving (1.15). One possibility is to make a careful counting of the number of particles leaving the region $\{k : |k| \le \delta\}$ towards $\{k : |k| > \delta\}$, as well as the particles entering into $\{k : |k| \le \delta\}$ from $\{k : |k| > \delta\}$, under the assumption (1.14). An alternative way of deriving (1.15), analogous to the method used in [2] is to approximate the singular behaviour $k^{-7/6}$ by the less singular behaviour $k^{-7/6+\delta}$, $\delta > 0$ and compute the rate of change in the number of particles. After deriving some asymptotics for the arising integrals we obtain that

$$\frac{d}{dt}\left(\int_{|k_1|\le k} \sqrt{k_1}\, f(k_1,t)\,dk_1\right) = -\frac{(a(t))^3}{3} U'(7/6) + \mathcal{O}(k^{1/10}) \quad \text{as } k \to 0$$

where the last term is uniform on δ for $0 < \delta = 7/6 - \nu$ sufficiently small. Taking the limit $\delta \to 0$ the result follows (cf. [2]).

The presence of a nonzero flux of particles towards the particles of zero momentum makes tempting to think that the solutions constructed in this paper could provide some information about the dynamic growth of Bose-Einstein condensates. However, this does not seem to be the case since the zero momentum particles would not interact at all with the particles outside the condensate. Actually, a more careful analysis yields more complicated models (cf. [14, 15, 10, 1]) where the condensate interacts with the particles that are not in the condensate. Some of the models proposed in these papers will be studied more carefully elsewhere.

There exist other kinetic equations describing fluxes of some physical quantity in some mathematical space (momentum, energy or others). One of the most typical examples is the case of gelation in coagulation processes described by means of Smoluchovski equation (cf. [11]). Actually the solutions obtained in this paper have several analogies with the explicit examples that describe gelation in such processes. Other physically relevant cases arise in the theory of weak turbulence that can be applied to describe the distribution of energy in fields of gravity waves, capillary waves, Langmuir waves in plasmas, acoustic waves or others. A detailed description of these examples can be found in [18]. A particularly simple example of solutions behaving like those found in this paper have been constructed for the Kompaneets equation that describes the energy of photons in plasma physics ([6]).

In all these cases, there exists a stationary solution of the corresponding kinetic equation of the form $f(k) = k^{-\beta}$, that plays a role analogous to the distribution $k^{-7/6}$ in our case. Physically, such solutions describe a flux of some physical quantity (particles, energy or others...) from high to small values of the quantity or viceversa, like in the classical Kolmogorov theory of turbulence.

We are not aware of any situation where the solutions constructed in this paper could have any clear physical meaning. However, we think that the mathematical methods employed in their construction can be used to treat some of the physical examples mentioned above.

1.2 Mathematical motivation

From the mathematical point of view, this paper is the continuation of the previous work [5]. In that paper we studied the linear problem that results linearising the leading term in the collision integral $\widetilde{Q}$ defined in (1.9)-(1.10). The paper [5] contains a detailed description of the fundamental solution associated to such linear problem. In this paper we construct singular solutions which behave like $k^{-7/6}$ near the origin, estimating carefully the nonlinear parts in the equation (2.3) in suitable functional spaces.

The solutions constructed in this paper are, as far as we are aware, the first example of singular solutions of a nonlinear kinetic equation with precise singular behaviour for a general initial data that has been rigorously obtained. Indeed, the solutions that we obtain have the precise asymptotic behaviour $f \sim a(t)\, k^{-7/6}$ as $k \to 0$. There is of course a large literature devoted to the study of bounded solutions of kinetic equations of Boltzmann type. On the other hand, X. Lu has recently proved the global existence of weak solutions for the Uehling Uhlenbeck equation cf. ([12, 13]). Moreover these papers also describe the long time asymptotics towards the stationary solutions as $t \to \infty$.

One of the mathematical consequences of our analysis that seems worth to mention is the presence of some kind of regularizing effects for the problem (1.1), (1.2). At a first glance this could seem surprising, because the structure of this equation suggests a “hyperbolic” non regularizing behaviour for its solutions. These regularizing effects are, however, restricted to the values of f at the particular point $k = 0$. Some typical examples of the kind of "smoothing effects" associated to this equation are Theorem 3.2 and Lemma 3.21 in Subsection 3.5 below. The estimates for $\frac{\partial a}{\partial t}(t)$ when (1.14) holds, resemble more a typical estimate for parabolic than for hyperbolic equations. Actually, a large part of the methods used in the proofs of our results are very similar to the standard semigroup arguments for parabolic equations. On the other hand, (3.27) indicates that such regularizing effects do not take place away from the origin. Indeed, the presence of the Dirac mass term shows that the smoothness of the initial data does not increase if $k \neq 0$.

Finally, let us notice that, most likely, the solutions obtained in this paper cannot be extended globally in time. Indeed, the numerical calculations in [7, 10, 14, 15] suggest that the regular solutions of the UU equation might blow up in finite time and it would not be surprising to find the same type of behaviour for the singular solutions derived in this paper.

2 Outline of the paper.

Our goal is to obtain an existence and uniqueness theory for singular solutions of the equation

$$\frac{\partial f}{\partial t}(t,k) = Q(f)(t,k) \tag{2.1}$$

$$f(0,k) = f_0(k) \tag{2.2}$$

where $Q(f)$ is defined as in (1.1)-(1.7). The initial data $f_0 \geq 0$ is assumed to satisfy the following conditions:

$$|f_0(k) - A\,k^{-7/6}| \leq \frac{B}{k^{7/6-\delta}}, \quad 0 \leq k \leq 1, \tag{2.3}$$

$$|f_0'(k) + \frac{7}{6}A\,k^{-13/6}| \leq \frac{B}{k^{13/6-\delta}}, \quad 0 \leq k \leq 1 \tag{2.4}$$

$$f_0(k) \leq B\frac{e^{-Dk}}{k^{7/6}}, \quad k \geq 1 \tag{2.5}$$

for some positive constants A, B, D and δ . The key assumption on $f_0(k)$ is that it behaves like the stationary solution $k^{-7/6}$ near the origin

The main result that we prove in this paper is the following.

Theorem 2.1 *For any f_0 satisfying (2.3)-(2.5), there exists a unique solution $f \in$ $\mathbf{C}^{1,0}((0,T)\times(0,+\infty))$ of (2.1), (2.2) as well as a function $a(t)$, satisfying:*

$$0 \leq f(t,k) \leq L\frac{e^{-Dk}}{k^{7/6}}, \quad \text{if } k > 0,\ t \in (0,T), \tag{2.6}$$

$$|f(t,k) - a(t)\, k^{-7/6}| \leq L\, k^{-7/6+\delta/2}, \quad k \leq 1, \;\; t \in (0,T), \tag{2.7}$$
$$|a(t)| \leq L, \quad \textit{for } t \in (0,T), \tag{2.8}$$

for some positive constant L and for some $T = T(A, B, \delta) > 0$.

Remark 2.2 *The space of functions $\mathbf{C}^{1,0}((0,T) \times (0,+\infty))$ is the set of functions which are continuously differentiable with respect to the first variable in $(0,+\infty)$ and continuous with respect to the second variable on $(0,\infty)$.*

In order to construct the desired solution we will argue as follows. It is convenient to consider first the problem (2.1) (2.2) replacing the kernel $W(k_1,k_2,k_3,k_4)$ by the truncated kernel

$$W_{M,M'}(k_1,k_2,k_3,k_4) = W(k_1,k_2,k_3,k_4)\, \chi\left(\frac{|k_3 - k_4|}{M}\right) \chi\left(\frac{|k_1|}{M'}\right) \tag{2.9}$$

where M and M' are large positive constants, $\chi(z) = 1$ if $0 \leq z \leq 1$, $\chi(z) = 0$ if $z > 1$. Similar cutoffs are often used in the study of other kinetic equations (cf. [4]). The reason for this cutoff in our case is to control the "Boltzmann like", quadratic terms in f in (1.4), that otherwise would yield divergences in some of the terms arising later. Using this truncation, the problem (2.1)-(2.2) becomes the truncated problem:

$$\frac{\partial f}{\partial t}(t,k) \;\; = \;\; Q_{M,M'}(f)(t,k) \tag{2.10}$$

$$f(0,k) \;\; = \;\; f_0(k) \tag{2.11}$$

where

$$Q_{MM'}\,(f)\,(k_1) = \int_{D(k_1)} W_{M,M'}\,(k_1,k_2,k_3,k_4)\, q(f) dk_3 dk_4 \tag{2.12}$$

Notice that f also depends on M and M' but, for the sake of simplicity, we will not write this dependence explicitly.

As a first step, we will obtain solutions of (2.10)-(2.12) in the form:

$$f(k,t) = \lambda(t)\, f_0(k) + g(k,t) \tag{2.13}$$

where $\lambda(t)$ will be uniquely chosen by means of the condition

$$\lim_{k \to 0} k^{7/6}\, g(t,k) = 0, \quad \forall t > 0, \tag{2.14}$$

that means that g is less singular near the origin than $k^{-7/6}$. Moreover we will assume that $\lambda(0) = 1$, whence (cf. (2.3))

$$g(0,k) = 0, \quad k \geq 0. \tag{2.15}$$

We introduce the notation:

$$q(f_0 + g) = q(f_0) + \ell(f_0, g) + n(f_0, g) \tag{2.16}$$

where $\ell(f_0, g)$ is a linear function on g and $n(f_0, g)$ contains the quadratic and higher order terms on g. The equation (2.10) might then be written as follows:

$$\frac{\partial g}{\partial t}(t, k_1) = \mathcal{L}_k(\lambda(t)\, f_0, g)(k_1, t) + \mathcal{R}_1(t, k_1) + \mathcal{R}_2(t, k_1, g) - \lambda'(t)\, f_0 \tag{2.17}$$

where, for $t > 0, k_1 > 0$,

$$\mathcal{L}_k(\lambda(t)\, f_0, g)(k_1, t) = \int_{D(k_1)} W_{M,M'}(k_1, k_2, k_3, k_4)\, \ell(\lambda(t) f_0, g)\, dk_3\, dk_4 \tag{2.18}$$

$$\mathcal{R}_1(t, k_1) = \int_{D(k_1)} W_{M,M'}(k_1, k_2, k_3, k_4)\, q(\lambda(t)\, f_0)\, dk_3\, dk_4 \tag{2.19}$$

$$\mathcal{R}_2(t, k_1, g) = \int_{D(k_1)} W_{M,M'}(k_1, k_2, k_3, k_4)\, n(\lambda(t)\, f_0, g)\, dk_3\, dk_4 \tag{2.20}$$

It may be convenient to reformulate the problem (2.10)-(2.12) using the new time variable

$$\tau = \int_0^t \lambda^2(s)\, ds. \tag{2.21}$$

Then, the problem (2.10)-(2.12) becomes:

$$\begin{aligned} \frac{\partial g}{\partial \tau}(\tau, k_1) &= \mathcal{L}_{k,2}(f_0, g)(k_1, \tau) + \frac{1}{\lambda(\tau)} \mathcal{L}_{k,1}(f_0, g)(k_1, \tau) + \\ &\quad + \frac{1}{\lambda^2(\tau)} \left(\mathcal{R}_1(\tau, k_1) + \mathcal{R}_2(\tau, k_1, g) \right) - \lambda_\tau\, f_0(k_1) && (2.22) \\ g(0, k_1) &= 0 && (2.23) \end{aligned}$$

where, with some abuse of notation, we still denote $g(\tau, k_1) \equiv g(t, k_1)$, $\lambda(\tau) = \lambda(t)$, $\lambda_\tau = \frac{\lambda'(t)}{\lambda^2(t)}$, and $\mathcal{L}_{k,2}(f_0, \tilde{g}_1)$ is quadratic with respect to f_0 and $\mathcal{L}_{k,1}(f_0, \tilde{g}_1)(k_1, t)$ is linear with respect to f_0. Notice that, as long as $0 < c_1 \le \lambda(\tau) \le c_2$ the two equations (2.22) and (2.17) are equivalent, or more precisely, a solution of (2.17) with the regularity given in Theorem 2.1 exists if and only if there exists a solution of (2.22) with the same regularity.

Our strategy in order to solve the problem (2.17), (2.15) and (2.14) is the following. It turns out that the most relevant terms to describe the asymptotics of $g(k, t)$ as $k \to 0$ are $\frac{\partial g}{\partial \tau}$ and $\mathcal{L}_k(\lambda(\tau)\, f_0, g)$. If only these terms are kept in the equation, we obtain a linear problem that can be analysed using the results of [5]. This is made in Section 3. The reason that the term $\mathcal{R}_1$ is less relevant than the linear terms in (2.16) is that f_0 behaves like the stationary solution $k^{-7/6}$ near the origin and this

yields a cancelation in the integral term in (2.19), and as a consequence this term is smaller than $\mathcal{L}_k(\lambda(t)\, f_0, g)$ as $k \to 0$. On the other hand, the term $\mathcal{R}_2$ only contains quadratic terms in g and due to (2.14) its contribution is also smaller than that due to the linear terms.
The solution of (2.15), (2.17) can be writen using the results for the linear semi-groups in Section 3 by means of the variation of constants formula. In particular, such formula can be used to compute the limit $\lim_{k\to 0} k^{7/6}\, g(t,k)$. Then, the condition (2.14) becomes an integro-differential equation for λ that is solved under suitable regularity assumptions on the initial data f_0 (cf. Section 4).
Moreover, we obtain uniform estimates on λ and g for M and M' suficiently large (cf. Section 5). Using these estimates it is not hard to take the limit as M and M' go to infinity to obtain a solution to (2.1)-(2.2). Similar arguments also provide the uniqueness in the class of functions under consideration.

3 On the linearized equation.

3.1 Functional framework and main results.

In this Section we study the solutions of the following Cauchy problem:

$$\frac{\partial h}{\partial \tau} = \mathcal{L}_{k,2}(f_0, h)(k_1, \tau) + \frac{1}{\lambda(\tau)} \mathcal{L}_{k,1}(f_0, h)(k_1, \tau) + \nu(k_1, \tau) \tag{3.1}$$

$$h(0,k) = h_0(k). \tag{3.2}$$

for some given function ν. To this end we rewrite (3.1) in a more convenient manner. We define the functions:

$$\tilde{q}(f) = f_3\, f_4\, (f_1 + f_2) - f_1\, f_2\, (f_3 + f_4) \tag{3.3}$$

$$r(f) = f_3\, f_4 - f_1\, f_2 \tag{3.4}$$

as well as

$$\tilde{q}(f_0 + g) = \tilde{q}(f_0) + \tilde{\ell}(f_0, g) + \tilde{n}(f_0, g) \tag{3.5}$$

$$r(f_0 + g) = r(f_0) + s(f_0, g) + r(g) \tag{3.6}$$

where $\tilde{\ell}$ and s contain only linear terms on g. Notice that, since $q(f) = \tilde{q}(f) + r(f)$, we have

$$\ell(f_0, g) = \tilde{\ell}(f_0, g) + s(f_0, g).$$

For further reference, it is convenient to define the operator:

$$\widetilde{\mathcal{L}}_k(k^{-7/6}, g)(k_1, t) = \int_{D(k_1)} W(k_1, k_2, k_3, k_4)\, \tilde{\ell}(k^{-7/6}, g)\, dk_3\, dk_4. \tag{3.7}$$

A detailed (and complicated) expression of $\tilde{\ell}(k^{-7/6}, g)$ can be found in [Ref.[5], Eq.(2.2) for $q_l(F)$]. But in the present paper we do not use that expression. We now introduce some suitable functional spaces.

$$\mathbf{X}_{p,q,r}(T) = \left\{\varphi \in \mathcal{C}([0,T]), L^\infty_{loc}(\mathbb{R}^+) \cap \mathcal{C}(\mathbb{R}^+);\ t^{1-r}||\varphi||_{p,q} < +\infty\right\} \tag{3.8}$$

endowed with the norm

$$|||\varphi|||_{p,q,r} = \sup_{0\le t\le T} t^{1-r}||\varphi||_{p,q}, \tag{3.9}$$

$$||\varphi||_{p,q} = \sup_{0\le k\le 1}\{k^p|\varphi(k)|\} + \sup_{k\ge 1}\{k^q|\varphi(k)|\}. \tag{3.10}$$

where p, q, r are three arbitrary real numbers. Since we will use these spaces repeatedly with $r = 1$, we write them using, by convenience, the particular notation:

$$\mathbf{Y}_{p,q}(T) := \mathbf{X}_{p,q,1}(T) = \left\{\varphi \in \mathcal{C}([0,T]), L^\infty_{loc}(\mathbb{R}^+) \cap \mathcal{C}(\mathbb{R}^+);\ |||\varphi|||_{p,q} < +\infty\right\} \tag{3.11}$$

where

$$|||\varphi|||_{p,q} := |||\varphi|||_{p,q,1} = \sup_{0\le \tau\le T} ||\varphi(\tau,\cdot)||_{p,q}.$$

Using the homogeneity of $\tilde{\ell}$ we can rewrite (3.1), as:

$$h_\tau = \widetilde{\mathcal{L}}_k(k^{-7/6}, h)(k_1,\tau) + \mathcal{U}(k;\lambda,h) + \nu(k,\tau) \tag{3.12}$$

where,

$$\begin{aligned}
\mathcal{U}(k_1;\lambda,h) &= \mathcal{U}_1(k_1;\lambda,h) + \mathcal{U}_2(k_1;\lambda,h) + \mathcal{U}_3(k_1;\lambda,h)\\
\mathcal{U}_1(k_1;\lambda,h) &= \int_{D(k_1)} W_{M,M'}\left(\tilde{\ell}(f_0,h) - \tilde{\ell}(k^{-7/6},h)\right) dk_3\, dk_4 \qquad (3.13)\\
\mathcal{U}_2(k_1;\lambda,h) &= \lambda(\tau)^{-1}\int_{D(k_1)} W_{M,M'}\, s(f_0,h)\, dk_3\, dk_4 \qquad (3.14)\\
\mathcal{U}_3(k;\lambda,h) &= \int_{D(k_1)} (W_{M,M'} - W)\ \tilde{\ell}(k^{-7/6},h)\, dk_3\, dk_4 \qquad (3.15)
\end{aligned}$$

We will say that a function h solves the equation (3.12) with initial data $h(0,k) = h_0(k)$ in the integral sense if the following integral equality holds:

$$\begin{aligned}
h(\tau,k) &= \int_0^\infty G(\tau,k,k_0)\, h_0(k_0)\, dk_0\\
&+ \int_0^\tau ds \int_0^\infty dk_0\, G(\tau-s,k,k_0)\, [\mathcal{U}(k,\lambda(s),h(s)) + \nu(k,s)], \qquad (3.16)
\end{aligned}$$

where $G(\tau,k,k_0)$ is the Green's function associated to the Cauchy problem:

$$\begin{aligned}
\frac{\partial h}{\partial \tau} &= \widetilde{\mathcal{L}}_k(k^{-7/6}, h) \qquad (3.17)\\
h(0,k) &= \delta(k-k_0) \qquad (3.18)
\end{aligned}$$

that was obtained in [5] whose detailed properties are recalled in the Theorem 3.5 below. The main results proved in this Section are the following.

Theorem 3.1 *Suppose that the function $\lambda(\tau)$ satisfies,*

$$\lambda(0)=1, \quad and \quad \frac{1}{2}\leq\lambda(\tau)\leq 2,\ \ \forall\tau\in[0,1] \tag{3.19}$$

that $||h_0||_{7/6,\beta}<+\infty$ and that $\nu\in\mathbf{Y}_{\alpha,\beta}(T')$ for some $T'>0$, where $\alpha=3/2-\delta$ and $\beta=11/6-\delta$ with $\delta>0$ sufficiently small.
Then, for any $M>1$ and $M'>1$ there exists $T>0$ and a unique solution h of (3.1), (3.2) in the integral sense in the space $\mathbf{Y}_{7/6,\beta}(T)$. Moreover

$$|||h|||_{7/6,\beta}\leq C_{M,M'}\left(||h_0||_{7/6,\beta}+T^{3\delta}\,|||\nu|||_{\alpha,\beta}\right). \tag{3.20}$$

On the other hand, there exists a function $a\in L^{\infty}([0,T])$ such that,

$$||h-a(\tau)k_1^{-7/6}\chi_{\{0\leq k_1\leq 1\}}||_{7/6-\delta/2,\beta}\leq C_{M,M'}\left(\tau^{-3\delta/2}||h_0||_{7/6,\beta}+\tau^{3\delta/2}|||\nu|||_{\alpha,\beta}\right). \tag{3.21}$$

$$|a(\tau)|\leq C_{M,M'}\left(||h_0||_{7/6,\beta}+\tau^{3\delta}|||\nu|||_{\alpha,\beta}\right). \tag{3.22}$$

Theorem 3.2 *Suppose that (3.19) holds. Suppose that $||h_0||_{\alpha,\beta}<+\infty$ and that $|||\nu|||_{\alpha,\beta,\gamma}<+\infty$ where $\alpha=3/2-\delta$ and $\beta=11/6-\delta$ with $\delta>0$, $\gamma>0$ sufficiently small.*
Then, for any $M>1$ and $M'>1$ there exists $T>0$ sufficiently small and a unique solution h of (3.1), (3.2) in the integral sense for $0<\tau<T$ such that

$$||h(\tau,\cdot)||_{7/6,\beta}\leq\frac{C}{\tau^{1-3\delta}}||h_0||_{\alpha,\beta}+C_{M,M'}T^{\gamma}\frac{|||\nu|||_{\alpha,\beta,\gamma}}{\tau^{1-3\delta}}.$$

On the other hand, there exists a function $a(\tau)$ such that,

$$\begin{aligned}||h-a(\tau)k_1^{-7/6}\chi_{0\leq k_1\leq 1\}}||_{7/6-\delta/2,\beta}\leq C_{M,M'}(&\tau^{-1+9\delta/2}||h_0||_{\alpha,\beta}+\\ &+|||\nu|||_{\alpha,\beta,\gamma}\tau^{-1+\gamma+3\delta/2}).\end{aligned} \tag{3.23}$$

$$|a(\tau)|\leq C_{M,M'}\,(\tau^{-1+6\delta}||h_0||_{\alpha,\beta}+|||\nu|||_{\alpha,\beta,\gamma}\tau^{-1+\gamma+3\delta}). \tag{3.24}$$

Remark 3.3 *The main difference between both Theorems is that Theorem 3.1 requires stronger boundedness assumptions on the initial data h_0 as $k\to 0$.*

Remark 3.4 *The existence time T in the Theorems above could depend, in principle, on $M$and M'. It will be shown in Section 5 that it is possible to derive uniform lower estimates for T if M and M' are large enough.*

The key ingredient in the proof of Theorem 3.1 is the description of the solution of the linear problem (3.17) (3.18) that we recall here for the reader' s convenience.

Theorem 3.5 *(cf. [5]) For each $k_0 > 0$ there exists a unique solution of (3.17)-(3.18) in the class of measures of the form:*

$$G(\tau, k, k_0) \quad = \quad \alpha(\tau)\delta(k - k_0) + H(\tau, k, k_0)$$

where,

$$\begin{aligned} H(\tau, \cdot, k_0) &\in L^\infty_{loc}(\mathbb{R}^+), \\ |H(\tau, k, k_0)|| &\leq \frac{C}{k^{7/6}}, \quad k \leq k_0/2, \\ |H(\tau, k, k_0)| &\leq \frac{C}{k^{11/6}}, \quad k \geq 2k_0 \\ |H(\tau, k, k_0)| &\leq \frac{C}{|k - k_0|^{5/6}}, \quad |k - k_0| \leq \frac{k_0}{2}. \end{aligned} \tag{3.25}$$

Moreover, $G(\tau, k, k_0)$ has the self similar form:

$$G(\tau, k, k_0) \quad = \quad \frac{1}{k_0} G(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}, 1) \tag{3.26}$$

and the function $G(\tau, k, 1)$ satisfies the following estimates. For $k \in (0, 2)$ we have

$$G(\tau, k, 1) = e^{-a\,\tau}\delta(k - 1) + \sigma(\tau)\, k^{-7/6} + \mathcal{R}_1(\tau, k) + \mathcal{R}_2(\tau, k), \tag{3.27}$$

where $\sigma \in \mathbf{C}[0, +\infty)$ satisfies:

$$\sigma(\tau) = \begin{cases} A\,\tau^4 + \mathcal{O}(\tau^{4+\varepsilon}) & \text{as } \tau \to 0^+, \\ \mathcal{O}(\tau^{-(3v_0 - 5/2)}) & \text{as } \tau \to +\infty \end{cases} \tag{3.28}$$

where $\mathcal{R}_1$, $\mathcal{R}_2$ might be estimated as

$$\mathcal{R}_1(\tau, k) \equiv 0 \quad \text{for } |k - 1| \geq \frac{1}{2},$$

$$|\mathcal{R}_1(\tau, k)| \leq C \frac{e^{-(a-\varepsilon)\tau}}{|k - 1|^{5/6}} \quad \text{for } |k - 1| \leq \frac{1}{2}, \tag{3.29}$$

$$\mathcal{R}_2(\tau, k) \leq C\psi_1(\tau) \left(\frac{\tau^3}{k}\right)^{\tilde{b}} \tag{3.30}$$

with

$$\psi_1(\tau) = \begin{cases} \dfrac{1}{\tau^{5/2+\varepsilon}} & \text{for } 0 \leq \tau \leq 1 \\ \dfrac{1}{\tau^{3v_0-\varepsilon}} & \text{for } \tau > 1. \end{cases} \tag{3.31}$$

On the other hand, for $k > 2$ we have:

$$G(\tau, k, 1) \leq C\psi_2(\tau) \left(\frac{\tau^3}{k}\right)^{\frac{11}{6}} \tag{3.32}$$

$$\psi_2(\tau) = \begin{cases} \dfrac{1}{\tau^{\frac{9}{2}+\varepsilon}} & \text{for } 0 \leq \tau \leq 1 \\ \dfrac{1}{\tau^{1+3v_0-\varepsilon}} & \text{for } \tau > 1, \end{cases} \tag{3.33}$$

In these formulae, $A \in \mathbb{R}$, $\varepsilon > 0$ is an arbitrarily small number, $\tilde{b}$ is an arbitrary number in the interval $(1, 7/6)$, and $v_0 = 1.84020\cdots$ The constant C depends on ε and $\tilde{b}$ but is independent on k_0 and τ.

Remark 3.6 *The constants $\tilde{b}$, v_0 and ε will have the same meaning throughout the rest of the paper.*

Remark 3.7 *Notice that, since the right hand sides of (3.31) and (3.33) are monotonically decreasing, we can assume without loss of generality that the functions ψ_1 and ψ_2 are globally decreasing in τ, something that will be made from now on.*

Remark 3.8 *Although not explicitly stated among the results in [5], the function $G(t, k, k_0)$ is differentiable with respect to t, for $k > 0$, $k_0 > 0$ and $t > 0$ as it can be seen using the explicit representation formula of G obtained in [5], cf. formulas (4.17), (4.19) and (4.25) therein. Moreover, the function $\tau \left|\frac{\partial G}{\partial \tau}\right|$ satisfy the same estimates as G.*

3.2 Some estimates for the semigroup generated by $\widetilde{\mathcal{L}}_k$.

The two Lemmas in this subsection provide some estimates for the semigroup generated by $\widetilde{\mathcal{L}}_k$ with initial data bounded near the origin or at infinity by power laws.

Lemma 3.9 *Suppose that φ solves*

$$\frac{\partial \varphi}{\partial \tau} = \widetilde{\mathcal{L}}_k(k^{-7/6}, \varphi)$$
$$\varphi(0, k) = \varphi_0(k).$$

where

$$|\varphi_0(k)| \leq k^{-\alpha} \chi_{\{k \leq 1\}}, \tag{3.34}$$

with $\alpha \in [7/6, 3/2)$. Then, there exists a function $a \in L^\infty([0, 1])$ such that, for any $\tau \in [0, 1]$:

$$|\varphi(\tau, k) - a(\tau)\, k^{-7/6}| \leq C\tau^{-3\alpha}\Phi(y), \qquad \text{for } 0 \leq k \leq 2, \tag{3.35}$$

$$|a(\tau)| \leq C\, \tau^{7/2-3\alpha}, \tag{3.36}$$

where $y = k\,\tau^{-3}$ and

$$\Phi(y) = \min\{y^{-\tilde{b}}, y^{-7/6}\}. \tag{3.37}$$

On the other hand:

$$|\varphi(\tau,k)| \leq C\, y^{-11/6}\, \tau^{-9/2-\varepsilon}, \qquad \text{for } k > 2 \tag{3.38}$$

for any $\tau \in [0,1]$, and where ε is as in Theorem 3.5 .

Proof. We assume in the rest of the proof that $0 \leq \tau \leq 1$. Using the fundamental solution G described in Theorem 3.5 as well as the Remark 3.8, we can write

$$\begin{aligned}\varphi(\tau,k) &= \int_0^1 \frac{1}{k_0} G(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}, 1)\, \varphi_0(k_0)\, dk_0 \\ &= \int_0^{\min(k/2,\,1)} \cdots\, dk_0 + \int_{\min(k/2,\,1)}^1 \cdots\, dk_0 \equiv I_1 + I_2.\end{aligned}$$

We first estimate I_1. Using (3.32) we have:

$$\begin{aligned}|I_1| &\leq C \left(\frac{\tau^3}{k}\right)^{11/6} \int_0^{\min(k/2,1)} \psi_2\left(\frac{\tau}{k_0^{1/3}}\right) k_0^{-(\alpha+1)}\, dk_0 \\ &= C \left(\frac{\tau^3}{k}\right)^{11/6} \tau^{-3\,\alpha} \int_0^{\min(k/2,1)\tau^{-3}} \psi_2\left(\frac{1}{\xi^{1/3}}\right) \xi^{-(\alpha+1)}\, d\xi.\end{aligned}$$

Using that ψ_2 is monotonically decreasing we deduce that

$$|I_1| \leq C \left(\frac{\tau^3}{k}\right)^{11/6} \psi_2\left(\frac{\tau}{\min(k/2,1)^{1/3}}\right) \min(k/2,1)^{-\alpha}. \tag{3.39}$$

Combining (3.39) and (3.33) we obtain

$$|I_1| \leq C\,\tau^{-3\,\alpha}\, \min\{y^{v_0-3/2-\alpha-\epsilon/3}, y^{-\alpha-1/3+\varepsilon/3}\}, \quad 0 < k \leq 2,$$

$$|I_1| \leq C\,\tau^{-9/2-\epsilon}\, y^{-11/6}, \quad k \geq 2. \tag{3.40}$$

We now estimate the term I_2. By definition, $I_2 = 0$ for $k > 2$. On the other hand, using (3.27) we can rewrite I_2 for $0 \leq k \leq 2$ as:

$$\begin{aligned}I_2 &= a(\tau)\, k^{-7/6} + \varphi_0(k)\, e^{-\frac{a\tau}{k^{1/3}}} \chi_{\{k\leq 1\}} + \int_0^{k/2} \sigma(\tau/k_0^{1/3}) \left(\frac{k_0}{k}\right)^{7/6} \varphi_0(k_0) \frac{dk_0}{k_0} \\ &+ \int_{k/2}^1 \mathcal{R}_1(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0})\, \varphi_0(k_0) \frac{dk_0}{k_0} + \int_{k/2}^1 \mathcal{R}_2(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0})\, \varphi_0(k_0) \frac{dk_0}{k_0} \\ &\equiv a(\tau)\, k^{-7/6} + I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4}.\end{aligned} \tag{3.41}$$

where

$$a(\tau) \;=\; \int_0^1 \sigma(\tau/k_0^{1/3})k_0^{7/6}\varphi_0(k_0)\frac{dk_0}{k_0}.$$

Therefore, using (3.34) and (3.28)

$$|a(\tau)| \le \tau^{7/2-3\alpha}\int_0^{\frac{1}{\tau^3}} \sigma(\xi^{-1/3})\,\xi^{1/6-\alpha}\,d\xi \le C\,\tau^{7/2-3\alpha} \quad \text{for } 0\le\tau\le 1. \tag{3.42}$$

Using again (3.34), we can estimate the second term in the right hand side of (3.41) as

$$|I_{2,1}| \le \tau^{-3\alpha}y^{-\alpha}\,e^{-ay^{-1/3}} \tag{3.43}$$

A similar argument yields,

$$|I_{2,2}| \le \tau^{-3\alpha}y^{-7/6}\int_0^{y/2}\sigma(\xi^{-1/3})\xi^{1/6-\alpha}d\xi. \tag{3.44}$$

$$|I_{2,3}| \le C\,\tau^{-3\alpha}\int_{y/2}^{3y/2}\frac{e^{-(a-\varepsilon)\frac{1}{\xi^{1/3}}}}{|y-\xi|^{5/6}}\xi^{-1/6-\alpha}d\xi \tag{3.45}$$

$$|I_{2,4}| \le C\tau^{-3\alpha}y^{-\tilde{b}}\int_{y/2}^{\infty}\psi_1(\frac{1}{\xi^{1/3}})\frac{d\xi}{\xi^{1+\alpha}}. \tag{3.46}$$

The right hand sides in the formulas (3.43)-(3.46) have a self similar structure of the form $\tau^{-3\alpha}\Theta(y)$, $y\equiv k/\tau^3$. Therefore it only remains to estimate the different functions Θ for $y\to 0$ and $y\to\infty$. The corresponding functions Θ in (3.43) and (3.45) have an exponential decay as $y\to 0$. Using (3.28) and (3.31) it follows that the contributions of the functions Θ in (3.44) and (3.46) behave respectively like $y^{v_0-5/6-\alpha}$ and $y^{-\tilde{b}}$ as $y\to 0$. Since $v_0-5/6-\alpha>-\tilde{b}$, and $\tilde{b}>1$ it follows that all the terms in (3.43)-(3.46) might be bounded as $C\,\tau^{-3\alpha}\,y^{-\tilde{b}}$ when $y\to 0$. On the other hand, the functions Θ might be estimated in an analogous manner for $y\to\infty$. In particular, the functions Θ in (3.43) and (3.45) are bounded like $C\,y^{-\alpha}$ as $y\to\infty$. The corresponding function Θ in (3.44) and (3.46) are bounded by $C\,y^{-7/6}$ and $y^{5/6-\tilde{b}+\varepsilon/3}y^{-\alpha}$ respectively as $y\to\infty$. Since $\alpha\ge 7/6$ and $5/6-\tilde{b}+\varepsilon/3<0$, all the terms in (3.43)-(3.46) are bounded as $Cy^{-7/6}$ as $y\to+\infty$. Combining the estimates obtained for the different functions Θ for large and small values of y we obtain (3.35). Finally (3.36) follows from (3.42) and (3.38) is a consequence of (3.40). □

Lemma 3.10 *Suppose that φ solves*

$$\varphi_\tau = \widetilde{\mathcal{L}}_k(k^{-7/6},\varphi)$$
$$\varphi(0,k) = \varphi_0(k).$$

where

$$|\varphi_0(k)| \leq k^{-\beta}\chi_{\{k\geq 1\}}, \tag{3.47}$$

with $\beta = 11/6-\delta$, $\delta > 0$ *small enough. Then, for* $\tau \in [0,1]$, *the following inequalities hold:*

$$|\varphi(\tau,k) - \beta(\tau)\, k^{-7/6}| \leq C\, k^{-\beta}\chi_{\{k\geq 1\}} + C\,\tau^{-5/2-\epsilon}\, y^{-\tilde{b}} \quad 0 \leq k \leq 2 \tag{3.48}$$

where $y = k\,\tau^{-3}$ *and*

$$|\beta(\tau)| \leq C\,\tau^4. \tag{3.49}$$

Moreover

$$|\varphi(\tau,k)| \leq Ck^{-\beta}, \quad k \geq 2. \tag{3.50}$$

Proof. Using the fundamental solution G described in Theorem 3.5 as well as the Remark 3.8 we can write

$$\begin{aligned}\varphi(\tau,k) &= \int_1^\infty \frac{1}{k_0} G(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}, 1)\, \varphi_0(k_0)\, dk_0 \\ &= \int_1^{\max(k/2,\,1)} \cdots\, dk_0 + \int_{\max(k/2,1)}^\infty \cdots\, dk_0 \equiv J_1 + J_2.\end{aligned}$$

We first estimate J_1. Using (3.26), (3.32) and (3.47) we obtain

$$\begin{aligned}|J_1| &\leq C\chi_{\{k\geq 2\}} y^{-11/6} \int_1^{k/2} \psi_2(\frac{\tau}{k_0^{1/3}}) \frac{dk_0}{k_0^{1+\beta}} \\ &\leq C\tau^{-3\beta}\chi_{\{k\geq 2\}} y^{-11/6} \int_0^{y/2} \psi_2(\xi^{-1/3}) \frac{d\xi}{\xi^{1+\beta}}.\end{aligned} \tag{3.51}$$

On the other hand $J_1 = 0$ for $k < 2$.
We now estimate J_2. Using (3.27) we can rewrite J_2 for $0 \leq k \leq 2$ as

$$\begin{aligned} J_2 - \beta(\tau)\, k^{-7/6} &= \varphi_0(k)\, e^{-\frac{a\tau}{k^{1/3}}} \chi_{\{k\geq 1\}} + \\ &+ \int_1^\infty \mathcal{R}_1(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0})\, \varphi_0(k_0) \frac{dk_0}{k_0} + \int_1^\infty \mathcal{R}_2(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0})\, \varphi_0(k_0) \frac{dk_0}{k_0}, \\ &\equiv J_{2,1} + J_{2,2} + J_{2,3},\end{aligned}$$

where

$$\beta(\tau) = \int_1^\infty \sigma(\tau/k_0^{1/3}) k_0^{7/6} \varphi_0(k_0) \frac{dk_0}{k_0}.$$

Taking into account (3.28) and (3.47) we arrive at

$$|\beta(\tau)| \leq C\tau^4. \tag{3.52}$$

On the other hand using again (3.47) as well as (3.29) and (3.30) we obtain,

$$|J_{2,1}| \leq C\,\tau^{-3\,\beta}\,y^{-\beta}\,e^{-a\,y^{-1/3}}\chi_{\{k\geq 1\}}.$$

$$|J_{2,2}| \ \leq C \int_{2k/3}^{2k} \chi_{\{k_0\geq 1\}} \frac{e^{-(a-\varepsilon)\tau/k_0^{1/3}}}{|k/k_0-1|^{5/6}} \frac{dk_0}{k_0^{1+\beta}}$$

that vanishes for k small enough. The term with $\mathcal{R}_2$ gives, for $k \leq 2$

$$|J_{2,3}| \leq Cy^{-\tilde{b}} \int_1^{\infty} \psi_1(\frac{\tau}{k_0^{1/3}}) \frac{dk_0}{k_0^{1+\beta}} \leq C\tau^{-5/2-\varepsilon}\,y^{-\tilde{b}}. \tag{3.53}$$

Combining (3.52) and (3.53), (3.48) follows.

We now estimate J_2 for $k \geq 2$. To this end we rewrite J_2 as

$$\begin{aligned} J_2 \ &= \ \varphi_0(k)\,e^{-\frac{a\tau}{k^{1/3}}} + \int_{k/2}^{\infty} \sigma(\tau/k_0^{1/3}) \left(\frac{k_0}{k}\right)^{7/6} \varphi_0(k_0)\frac{dk_0}{k_0} \\ &+ \ \int_{k/2}^{\infty} \left(\mathcal{R}_1(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}) + \mathcal{R}_2(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0})\right) \varphi_0(k_0)\frac{dk_0}{k_0}. \end{aligned}$$

Using (3.28), (3.29) and (3.30) we deduce

$$\begin{aligned} &|J_2| \leq Ck^{-\beta} + Ck^{-7/6} \int_{k/2}^{\infty} \sigma(\tau/k_0^{1/3})\,k_0^{7/6-\beta-1}\,dk_0 + \\ &+ \left(\frac{\tau^3}{k}\right)^{\tilde{b}} \int_{k/2}^{\infty} \psi_1\left(\frac{\tau}{k_0^{1/3}}\right) k_0^{-\beta-1} dk_0 + C\int_{2k/3}^{2k} \frac{k_0^{-\beta-1/6}}{|k-k_0|^{5/6}}\,dk_0. \end{aligned}$$

Using a re-scaling argument, the last integral term can be estimated as $k^{-\beta}$. Therefore:

$$|J_2| \leq C\tau^{-3\beta}\Theta(y), \qquad y = k\,\tau^{-3} \tag{3.54}$$

where

$$\begin{aligned} &\Theta(y) := y^{-\beta} + y^{-7/6} \int_{y/2}^{\infty} \sigma(\xi^{-1/3})\,\xi^{7/6-\beta-1}\,d\xi + \\ &+y^{-\tilde{b}} \int_{y/2}^{3y/2} \psi_1(\frac{1}{\xi^{1/3}}) \frac{\xi^{-\beta-1/6}}{|y-\xi|^{5/6}}\,d\xi. \end{aligned} \tag{3.55}$$

Using (3.28), (3.31) and (3.33) it follows that, for large values of y, the second term on the right hand side of (3.55) can be bounded as $C\,y^{-4/3-\beta}$, and the third one as $C\,y^{-\tilde{b}+5/6+\varepsilon/3}\,y^{-\beta}$. Therefore $\Theta(y) \leq C\,y^{-\beta}$ for $y > 1$. On the other hand combining (3.33) and (3.51) it follows that $|J_1| \leq \tau^{-3\,\beta}\,y^{-11/6}$ for $y > 1$. Using then (3.54), as well as the fact that for $k \geq 2$ and $0 \leq \tau \leq 1$ we have $y \geq 2$, the estimate (3.50) follows. □

We now derive similar results for the non homogeneous equation.

Proposition 3.11 *Let us define*

$$\theta \equiv \sup_{0\le\tau\le T}\left(\sup_{0\le k\le 1}\{k^{\alpha}|\mu(\tau,k)|\} + \sup_{k\ge 1}\{k^{\beta}|\mu(\tau,k)|\}\right). \tag{3.56}$$

where $\alpha = 3/2-\delta$, $\beta = 11/6-\delta$ *with* $\delta>0$ *arbitrarily small. Suppose that* $0\le T\le 1$. *Then, there exists a function* $y\in L^{\infty}([0,T])$ *and a constant* $C>0$ *independent on* θ *and* T *such that the solution in the integral sense of*

$$\frac{\partial h}{\partial \tau} = \widetilde{\mathcal{L}}_k(k^{-7/6},h) + \mu(\tau,k_1),$$
$$h(0,k_1)=0.$$

satisfies

$$|h(\tau,k_1) - y(\tau)\,k_1^{-7/6}| \le C\,\theta\,\tau^{3\delta/2}\,k_1^{-7/6+\delta/2} \;\; \text{for } 0\le k\le 1. \tag{3.57}$$
$$|h(\tau,k_1)| \le C\,\theta\,\tau\,k_1^{-\beta} \quad \text{for } k>1 \tag{3.58}$$

where

$$|y(\tau)| \le C\,\theta\,\tau^{3\delta}, \tag{3.59}$$

for $0\le\tau\le T$.

Proof. The idea is to use the estimates derived in Lemma 3.9 with $\alpha = 3/2-\delta$ and Lemma 3.10. Combining (3.36) and (3.49), with the variation of constant formula we obtain (3.59). On the other hand,

$$\begin{aligned}
|h(\tau,k_1) - y(\tau)\,k_1^{-7/6}| &\le C_M\int_0^{\tau}(\tau-s)^{-3\alpha}\Phi\Big(\frac{k}{(\tau-s)^3}\Big)ds \\
&+ \frac{C}{k_1^{\tilde b}}\int_0^{\tau}(\tau-s)^{3\tilde b-5/2-\varepsilon}ds + C\,\theta\,\tau\,k^{-\beta}\chi\{k\ge 1\} \\
&= Ck_1^{1/3-\alpha}\int_0^{\tau/k_1^{1/3}} u^{-3\alpha}\Phi(u^{-3})du + Ck_1^{-\tilde b}\tau^{3\tilde b-3/2-\varepsilon} \\
&+ C\,\theta\,\tau\,k^{-\beta}\chi\{k\ge 1\}.
\end{aligned} \tag{3.60}$$

Then, using (3.37) we deduce that $\int_0^{\tau/k_1^{1/3}} u^{-3\alpha}\Phi(u^{-3})du$ is convergent as $k_1\to 0$ and it behaves like $(k_1/\tau^3)^{-\delta}$ as $\tau^3/k_1\to 0$. Therefore (3.58) follows.
To obtain (3.57), we use (3.56) as well as the estimates (3.35), (3.50) in Lemmas 3.9 and 3.10. □

3.3 Estimates for the higher order terms.

Let us rewrite by convenience the equation (3.12) in the form,

$$h_\tau = \widetilde{\mathcal{L}}_k(k^{-7/6}, h)(k_1, \tau) + \mathcal{U}(k; \lambda, h) + \nu(k_1, \tau). \tag{3.61}$$

In this subsection we obtain some technical estimates for the terms $\mathcal{U}$ that are linear on h but less singular near the origin than $\widetilde{\mathcal{L}}_k(k^{-7/6}, h)(k_1, \tau)$. These estimates are written in terms of suitable functional norms of the function h itself. The results in this subsection will allow to prove Theorem 3.1 by means of a standard fixed point argument.

We rewrite $\tilde{q}$ and r in (3.3), (3.4) as

$$\begin{aligned} \tilde{q}(f) &= f_1 \tilde{q}_1(f) + \tilde{q}_2(f) \\ r(f) &= r_1(f) - f_1\, f_2 \end{aligned} \tag{3.62}$$

where

$$\begin{aligned} \tilde{q}_1(f) &= f_3\, f_4 - f_2\, f_3 - f_2\, f_4 && (3.63) \\ \tilde{q}_2(f) &= f_2\, f_3\, f_4 && (3.64) \\ r_1(f) &= f_3\, f_4. \end{aligned}$$

Notice that the functions $\tilde{q}_1(f)$, $\tilde{q}_2(f)$, $r(f)$ do not depend on f_1. On the other hand, we introduce the linearisations of these functions by means of:

$$\begin{aligned} \tilde{q}_i(f_0 + g) &= \tilde{q}_i(f_0) + \tilde{\ell}_i(f_0, g) + \tilde{n}_i(f_0, g), \quad i = 1, 2 && (3.65) \\ r_1(f_0 + g) &= r_1(f_0) + s_1(f_0, g) + r_1(g). && (3.66) \end{aligned}$$

where $\tilde{\ell}_i$ and s_1 only contain linear terms on g. Combining (3.5), (3.6) and (3.65), (3.66), we obtain

$$\tilde{\ell}(f_0, g) = \tilde{q}_1(f_0)\, g_1 + \Big[f_{0,1} \tilde{\ell}_1(f_0, g) + \tilde{\ell}_2(f_0, g) \Big] \tag{3.67}$$

$$s(f_0, g) = -g_1\, f_{0,2} + s_1(f_0, g) - f_{0,1}\, g_2 \tag{3.68}$$

($f_{0,i} \equiv f_0(k_i)$.) Using (3.67) and (3.68) we can rewrite $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ in (3.13), (3.14) as

$$\begin{aligned} \mathcal{U}_1 &= h_1\, \mathcal{U}_{1,1} + \mathcal{U}_{1,2} \\ \mathcal{U}_2 &= h_1\, \mathcal{U}_{2,1} + \mathcal{U}_{2,2} \\ \mathcal{U}_3 &= h_1\, \mathcal{U}_{3,1} + \mathcal{U}_{3,2} \end{aligned}$$

where

$$\mathcal{U}_{1,1} = \int_{D(k_1)} W_{M,M'}(k_1, k_2, k_3, k_4)\, \left(\tilde{q}_1(f_0) - \tilde{q}_1(k^{-7/6}) \right)\, dk_3\, dk_4$$

$$
\begin{aligned}
\mathcal{U}_{1,2} &= \int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)\,\Big(f_{0,1}\tilde{\ell}_1(f_0,h) - k_1^{-7/6}\tilde{\ell}_1(k^{-7/6},h) \\
&\qquad\qquad + \tilde{\ell}_2(f_0,h) - \tilde{\ell}_2(k^{-7/6},h)\Big)\,dk_3\,dk_4 \qquad (3.69)\\
\mathcal{U}_{2,1} &= -\lambda(\tau)^{-1}\int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)\,f_{0,2}\,dk_3\,dk_4\\
\mathcal{U}_{2,2} &= \lambda(\tau)^{-1}\int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)\,\left(s_1(f_0,h) - f_{0,1}\,h_2\right)\,dk_3\,dk_4\\
\mathcal{U}_{3,1} &= \int_{D(k_1)} (W_{M,M'} - W)(k_1,k_2,k_3,k_4)\,\tilde{q}_1(k^{-7/6})\,dk_3\,dk_4\\
\mathcal{U}_{3,2} &= \int_{D(k_1)} (W_{M,M'} - W)\,\Big(k_1^{-7/6}\tilde{\ell}_1(k^{-7/6},h) + \tilde{\ell}_2(k^{-7/6},h)\Big)\,dk_3\,dk_4
\end{aligned}
$$

The dependence of the functions $\mathcal{U}_{i,j}$ with respect to their arguments will not be explicitely written unless it is necessary. As a general rule we will only write the dependence on the variables that are relevant in the argument.

Lemma 3.12 *There exists a positive constant C, depending only on A, B, D, δ in (2.3)- (2.5) such that, for all (k_1, k_2, k_3, k_4) satisfying $k_2 = k_3 + k_4 - k_1$, there holds*

$$
|\tilde{q}_1(f_0) - \tilde{q}_1(A\,k^{-7/6})| \le C\left(\frac{k_3^\delta + k_4^\delta}{k_3^{7/6}\,k_4^{7/6}} + \frac{k_2^\delta + k_4^\delta}{k_2^{7/6}\,k_4^{7/6}} + \frac{k_2^\delta + k_3^\delta}{k_2^{7/6}\,k_3^{7/6}}\right).
$$

Proof. Notice that,

$$
|\tilde{q}_1(f_0) - \tilde{q}_1(A\,k^{-7/6})| \le \sum_{i,j=2;\ i<j}^{4} \left| f_{0,i}\,f_{0,j} - k_i^{-7/6}\,k_j^{-7/6}\right|
$$

and since

$$
|f_{0,j} - A\,k_j^{-7/6}||f_{0,i}| \le C k_j^{-7/6+\delta}\,k_i^{-7/6}, \quad |f_{0,i} - A\,k_i^{-7/6}||k_j^{-7/6}| \le C k_i^{-7/6+\delta}\,k_j^{-7/6}
$$

the result follows. □

Lemma 3.13 *There exists a positive constant C as in Lemma 3.12 such that*

$$
|\mathcal{U}_{1,1}| \le \frac{C}{k_1^{1/3-\delta}}
$$

Proof. The result follows using Lemma 3.12, rescaling the variables of integration as $k_3 = k_1\xi_3, k_4 = k_1\xi_4$ and using the expression of W. □

Lemma 3.14 *There exists a positive constant C as in Lemma 3.12 such that*

$$|f_{0,1}\tilde{\ell}_1(f_0,h) - A\,k_1^{-7/6}\,\tilde{\ell}_1(A\,k^{-7/6,},h)| \leq$$
$$C\left[k_1^{-7/6+\delta}\sum_{i,j=2;\;i\neq j}^{4} k_i^{-7/6}\zeta(k_j) + k_1^{-7/6}\sum_{i,j=2;\;i\neq j}^{4} k_i^{-7/6+\delta}\zeta(k_j)\right]\,|||h|||_{7/6,\beta} \tag{3.70}$$

where $\zeta(k) = k^{-7/6}$ *if* $0 \leq k \leq 1$ *and* $\zeta(k) = k^{-11/6+\delta}$ *if* $k \geq 1$.

Proof. We write

$$f_{0,1}\tilde{\ell}_1(f_0,h) - A\,k_1^{-7/6}\,\tilde{\ell}_1(A\,k^{-7/6,},h) = \left(f_{0,1} - A\,k_1^{-7/6}\right)\tilde{\ell}_1(f_0,h) + k_1^{-7/6}\left(\tilde{\ell}_1(f_0,h) - \tilde{\ell}_1(A\,k^{-7/6,},h)\right)$$

The first term is estimated using

$$\left|\left(f_{0,1} - A\,k_1^{-7/6}\right)\tilde{\ell}_1(f_0,h)\right| \leq C\,k_1^{-7/6+\delta}\,|||h|||_{7/6,\beta}\sum_{i,j=2;\;i\neq j}^{4} k_i^{-7/6}\zeta(k_j).$$

The second term is estimated as in Lemma (3.12). □

Lemma 3.15 *There exists a positive constant* $C = C(A,B,D,\delta)$ *such that*

$$|\tilde{\ell}_2(f_0,h) - \tilde{\ell}_2(A\,k^{-7/6},h)| \leq C\,|||h|||_{7/6,\beta}\sum_{i,j,\ell=2;\;i\neq j, i\neq\ell, j\neq\ell}^{4} k_i^{-7/6}\,k_j^{-7/6}\left(k_i^{\delta} + k_j^{\delta}\right)\zeta(k_\ell) \tag{3.71}$$

Proof. Formula (3.71) is a consequence of the definition of $\tilde{\ell}_2$ as well as of (2.3)-(2.5).

Lemma 3.16 *There exists a positive constant* $C_M \equiv C(A,B,D,\delta,M)$*, independent of* M'*, such that the following estimates hold*

$$|\mathcal{U}_{1,2}(h) - \mathcal{U}_{1,2}(\tilde{h})| \leq \frac{C_M}{k_1^{3/2-\delta}}\,|||h-\tilde{h}|||_{7/6,\beta} \;\; for \quad 0 \leq k_1 \leq 1, \tag{3.72}$$
$$|\mathcal{U}_{1,2}(h) - \mathcal{U}_{1,2}(\tilde{h})| \leq \frac{C_M}{k_1^{17/6-\delta}}\,|||h-\tilde{h}|||_{7/6,\beta} \quad for \quad k_1 > 1. \tag{3.73}$$

Proof. Let us suppose by simplicity that $\tilde{h} \equiv 0$, since the argument in the general case is similar. Using (3.70), (3.71) in (3.69) we deduce:

$$|\mathcal{U}_{1,2}| \leq C\int_{D(k_1)} W_{M,M'}\left\{\sum_{i,j=1;i\neq j}^{4}\;\sum_{\ell=2;\ell\neq i,\ell\neq j}^{4} k_i^{-7/6}\,k_j^{-7/6}(k_i^{\delta}+k_j^{\delta})\zeta(k_\ell)\right\}dk_3\,dk_4 \tag{3.74}$$

In order to obtain (3.72), we bound $\zeta(k_\ell)$ by $k_\ell^{-7/6}$ in (3.70) and (3.71). Using the rescaling $k_j = k_1\xi_j$ for $j = 2, 3, 4$ the result follows. For $k_1 > 1$ the largest contribution to the integral in (3.74) is due to the terms where $\ell = 2$. On the other hand, due to the cutoff in $W_{M,M'}$, k_3 and k_4 are of order of k_1 for k_1 large. Due to this, the corresponding integral can be estimated by $k_1^{-1/2}k_1^{-7/3+\delta}$ and this yields (3.73). □

Lemma 3.17 *For all $\varepsilon > 0$ arbitrarily small there exists a positive constant C_M with the same dependences as in Lemma 3.16 and depending also on ε such that*

$$|\mathcal{U}_{2,1}| \le \frac{C_M}{k_1^{1/2}} \quad \textit{for} \quad \forall k_1 > 1.$$

Proof. Using that $W \le \sqrt{k_2}/\sqrt{k_1}$ and that the kernel $W_{M,M'}$ is not zero only if $|k_3 - k_4| \le M$ the result follows.

Lemma 3.18 *There exists a positive constant C_M as in Lemma 3.17 such that*

$$|\mathcal{U}_{2,2}(h)| \le \frac{C_M}{k_1^{5/3}} \,|||h|||_{7/6,\beta}, \quad \textit{for } 0 \le k_1 \le 1 \tag{3.75}$$

and

$$|\mathcal{U}_{2,2}(h)| \le \frac{C_M}{k_1^{\beta+1/2}} \,|||h|||_{7/6,\beta}, \quad \textit{for } k_1 \le 1. \tag{3.76}$$

Proof. When $0 \le k_1 \le 1$, the term due to $s_1(f_0, h)$ might be estimated as

$$|s_1(f_0, h)| \le \sum_{i,j=2, i\neq j}^{4} k_i^{-7/6} k_j^{-7/6}.$$

The corresponding estimate follows using the rescaling $k_j = k_1\,\xi_j$, $j = 2, 3, 4$. On the other hand, the term in $\mathcal{U}_{2,2}$ containing $f_{0,1}\,h_2$ can be estimated, after integrating in k_3, k_4, as $C_M k_1^{-7/6}\,k_1^{-1/2}$ for $k_1 \le 1$. In order to make this integration it is convenient to change the integral variables from k_3, k_4 to $k_2, k_3 - k_4$. Then the function $W_{M,M'}$ is estimated by 1 for $k_2 \ge k_1$ and $\sqrt{k_2}/\sqrt{k_1}$ for $k_2 \le k_1$. Whence estimate (3.75) follows. On the other hand, in order to derive the estimate for $k_1 > 1$, we use the fact that due to the cutoff k_3 and k_4 are of order of k_1. The contribution in $\mathcal{U}_{2,2}$ due to the term $f_{0,1}\,h_2$ might be estimated by $C_M e^{-Dk_1}$ after integration in k_3, k_4. To estimate the remaining terms in $\mathcal{U}_{2,2}$ we use the fact that

$$|s_1(f_0, h)| \le \sum_{i,j=3; i\neq j}^{4} f_{0,i}\,h_j.$$

For $k_1 > 1$, the largest contribution to $\mathcal{U}_{2,2}$ is due to the the terms with $i = 2$. The resulting contribution can be bounded as $k_1^{-\beta-1/2}$, whence (3.76) follows. □

Lemma 3.19 *There exists a positive constant C_M as in Lemma 3.17 such that*

$$|\mathcal{U}_{3,1}(h)| \leq C_M, \quad \textit{for } 0 \leq k_1 \leq 1 \tag{3.77}$$

and

$$|\mathcal{U}_{3,1}(h)| \leq \frac{C_M}{k_1^{1/3}}, \quad \textit{for } k_1 \geq 1.$$

Proof. The estimate (3.77) for $0 \leq k_1 \leq 1$ follows using that, due to the cutoff, the domain of integration is contained in a fixed domain independent of k_1. For $k_1 \geq 1$, we estimate $|W_{M,M'} - W|$ by $2W$ and use in the resulting integral the rescaling $k_j = k_1 \xi_j$, $j = 2, 3, 4$. □

Lemma 3.20 *There exists a positive constant C_M as in Lemma 3.17 such that*

$$|\mathcal{U}_{3,2}(h)| \leq \frac{C_M}{k_1^{7/6}} \, |||h|||_{7/6,\beta}, \quad \textit{for } 0 \leq k_1 \leq 1$$

and

$$|\mathcal{U}_{3,2}(h)| \leq \frac{C_M}{k_1^{1/3+\beta}} \, |||h - \tilde{h}|||_{7/6,\beta}, \quad \textit{for } k_1 \geq 1.$$

Proof. The proof is essentially similar to that of the previous Lemma. □

3.4 Proof of the Theorems 3.1 and 3.2.

We can reformulate the original problem (3.1), (3.2) as a fixed point problem. To this end we use the variation of constants formula in (3.2) (3.61) to obtain

$$\begin{aligned} h(\tau, k_1) &= \int_0^\infty G(\tau, k_1, \xi)\, h_0(\xi) + \int_0^\tau ds \int_0^\infty d\xi\, G(\tau - s, k_1, \xi)\, \mathcal{U}(\xi; \lambda(s), h(s, \xi)) \\ &+ \int_0^\tau ds \int_0^\infty d\xi\, G(\tau - s, k_1, \xi)\, \nu(s, \xi) \equiv \mathcal{T}(h)(\tau, k_1). \end{aligned} \tag{3.78}$$

where $G(\tau, k_1, \xi)$ is the fundamental solution of the problem (3.17)-(3.18) described in Theorem 3.5.

Proof of Theorem 3.1. The Theorem will follow by proving that the operator $\mathcal{T}$ defined in (3.78) is contractive in the space $\mathbf{Y}_{7/6,\beta}(T)$ for $T > 0$ small enough.

To this end, notice that using Lemma 3.13, as well as Lemmas 3.16 - 3.20, we obtain

$$|\sum_{j=1}^{3} (h_1 \mathcal{U}_{j,1} + \mathcal{U}_{j,2})\, (\xi; \lambda(s), h(s, \xi)) + \nu(s, \xi)| \leq$$

$$C_M\,\xi^{-3/2+\delta}\left(|||h|||_{7/6,\beta}+|||\nu|||_{\alpha,\beta}\right)\ \text{for}\quad 0\le\xi\le 1 \tag{3.79}$$

$$|\sum_{j=1}^{3}\left(h_1\,\mathcal{U}_{j,1}+\mathcal{U}_{j,2}\right)(\xi;\lambda(s),h(s,\xi))+\nu(s,\xi)|\le$$

$$C_M\,\xi^{-\beta-1/3+\delta}\left(|||h|||_{7/6,\beta}+|||\nu|||_{\alpha,\beta}\right)\ \text{for}\quad \xi>1 \tag{3.80}$$

Combining these estimates with Proposition 3.11 we obtain

$$|||\mathcal{T}(h-\tilde{h})|||_{7/6,\beta}\le C_M\,T^{3\delta/2}|||h-\tilde{h}|||_{7/6,\beta}$$

where C_M is a positive constant as in Lemma 3.16. The existence and uniqueness parts in Theorem 3.1 for small T follow by means of a standard fixed point argument. On the other hand, combining (3.79) and (3.80) with Proposition 3.11, we arrive at

$$|||\mathcal{T}(h)|||_{7/6,\beta}\le C_M\left(||h_0||_{7/6,\beta}+T^{3\delta/2}\,|||h|||_{7/6,\beta}\right)+T^{3\delta/2}\,|||\nu|||_{7/6,\beta} \tag{3.81}$$

that yields the estimate (3.20).
The proof of (3.21) (3.22) follows from Proposition 3.11 that yields an estimate for the contribution due to the term ν, as well as Lemma 3.9 with $\alpha=7/6$ that provides bounds for the contribution due to h_0. □

Proof of Theorem 3.2. The proof of Theorem 3.2 is very similar to the one of Theorem 3.1 although we must use the functional space $\mathbf{X}_{7/6,\beta,3\,\delta}(T)$. We first rewrite the equation as

$$h_\tau=\mathcal{L}_k(\lambda(\tau)\,A\,k^{-7/6},h)+\mu(k_1,\tau)+\nu \tag{3.82}$$

where

$$\mu(\tau,k_1)=\mathcal{L}_k(\lambda(\tau)f_0,h)-\mathcal{L}_k(\lambda(\tau)\,A\,k^{-7/6},h). \tag{3.83}$$

Arguing then as in the proofs of formulas (3.79) and (3.80), we first obtain

$$||\mu(\tau,\cdot)||_{3/2-\delta,\beta}\le\frac{C}{\tau^{1-3\delta}}|||h|||_{7/6,\beta,3\,\delta},\quad 0\le\tau\le T.$$

We use now the usual fix point argument. Given h in $\mathbf{X}_{7/6,\beta,3\,\delta}(T)$ we define μ as in (3.83) and then solve (3.82) with $h(0,k_1)=h_0(k_1)$. This defines an operator $\mathcal{T}(h)$. Using the variation of constants formula as well as Lemmae 3.9, 3.10 we obtain

$$\begin{aligned}||\mathcal{T}(h)(\tau,\cdot)||_{7/6,\beta}\ &\le\ C||h_0||_{7/6,\beta}+C\int_0^\tau\frac{ds}{(\tau-s)^{1-3\delta}}\times\\&\qquad\left\{\frac{|||h|||_{7/6,\beta,3\,\delta}}{s^{1-3\delta}}+\frac{|||\nu|||_{\alpha,\beta,\gamma}}{s^{1-\gamma}}\right\}\\&\le\ C||h_0||_{7/6,\beta}+C\,T^{3\delta}\frac{|||h|||_{7/6,\beta,3\,\delta}}{\tau^{1-3\delta}}+CT^{\gamma}\frac{|||\nu|||_{\alpha,\beta,\gamma}}{\tau^{1-3\delta}}\end{aligned}$$

and similarly,

$$|||\mathcal{T}(h_1 - h_2)|||_{7/6,\beta,3\,\delta} \;\; \leq \;\; CT^{3\delta}|||h_1 - h_2|||_{7/6,\beta,3\,\delta}.$$

The existence and uniqueness of solution of (3.1), (3.2) in the space $\mathbf{X}_{7/6,\beta,3\,\delta}(T)$ follows for $T > 0$ sufficiently small by the usual contraction argument. Finally, (3.23) and (3.24) follow by a small modification of the proof of Proposition 3.11. More precisely, if $\tilde{h}$ is a solution of (3.82) with initial data $\tilde{h}_0(k) = 0$ then, arguing as in the derivation of (3.60), we have

$$\begin{aligned} |\tilde{h}(\tau, k_1) - y(\tau)\, k_1^{-7/6}| \;\; &\leq \;\; C_M \int_0^\tau (\tau - s)^{-3\alpha} \Phi(\frac{k}{(\tau - s)^3})\, j(s) ds \\ &+ \;\; \frac{C}{k_1^{\tilde{b}}} \int_0^\tau (\tau - s)^{3\tilde{b} - 5/2 - \varepsilon}\, j(s) ds \end{aligned} \tag{3.84}$$

where

$$j(s) \equiv \left\{ \frac{||h||_{\alpha,\beta,\delta}}{s^{1-3\delta}} + \frac{|||\nu|||_{\alpha,\beta,\gamma}}{s^{1-\gamma}} \right\} \quad \text{and} \quad \tilde{b} = \frac{7}{6} - \frac{\delta}{2}.$$

We can now estimate the first term in the right hand side of this inequality splitting the integral in the intervals $(0, \tau/2)$ and $(\tau/2, \tau)$. In the second one, we can bound $s^{-1+3\delta}$ and $s^{-1+\gamma}$ by $C\tau^{-1+3\delta}$ and $C\tau^{-1+\gamma}$ respectively, and estimate the remaining integral as in (3.81). This gives eventually, for $0 \leq \tau \leq T$ and $0 \leq k_1 \leq 1$:

$$\begin{aligned} &\int_{\tau/2}^\tau (\tau - s)^{-3\alpha} \Phi(\frac{k}{(\tau - s)^3})\, j(s) ds \leq \\ &\qquad C(||h||_{\alpha,\beta,\delta} \tau^{-1+9\delta/2} + |||\nu|||_{\alpha,\beta,\gamma} \tau^{-1+\gamma+3\delta/2})\, k_1^{-7/6+\delta/2} \end{aligned}$$

On the other hand, the contribution due to the integral for $0 \leq s \leq \tau/2$ is estimated using the monotonicity of the function Φ defined in (3.37). Then

$$\begin{aligned} &\int_0^{\tau/2} (\tau - s)^{-3\alpha} \Phi(\frac{k}{(\tau - s)^3})\, j(s)\, ds \leq C \frac{1}{\tau^{3\alpha}} \Phi(\frac{k}{\tau^3}) \int_0^{\tau/2} j(s)\, ds \\ &\leq C(||h_0|| \tau^{-1+9\delta/2} + |||\nu|||_{\alpha,\beta,\gamma} \tau^{-1+\gamma+3\delta/2})\, k_1^{-7/6+\delta/2}. \end{aligned}$$

The second integral in the right hand side of (3.84) is estimated using similar arguments. Finally, the bound (3.22) for $a(\tau)$ follows as in Proposition 3.11, using (3.36) and (3.49). □

3.5 Some regularity results for the time derivatives.

We now prove some regularity properties with respect to the initial time for the function $a(\tau)$ whose existence is asserted in (3.21) that will be needed later.

Lemma 3.21 *Let us suppose that f_0 satisfies (2.3), (2.4), (2.5) and $1/2 \leq \lambda(\tau) \leq 1$ for $\overline{\tau} \leq \tau \leq T$. Let us denote as H the unique solution of the problem*

$$\frac{\partial H}{\partial \tau}(\tau, \overline{\tau}, k) = \mathcal{L}_{k,2}(f_0, H(\tau, \overline{\tau})) + \frac{1}{\lambda(\tau)} \mathcal{L}_{k,1}(f_0, H(\tau, \overline{\tau})) \tag{3.85}$$

$$\text{for} \quad \overline{\tau} \leq \tau \leq T,$$

$$H(\overline{\tau}, \overline{\tau}) = f_0. \tag{3.86}$$

in $\mathbf{Y}_{7/6,\beta}(T)$. Suppose also that

$$|\lambda'(\tau)| \leq C, \quad 0 \leq \tau \leq T. \tag{3.87}$$

Then, the function $a(\tau, \overline{\tau})$ defined as

$$a(\tau, \overline{\tau}) = \lim_{k_1 \to 0} k_1^{7/6} H(\tau, \overline{\tau}, k_1), \tag{3.88}$$

satisfies:

$$a(\overline{\tau}^+, \overline{\tau}) = A \tag{3.89}$$

$$\left|\frac{\partial}{\partial \overline{\tau}}\, a(\tau, \overline{\tau})\right| \leq C\, (\tau - \overline{\tau})^{-1+3\delta}, \quad \overline{\tau} \leq t \leq T. \tag{3.90}$$

$$\left|\left|\frac{\partial H}{\partial \overline{\tau}}\right|\right|_{7/6,\beta} \leq \frac{C\, \sup_{0 \leq \tau \leq T} |\lambda(\tau)|}{(\tau - \overline{\tau})^{1-3\delta}}, \tag{3.91}$$

$$\left|\left|\frac{\partial H}{\partial \tau}\right|\right|_{7/6,\beta} \leq \frac{C}{(\tau - \overline{\tau})^{1-3\delta}} \left\{ \sup_{0 \leq \tau \leq T} |\lambda(\tau)| + \sup_{0 \leq \tau \leq T} |\lambda'(\tau)| \right\} \tag{3.92}$$

and

$$\left|\frac{\partial a}{\partial \tau}(\tau, \overline{\tau})\right| \leq \frac{C}{(\tau - \overline{\tau})^{1-3\delta}} \left\{ \sup_{0 \leq \tau \leq T} |\lambda(\tau)| + \sup_{0 \leq \tau \leq T} |\lambda'(\tau)| \right\} \tag{3.93}$$

for $\overline{\tau} \leq \tau \leq T$.

Finally, under the same assumptions,

$$||H(\tau, \overline{\tau}) - a(\tau, \overline{\tau})\, k^{-7/6}||_{7/6-\delta/2,\beta} \leq C \tag{3.94}$$

$$||\frac{\partial H}{\partial \tau}(\tau, \overline{\tau}) - \frac{\partial a}{\partial \tau}(\tau, \overline{\tau})\, k^{-7/6}||_{7/6-\delta/2,\beta} \leq \frac{C}{(\tau - \overline{\tau})^{1-3\delta/2}} \tag{3.95}$$

for $\overline{\tau} \leq \tau \leq T$.

Proof of Lemma 3.21. The existence and uniqueness of the solution H follows from Theorem 3.1 with $\nu = 0$. Using now (3.78) we obtain:

$$H(\tau,\overline{\tau},k_1) = \int_0^\infty G(\tau-\overline{\tau},k_1,\xi)\, f_0(\xi)\, d\xi \tag{3.96}$$

$$+ \int_{\overline{\tau}}^{\tau} ds \int_0^\infty d\xi G(\tau-s,k_1,\xi)\,\mathcal{U}(\xi;\lambda(s),H(s,\overline{\tau},\xi)). \tag{3.97}$$

Multiplying by $k_1^{7/6}$ and taking the limit as $k_1 \to 0$ we arrive at,

$$\begin{aligned} a(\tau,\overline{\tau}) &= \int_0^\infty \xi^{1/6}\,\sigma(\frac{\tau-\overline{\tau}}{\xi^{1/3}})\, f_0(\xi)\, d\xi \\ &+ \int_{\overline{\tau}}^{\tau} ds \int_0^\infty d\xi\, \xi^{1/6}\,\sigma(\frac{\tau-s}{\xi^{1/3}})\mathcal{U}(\xi;\lambda(s),H(s,\overline{\tau},\xi)) \end{aligned} \tag{3.98}$$

for all $\tau < \overline{\tau}$, where the convergence of the different integrals is ensured by the estimates (3.79) and (3.80). We now take the limit of (3.98) as $\tau \to \overline{\tau}$. To this end, we use in the first integral of the right hand side the change of variables $\xi = \zeta\,\tau^3$ and (2.3), whence

$$\lim_{\tau\to\overline{\tau}} \int_0^\infty \xi^{1/6}\,\sigma(\frac{\tau-\overline{\tau}}{\xi^{1/3}})\, f_0(\xi)\, d\xi = A \int_0^\infty \sigma(\zeta^{-1/3})\,\zeta^{-1}\, d\zeta. \tag{3.99}$$

Differentiating the identity $\widetilde{Q}(A\,k^{-7/6}) = 0$ (c.f. (1.8)) with respect to A we obtain $\mathcal{L}_{k,2}(A\,k^{-7/6},H_s) = 0$. Therefore, if $f_0(\xi) = \xi^{-7/6}$ and $\mathcal{U} = 0$ in (3.99) it would follow that $a(\overline{\tau},\tau) = A$, whence

$$\int_0^\infty \sigma(\zeta^{-1/3})\,\zeta^{-1}\, d\zeta = 1. \tag{3.100}$$

On the other hand, using (3.79) and (3.80) and Lemmas 3.9, 3.10 we deduce

$$\left| \int_0^\infty d\xi\, \xi^{1/6}\,\sigma(\frac{\tau-s}{\xi^{1/3}})\mathcal{U}(\xi;\lambda(s),H(s,\overline{\tau},\xi)) \right| \le C(\tau-s)^{-1+3\delta}.$$

Integrating this formula in the interval $(\overline{\tau},\tau)$, we derive an estimate for the second term of the right hand side of (3.98) in the form $C\,(\tau-\overline{\tau})^{-3\,\delta}$. Taking the limit $\tau \to \overline{\tau}$ and using (3.99), (3.100) we obtain (3.89).

The function $H(\tau,\overline{\tau},k)$ satisfies (3.85) in the classical sense. To check this we could differentiate formally in (3.97), after rewriting the second integral in the right hand side as:

$$\int_0^{\tau-\overline{\tau}} ds \int_0^\infty d\xi\, G(s,k_1,\xi)\,\mathcal{U}(\xi;\lambda(\tau-s),H(\tau-s,\overline{\tau},\xi))$$

and obtain:

$$\begin{aligned}\frac{\partial H}{\partial \tau}(\tau,\overline{\tau},k_1) &= \int_0^\infty \frac{\partial G}{\partial \tau}(\tau-\overline{\tau},k_1,\xi)\, f_0(\xi)\, d\xi + \int_0^\infty G(\tau-\overline{\tau},k_1,\xi)\,\mathcal{U}(\xi;\lambda(\overline{\tau}),f_0(\xi))\, d\xi \\ &+ \int_0^{\tau-\overline{\tau}} ds \int_0^\infty d\xi\, G(s,k_1,\xi)\left\{\frac{\partial \mathcal{U}}{\partial \lambda}\lambda'(\tau-s) + \frac{\partial \mathcal{U}}{\partial H}\frac{\partial H}{\partial \tau}(\tau-s,\overline{\tau},\xi)\right\}.\end{aligned}$$

Use of Gronwall's Lemma would then give that H is a classical solution of (3.85). To make this argument rigorously we have just replaced $\partial/\partial\tau$ by the incremental quotients and pass to the limit.

Let us first indicate the formal arguments that we will use to prove (3.90), (3.92), (3.93), (3.94) and (3.95). In order to prove (3.90) we differentiate (3.85) and (3.86) with respect to $\overline{\tau}$ to obtain:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial H}{\partial \overline{\tau}}\right)(\tau,\overline{\tau},k) = \mathcal{L}_k\left(\lambda(\tau)\, f_0, \left(\frac{\partial H}{\partial \overline{\tau}}\right)(\tau,\overline{\tau})\right), \tag{3.101}$$

$$\frac{\partial H}{\partial \overline{\tau}}(\overline{\tau},\overline{\tau}) = -\frac{\partial H}{\partial \tau}(\overline{\tau},\overline{\tau}) = -\mathcal{L}_k(\lambda(\overline{\tau})\, f_0, f_0) \tag{3.102}$$

Using (2.3) we obtain

$$||\mathcal{L}_k(\lambda(\overline{\tau})\, f_0, f_0)||_{\alpha,\beta} \leq C \tag{3.103}$$

with $\alpha = 3/2-\delta$ and $\beta = 11/6-\delta$. The estimate (3.90) is then a consequence of Theorem 3.2.

The analogous argument to prove (3.92) and (3.93) would be as follows. We notice first that, due to (3.87), to estimate the derivative of a function with respect to t is equivalent to estimate its derivative with respect to τ. Differentiating (3.85) with respect to τ, and using (3.101) and (3.102), we obtain that $\frac{\partial H}{\partial \tau}$ solves

$$\begin{aligned}\frac{\partial}{\partial \tau}\left(\frac{\partial H}{\partial \tau}\right) &= \mathcal{L}_{k,2}\left(f_0, \left(\frac{\partial H}{\partial \tau}\right)\right) + \\ &+ \frac{1}{\lambda(\tau)}\mathcal{L}_{k,1}\left(f_0, \left(\frac{\partial H}{\partial \tau}\right)\right) + \frac{\partial}{\partial \tau}\left(\frac{1}{\lambda(\tau)}\right)\mathcal{L}_{k,1}\left(f_0, H\right)\end{aligned} \tag{3.104}$$

$$\frac{\partial H}{\partial \tau}(\overline{\tau},\overline{\tau}) = \frac{1}{\lambda(\overline{\tau})}\mathcal{L}_k\left(\lambda(\overline{\tau})\, f_0, f_0\right). \tag{3.105}$$

Combining (2.3), (2.5), (3.87), as well as the fact that $H \in \mathbf{Y}_{7/6,\beta}(T)$, it follows that

$$\left|\left|\frac{\partial}{\partial \tau}\left(\frac{1}{\lambda(\tau)}\right)\mathcal{L}_{k,1}\left(f_0, H\right)\right|\right|_{\alpha,\beta} \leq C.$$

Applying Theorem 3.2 to (3.103) we deduce (3.92). Formula (3.93) follows from (3.22).

Analogously, in order to derive (3.94) we use the fact that the equation satisfied for $W = H - f_0$, that might be derived using (3.85), (3.86) is a linear equation with zero initial data and source terms bounded by $Ck_1^{-3/2+\delta}$ for $k_1 \leq 1$. Therefore (3.94) follows using variation of the constants as above and Theorem 3.1. The proof of (3.95) is similar, but using (3.104), (3.105) instead of (3.85), (3.86) and Theorem 3.2 instead of Theorem 3.1.

The above computations can be made rigorous replacing the derivatives $\partial/\partial\overline{\tau}$ and $\partial/\partial\tau$ by the corresponding incremental quotients. □

4 Solving the nonlinear truncated equation.

In this Section we prove the following result.

Theorem 4.1 *Suppose that f_0 satisfies (2.3), (2.5), then for any $M > 0$ and $M' > 0$ there exists $T = T(M, M') > 0$ and a unique solution of (2.10)-(2.12) of the form $f(t) = \lambda(t)\, f_0 + g(t)$ where $g \in \mathcal{C}[[0,T] \times (0,\infty)]$, $g \in \mathbf{Y}_{7/6-\delta/2,\beta}(T)$, $\beta = 11/6 - \delta$, $\delta > 0$ sufficiently small, and $\lambda \in \mathcal{C}[0,T] \cap \mathcal{C}^1(0,T)$. Moreover,*

$$|||g|||_{7/6-\delta,\beta}(T) \leq C_{M,M'}\, T^{\delta/2}. \tag{4.1}$$

Remark 4.2 *Notice that the condition $g \in \mathbf{Y}_{7/6-\delta,\beta}(T)$ implies that (2.14) holds.*

The idea of the proof of Theorem 4.1 is to use a fixed point argument for (2.17) under the constraint (2.14). We will obtain first a proof of the result in the τ variable instead of t because due to (2.21) both formulations are equivalent as long as $1/2 \leq \lambda \leq 2$. The statement in the t variable immediately follows due to the same reason. As a first step, we derive suitable estimates for the terms $\mathcal{R}_1$ $\mathcal{R}_2$ defined in (2.19), (2.20).

Lemma 4.3 *Suppose that f_0 satisfies (2.3), (2.5) and $1/2 \leq \lambda(\tau) \leq 2$ for $0 \leq \tau \leq T$ for some $T > 0$. Then the function $\mathcal{R}_1(\tau, k_1)$ defined in (2.19) satisfies:*

$$\sup_{0\leq\tau\leq T} |\mathcal{R}_1(\tau, k_1)| \leq \frac{C_M}{k_1^{3/2-\delta}}, \quad k_1 \leq 1, \tag{4.2}$$

$$\sup_{0\leq\tau\leq T} |\mathcal{R}_1(\tau, k_1)| \leq \frac{C_M}{k_1^{7/3-2\delta}}, \quad k_1 \geq 1. \tag{4.3}$$

where $C_M = C(A, B, D, \delta, M)$ is a positive constant independent of M'.

Proof of Lemma 4.3. Using the fact that $q(f) = \tilde{q}(f) + r(f)$ as well as (3.5) with $g = \lambda(\tau)(f_0 - Ak^{-7/6})$, we can rewrite $\mathcal{R}_1$ as

$$\begin{aligned}
\mathcal{R}_1(\tau, k_1) &= \int_{D(k_1)} W_{M,M'}\, \tilde{q}(\lambda(\tau)\, A\, k^{-7/6})\, dk_3\, dk_4 + \\
&+ \int_{D(k_1)} W_{M,M'}\, \tilde{\ell}(\lambda(\tau)\, A\, k^{-7/6},\ \lambda(\tau)\, (f_0 - A\, k^{-7/6}))\, dk_3\, dk_4 \\
&+ \int_{D(k_1)} W_{M,M'}\, \tilde{n}(\lambda(\tau)\, A\, k^{-7/6},\ \lambda(\tau)\, (f_0 - A\, k^{-7/6}))\, dk_3\, dk_4 \\
&+ \int_{D(k_1)} W_{M,M'}\, r(\lambda(\tau) f_0)\, dk_3\, dk_4 \\
&\equiv \mathcal{R}_{1,1} + \mathcal{R}_{1,2} + \mathcal{R}_{1,3} + \mathcal{R}_{1,4}. \qquad (4.4)
\end{aligned}$$

Since $W_{M,M'}$ is supported in the region $|k_3 - k_4| \le M$, the term $\tilde{q}(f)$ may be bounded by $C_M\, k_1^{-7/3} \min(1, k_2^{-7/6})$ (c.f. (3.3)). We then deduce, using that $W(k_1, k_2, k_3, k_4) \le \min(1, \sqrt{k_2}/\sqrt{k_1})$:

$$|\mathcal{R}_{1,1}| \le C_M\, k_1^{-7/3} \int_0^\infty \min(1, \frac{\sqrt{k_2}}{\sqrt{k_1}})\, \min(1, k_2^{-7/6})\, dk_2$$

Spliting the integral in the three regions $0 < k_2 < 1$, $1 < k_2 < k_1$ and $k_1 < k_2 < \infty$ we obtain:

$$|\mathcal{R}_{1,1}| \le \frac{C_M}{k_1^{5/2}}, \quad k_1 \ge 1. \qquad (4.5)$$

On the other hand, since

$$\int_{D(k_1)} W\, \tilde{q}(Ak^{-7/6})\, dk_3\, dk_4 = 0 \qquad (4.6)$$

we can rewrite $\mathcal{R}_{1,1}$ as

$$\mathcal{R}_{1,1} = \int_{D(k_1)} (W_{M,M'} - W)\ \tilde{q}(\lambda(\tau)\, A\, k^{-7/6})\, dk_3\, dk_4. \qquad (4.7)$$

Using that $W_{M,M'} - W$ vanishes for $|k_3 - k_4| < M$, we obtain

$$|\mathcal{R}_{1,1}| \le \frac{C_M}{k_1^{7/6}}, \quad k_1 \le 1. \qquad (4.8)$$

We consider now $\mathcal{R}_{1,2}$. Due to (2.3) the estimate $|\lambda(\tau)(f_0 - Ak^{-7/6})| \le Ck_1^{-7/6+\delta}$ holds for all $k_1 > 0$. Making the change of variables $k_3 = k_1\, \xi_3$, $k_4 = k_1\, \xi_4$, it follows that

$$|\mathcal{R}_{1,2}| \le \frac{C}{k_1^{3/2-\delta}}, \quad \text{for } k_1 \le 1. \qquad (4.9)$$

Arguing as in the derivation of (4.5) we obtain

$$|\mathcal{R}_{1,2}| \leq C_M e^{-Bk_1}, \quad \text{for } \ k_1 \geq 1. \tag{4.10}$$

Similar arguments yield

$$|\mathcal{R}_{1,3}| \leq \frac{C_M}{k_1^{5/2}}, \quad \text{for } \ k_1 \geq 1, \tag{4.11}$$

$$|\mathcal{R}_{1,3}| \leq \frac{C_M}{k_1^{3/2-2\delta}}, \quad \text{for } \ k_1 \leq 1, \tag{4.12}$$

as well as

$$|\mathcal{R}_{1,4}| \leq \frac{C_M}{k_1^{4/3}}, \quad \text{for } \ k_1 \leq 1, \tag{4.13}$$

$$|\mathcal{R}_{1,4}| \leq C_M \, e^{-Bk_1}, \quad \text{for } \ k_1 \geq 1. \tag{4.14}$$

Putting together (4.5) and (4.8)-(4.14), Lemma 4.3 follows. □

Lemma 4.4 *Suppose that $g \in \mathbf{Y}_{7/6-\delta/2,\beta}(T)$, for some $T > 0$, with β as in Theorem 4.1. Suppose that λ satisfies also the assumptions in Theorem 4.1 and $1/2 \leq \lambda(\tau) \leq 2$. Then the function $\mathcal{R}_2(\tau, k_1, g)$ defined by (2.20) satisfies*

$$\sup_{0 \leq \tau \leq T} |\mathcal{R}_2(\tau, k_1, g)| \leq \frac{C_M}{k_1^{3/2-\delta}}, \quad k_1 \leq 1 \tag{4.15}$$

$$\sup_{0 \leq \tau \leq T} |\mathcal{R}_2(\tau, k_1, g)| \leq \frac{C_M}{k_1^{\beta}}, \quad k_1 \geq 1, \tag{4.16}$$

where $C_M = C(A, B, D, \delta, M, |||g|||_{7/6-\delta/2,\beta})$ is uniformly bounded if $|||g|||_{7/6-\delta/2,\beta}$ is bounded and is independent of M'. Moreover, suppose that $g, \overline{g}$ are such that

$$|||g|||_{7/6-\delta/2,\beta} + |||\overline{g}|||_{7/6-\delta/2,\beta} \leq \rho \tag{4.17}$$

for some positive constant ρ. Then,

$$|||\mathcal{R}_2(\cdot,\cdot,g) - \mathcal{R}_2(\cdot,\cdot,\overline{g})|||_{3/2-\delta,\beta} \leq C_M \, |||g - \overline{g}|||_{7/6-\delta/2,\beta} \tag{4.18}$$

where $C_M = C(A, B, D, \delta, M, \rho)$

Remark 4.5 *Lemma 4.4 will play a crucial role in the forthcoming argument. The reason is that it states that the function* $\mathcal{R}_2(\tau, k_1, g)$ *is smaller near the origin than the leading linear term* $\mathcal{L}_{k,2}(f_0, g)(\tau, k_1)$ *in (2.22). Indeed, given* $g \in \boldsymbol{Y}_{7/6-\delta/2,\beta}(T)$*, it follows that* $\mathcal{L}_{k,2}(f_0, g)(\tau, k_1)$ *is pointwise bounded by* $Ck_1^{-3/2+\delta/2}$ *for* $0 < k_1 \leq 1$*. On the other hand, the term* $\mathcal{R}_2(\tau, k_1, g)$ *can be estimated by the smaller quantity* $Ck_1^{-3/2+\delta}$ *for* $0 < k_1 \leq 1$*. This additional smallness, that is due to the fact that* $\mathcal{R}_2(\tau, k_1, g)$ *is quadratic with respect to* g*, allow to handle this last term in a perturbative manner.*

Proof of Lemma 4.4 The function $n(\lambda(\tau)\, f_0, g)$ contains two types of terms depending on their homogeneity. Some of the terms are the ones in $\tilde{n}$ that are quadratic in g and linear in f_0. These terms can be estimated for $0 \leq k_1 \leq 1$ using (2.3) and $g \in Y_{7_6-\delta/2,\beta}(T)$. Using the change of variables $k_3 = k_1\, \xi_3$, $k_4 = k_1\, \xi_4$ we deduce an estimate of the form (4.15) for the contribution due to these terms. The remaining terms in $n(\lambda(\tau)\, f_0, g)$ are the ones in $r(\lambda(\tau)\, f_0, g)$. Their contribution can be estimated as $C_M k_1^{-7/6-\delta}$ when $k_1 \leq 1$ which is smaller than the right hand side of (4.15). Finally, (4.16) follows using the same arguments as in the proof of Lemma 4.3. Estimates for the differences (4.18) are obtained in the same way. □

Proof of Theorem 4.1. We recall that we look for a solution of the problem (2.22)-(2.23) of the form:

$$f(\tau, k) = \lambda(\tau)\, f_0(k) + g(\tau, k)$$

where $\lambda(\tau)$ will be prescribed imposing $g \in Y_{7/6-\delta/2,\beta}(T)$ for some $T > 0$. Moreover we also have $g(0, k) = 0$ for $k \geq 0$ (cf. (2.15)).

Let us introduce a suitable functional framework. We define the space

$$\Lambda(T) \equiv \{\lambda \in \mathbf{C}([0,T]) \cap \mathbf{C}^1(0,T) : |\lambda(\tau) - \lambda(0)| \leq \frac{1}{4},\ |\lambda'(\tau)| \leq C,\ 0 \leq \tau \leq T\} \tag{4.19}$$

endowed with the norm

$$||\lambda||_{1,\infty} = \sup_{0 \leq \tau \leq T} \{|\lambda(\tau)| + |\lambda'(\tau)|\}\,. \tag{4.20}$$

Let us introduce the following functional spaces:

$$\mathcal{W}(T) = \left\{ g \in \mathcal{Y}_{7/6-\delta/2,\beta}(T),\ \ \frac{\partial g}{\partial \tau} \in \mathcal{Y}_{7/6-\delta/2,\beta}(T) \right\} \tag{4.21}$$

with the norm,

$$||g||_{\mathcal{W}} = |||g|||_{7/6-\delta/2,\beta} + \left|\left|\left| \frac{\partial g}{\partial \tau} \right|\right|\right|_{7/6-\delta/2,\beta} \tag{4.22}$$

and $\mathcal{Z}(T) = \mathcal{W} \times \Lambda(T)$. We define an operator $\mathcal{T}$ from $\mathcal{Z}$ into itself as follows. Given $(g,\lambda) \in \mathcal{Z}$ let $\tilde{g}_1$ be the solution of:

$$\begin{aligned} \frac{\partial \tilde{g}_1}{\partial \tau}(\tau, k_1) &= \mathcal{L}_{k,2}(f_0, \tilde{g}_1)(k_1, \tau) + \frac{1}{\lambda(\tau)} \mathcal{L}_{k,1}(f_0, \tilde{g}_1)(k_1, \tau) + \\ &\quad + \frac{1}{\lambda^2(\tau)} \left(\mathcal{R}_1(\tau, k_1) + \mathcal{R}_2(\tau, k_1, g) \right) \qquad (4.23) \\ \tilde{g}_1(0) &= 0. \qquad (4.24) \end{aligned}$$

The function $\tilde{g}_1$ is uniquely defined due to Theorem 3.1. Moreover, the limit

$$b(\tau) \equiv b_{g,\lambda}(\tau) \equiv \lim_{k \to 0} k^{7/6} \tilde{g}_1(\tau, k) \qquad (4.25)$$

exists. We define the function $\tilde{\lambda}(t)$ as the solution of the integral equation

$$\tilde{\lambda}(\tau) \equiv a(\tau, 0) + \frac{1}{A} \int_0^\tau \frac{\partial a}{\partial \overline{\tau}}(\tau, \overline{\tau})\, \tilde{\lambda}(\overline{\tau})\, d\,\overline{\tau} - b(\tau) \equiv \mathcal{S}(\tilde{\lambda}), \qquad (4.26)$$

where a is defined by (3.88) in Lemma 3.21. Let us suppose for the moment that the function $\tilde{\lambda}(\tau)$ solution of (4.26) is well defined. We then define a function $\tilde{g}_2$ by means of

$$\tilde{g}_2(\tau, k) = \frac{1}{A} \left\{ H(\tau, \tau, k)\, \tilde{\lambda}(\tau) - H(\tau, 0, k)\, \tilde{\lambda}(0) - \int_0^\tau \frac{\partial H}{\partial \overline{\tau}}(\tau, \overline{\tau}, k)\, \tilde{\lambda}(\overline{\tau})\, d\overline{\tau} \right\} \quad (4.27)$$

where H is the solution of the problem (3.85), (3.86) whose existence and uniqueness is asserted in Lemma 3.21.
After all these preliminaries we define

$$\begin{aligned} &\mathcal{T} : \ \mathcal{Z} \to \mathcal{Z} \\ &\mathcal{T}(g, \lambda) = (\tilde{g}, \tilde{\lambda}) \qquad (4.28) \\ &\tilde{g} = \tilde{g}_1 + \tilde{g}_2. \end{aligned}$$

Notice that a fixed point of the operator $\mathcal{T}$ is a solution of the integral equation associated to the problem (2.22-(2.23). Moreover, we remark that the solution of such an integral equation solves the differential equation (2.22)-(2.23). Indeed, this follows from the differentiability of the function $\tilde{g}_2$ defined in (4.27) with respect to τ for $k > 0$. Such a regularity can be seen by differentiating formally the right hand side of (3.101) with respect to τ and using the regularity properties of the function H proved in Lemma 3.21 (cf. (3.91) and (3.92)).

We then proceed to check that the operator $\mathcal{T}$ is well defined. As a first step we derive a local well-posedness result for (4.26). To this end we first prove an auxiliary result. Let us denote as $\mathbf{T}(g; \lambda) = \tilde{g}_1$ the solution of (4.23), (4.24) and $\mathbf{S}(g; \lambda) = \mathbf{T}(g) - b_{g,\lambda}\, k^{-7/6}$. We then have the following

Lemma 4.6 *Suppose that* $\lambda \in \Lambda(T)$ *satisfies* $||\lambda||_{1,\infty} < \infty$ *with* $||\lambda||_{1,\infty}$ *defined in (4.20) and* $g, \partial g/\partial\tau \in \mathbf{Y}_{7/6-\delta/2,\beta}(T)$*. Then the function* $b(\tau)$ *defined in (4.25) satisfies*

$$|b(\tau)| + |b'(\tau)| \leq C\,\tau^{3\delta}, \quad 0 \leq \tau \leq T \tag{4.29}$$

Moreover

$$|b_{g,\lambda}(\tau) - b_{h,\mu}(\tau)| + |b'_{g,\lambda}(\tau) - b'_{h,\mu}(\tau)| \leq C\,\tau^{3\delta}\,(||\lambda-\mu||_{1,\infty} + ||g-h||_{\mathcal{W}}) \tag{4.30}$$

and

$$||\mathbf{S}(g;\lambda) - \mathbf{S}(h;\mu)||_{\mathcal{W}} \leq CT^{3\delta/2}(||g-h||_{\mathcal{W}} + ||\lambda-\mu||_{L^\infty(0,T)}) \tag{4.31}$$

for $0 \leq \tau \leq T$*, where* $C = C(A, B, D, \delta, M, M', d)$ *and* $d = |||g|||_{\mathcal{W}} + |||h|||_{\mathcal{W}} + ||\lambda||_{1,\infty} + ||\mu||_{1,\infty} + ||g-h||_{\mathcal{W}}$*.*

Proof of Lemma 4.6. The existence of the functions $\tilde{g}_1$ and $b(\tau)$ and the part of the estimate (4.29) for b is just a consequence of Theorem 3.1.

In order to estimate $b'(\tau)$, we differentiate (4.23) with respect to τ. The resulting equation has the form:

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\tilde{g}_1}{\partial\tau}\right) = \mathcal{L}_{k,2}(f_0, \frac{\partial\tilde{g}_1}{\partial\tau})(k_1,\tau) + \frac{1}{\lambda(\tau)}\mathcal{L}_{k,1}(f_0, \frac{\partial\tilde{g}_1}{\partial\tau})(k_1,\tau) + \mathcal{F}(k_1, g, \tilde{g}_1, \frac{\partial g}{\partial\tau}, \tau). \tag{4.32}$$

Arguing as in the proof of Lemmas 4.3 and 4.4, we deduce

$$\left|\left|\left|\mathcal{F}(k_1, g, \tilde{g}_1, \frac{\partial g}{\partial\tau}, \tau)\right|\right|\right|_{3/2-\delta,\beta} \leq C||g||_{\mathcal{W}} \tag{4.33}$$

The estimate for $b'(\tau)$ in (4.29) then follows from Theorem 4.1. Combining (4.18) and Theorem 3.1 we obtain:

$$|b_{g,\lambda} - b_{h,\mu}| \leq C\,T^{3\delta}\left(|||g-h|||_{7/6-\delta/2,\beta} + ||\lambda-\mu||_{L^\infty(0,T)}\right) \tag{4.34}$$

$$|||\mathbf{S}(g;\lambda) - \mathbf{S}(h;\mu)|||_{7/6-\delta/2,\beta} \leq C\,T^{3\delta/2}(|||g-h|||_{7/6-\delta/2,\beta} + ||\lambda-\mu||_{L^\infty(0,T)}). \tag{4.35}$$

Arguing as in the proof of (4.33) we arrive at

$$\left|\left|\left|\mathcal{F}(k_1, g, \mathbf{T}(g;\lambda), \frac{\partial g}{\partial\tau}, \tau) - \mathcal{F}(k_1, h, \mathbf{T}(h;\mu), \frac{\partial h}{\partial\tau}, \tau)\right|\right|\right|_{3/2-\delta,\beta} \leq C(||g-h||_{\mathcal{W}} + ||\lambda-\mu||_{L^\infty(0,T)}) \tag{4.36}$$

Using again Theorem 3.1 we deduce

$$|b'_{g,\lambda} - b'_{h,\mu}| \leq C\, T^{3\delta}(|||g-h|||_{\mathcal{W}} + ||\lambda - \mu||_{L^\infty(0,T)}), \quad (4.37)$$

$$\left|\left|\left|\frac{\partial}{\partial t}\mathbf{S}(g;\lambda) - \frac{\partial}{\partial t}\mathbf{S}(h;\mu)\right|\right|\right|_{7/6-\delta/2,\beta} \leq CT^{3\delta/2}(||g-h||_{\mathcal{W}} + ||\lambda-\mu||_{L^\infty(0,T)}). \quad (4.38)$$

This concludes the proof of the Lemma 4.6. □

We can now prove a local well-posedness result for (4.26).

Lemma 4.7 *For any $M > 0$ and $M' > 0$ there exists T such that, $T = T(A, B, D, \delta, M, M')$ and a unique $\lambda \in \boldsymbol{C}([0,T])$ solving (4.26) for $0 \leq \tau \leq T$. Moreover,*

$$|\lambda(\tau) - A| \leq C(A, B, D, \delta, M, M')\, T^{3\,\delta}, \quad 0 \leq \tau \leq T. \quad (4.39)$$

Proof of Lemma 4.7. We notice that the operator $\mathcal{S}$ defined in (4.26) maps $\mathcal{C}[0,T]$ onto $\mathcal{C}[0,T]$ and is contractive for T sufficiently small. Indeed, due to (4.30) and (3.90), we have

$$|\mathcal{S}(\lambda_1)(\tau) - \mathcal{S}(\lambda_2)(\tau)| \leq C\, q(T)\, T^{3\delta}||\lambda_1 - \lambda_2||_{1,\infty} \quad (4.40)$$

where $C = C(\delta)$ and

$$q(T) = \sup_{0 \leq \overline{\tau} \leq \tau \leq T} |(\tau - \overline{\tau})^{1-3\,\delta}\frac{\partial a}{\partial \overline{\tau}}(\tau, \overline{\tau})|. \quad (4.41)$$

Moreover:

$$\begin{aligned} ||\mathcal{S}(\lambda) - b(\cdot) - a(\cdot,0)||_\infty \leq q(T)\, T^{3\delta}(||b||_\infty \quad &+ \quad ||a(\cdot,0)||_\infty + \\ &+ \quad ||\lambda - b(\cdot) - a(\cdot,0)||_\infty) \end{aligned} \quad (4.42)$$

By Theorem 3.1 and Lemma 3.21, we have, for some $\tilde{T} = \tilde{T}(A, B, D, \delta, M, M') > 0$

$$||b(\cdot)||_\infty + ||a(\cdot,0)||_\infty + q(T) \leq C(A, B, D, \delta, M, M'), \quad 0 \leq T \leq \tilde{T}. \quad (4.43)$$

Therefore, a standard fixed point argument concludes the proof of the Lemma. □

Lemma 4.8 *The function $\tilde{\lambda}$ solution of the integral equation (4.26) satisfies:*

$$|\tilde{\lambda}(\tau)| \leq C\,(||a(\cdot,0)||_\infty + ||b||_\infty)\,, \quad 0 \leq \tau \leq T \quad (4.44)$$

$$|\tilde{\lambda}_\tau(\tau)| \leq C||b'||_\infty, \quad 0 \leq \tau \leq T \quad (4.45)$$

for $T > 0$ is sufficiently small.

Proof of the Lemma 4.8. The inequality (4.44) is a consequence of (3.90) and (4.26). On the other hand, in order to derive (4.45), we remark that integration by parts in (4.26) yields:

$$\frac{1}{A}\int_0^{\tau} a(\tau,\overline{\tau})\,\tilde{\lambda}'(\overline{\tau})\,d\overline{\tau} + b(\tau) = 0. \tag{4.46}$$

Differentiating this equation we obtain

$$\tilde{\lambda}'(\tau) + \frac{1}{A}\int_0^{\tau} \frac{\partial a}{\partial \tau}(\tau,\overline{\tau})\,\tilde{\lambda}'(\overline{\tau})\,d\overline{\tau} + b'(\tau) = 0 \tag{4.47}$$

that, combined with (3.93) gives (4.45). □

End of the Proof of Theorem 4.1. The proof reduces to show that the operator $\mathcal{T}$ defined in (4.28) is a contraction for T small enough. Notice that

$$\mathcal{T}(g,\lambda) = (\mathbf{T}(g) + \tilde{g}_2, \tilde{\lambda}). \tag{4.48}$$

Let us first show that $\mathbf{T}(g) + \tilde{g}_2 \in \mathcal{W}(T)$. Indeed, using (4.27) and (3.88) we obtain:

$$\lim_{k\to 0} k^{7/6}\,\tilde{g}_2(\tau,k) = \lambda(\tau) - a(\tau,0)\,\lambda(0) - \frac{1}{A}\int_0^{\tau} \frac{\partial a}{\partial \tau}(\overline{\tau},\overline{\tau})\,d\overline{\tau} \tag{4.49}$$

Combining (4.25), (4.26) and (4.49) it then follows that:

$$\lim_{k\to 0}\left(k^{7/6}(\mathbf{T}(g) + \tilde{g}_2)\right) = 0 \tag{4.50}$$

Then, the fact that, $\mathbf{T}(g) + \tilde{g}_2 \in \mathcal{W}(T)$ follows from (3.94), (3.95), (4.27) and (4.31). Moreover we also obtain:

$$||(\mathbf{T}(g) + \tilde{g}_2) - (\mathbf{T}(h) + \tilde{h}_2)||_{\mathcal{W}} \leq \frac{1}{4}\left(||g-h||_{\mathcal{W}} + ||\lambda - \mu||_{1,\infty}\right) \tag{4.51}$$

for $T > 0$ sufficiently small.

On the other hand, in order to keep track of the dependence of $a(\cdot,0)$ with respect to λ we denote as $H_\lambda(t,0)$ the solution of (3.85) and a_λ the function defined by (3.88) in Lemma 3.21. Lemma 4.8 then yields

$$||\tilde{\lambda} - \tilde{\mu}||_{1,\infty} \leq C\left(||b_{g,\lambda} - b_{h,\mu}||_{1,\infty} + ||a_\lambda(\cdot,0) - a_\mu(\cdot,0)||_{\infty}\right). \tag{4.52}$$

The first term of the right hand side of (4.52) has been estimated in (4.30). On the other hand, the second term might be estimated as follows. Consider

$$\begin{aligned}\frac{\partial}{\partial \tau}(H_\lambda - H_\mu) \;&=\; \mathcal{L}_{k,1}(f_0, H_\lambda - H_\mu) + \frac{1}{\lambda(\tau)}\mathcal{L}_{k,2}(f_0, H_\lambda - H_\mu) + \\ &\left(\frac{1}{\lambda(\tau)} - \frac{1}{\mu(\tau)}\right)\mathcal{L}_k(f_0, H_\mu)\end{aligned} \tag{4.53}$$

Using that

$$\left|\left|\left(\frac{1}{\lambda(\tau)}-\frac{1}{\mu(\tau)}\right)\mathcal{L}_{k,2}(f_0,H_\mu)(t)\right|\right|_{3/2-\delta,\beta}\leq C||\lambda-\mu||_\infty$$

and Theorem 3.1 we deduce that

$$||a_\lambda(\cdot,0)-a_\mu(\cdot,0)||_\infty\leq C\,||\lambda-\mu||_{\mathcal{W}}. \tag{4.54}$$

Combining (4.30) and (4.54) we obtain,

$$||\tilde{\lambda}-\tilde{\mu}||_{1,\infty}\leq\frac{1}{4}\left(||g-h||_{\mathcal{W}}+||\lambda-\mu||_{1,\infty}\right) \tag{4.55}$$

for $T>0$ sufficiently small. Formulae (4.49), (4.51) and (4.55) imply that $\mathcal{T}$ is a contractive operator, whence the operator $\mathcal{T}$ defined in (4.28) has a unique fixed point. Finally, changing to the time variable t using (2.21), Theorem 4.1 follows. □

Remark 4.9 *We notice that the dependence on M, M' of the different constants C that have been used in the proof of Theorem 4.1 is due to the dependence on M, M' of the terms $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{U}_k$, $k=1,2,3$ in (2.19), (2.20) and (3.13)-(3.15). This fact is relevant because in the next Section we will derive refined estimates on the solution f of (2.10)-(2.11) that, in particular, will provide estimates on the terms $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{U}_k$, $k=1,2,3$ independent on M, M'. This will make possible to show that the solution f constructed in Theorem 4.1 can be extended on a time interval independent on M, M'.*

5 The limit $M, M'\to\infty$.

5.1 Uniform bounds.

The aim of this subsection is to obtain uniform bounds on the solutions of the truncated nonlinear problem (2.10)-(2.12) with respect to the truncation parameters M and M'. The main result that we prove is an estimate of the form

$$0\leq f(t,k)\leq L\frac{e^{-Dk}}{k^{7/6}},\quad \text{if }\ k>0,\ t\in(0,T), \tag{5.1}$$

with L and T independent of M and M' and with D as in (2.5). We recall that, although the functions f depend on M and M' we will not write this dependence explicitly.

Notice that, due to (2.5) and (2.5)-(2.11), we have, for all $M>0$ and $M'>0$:

$$f(t,k)=f(0,k)\leq L\frac{e^{-Dk}}{k^{7/6}},\quad \text{for all }\ k>M'\ \text{ and }\ t>0, \tag{5.2}$$

whence (5.1) holds immediately for all $k>M'$. Our goal now is to extend the range of validity of this inequality to the values $k<M'$.

Due to the interaction between the regions of small and large values of k, it is not possible to obtain the estimate (5.1) without estimating also the function $f(t,k)$ for k of order one. More precisely, in the derivation of (5.1), we will also obtain the following

$$|f(t,k) - a(t)\, k^{-7/6}| \leq L\, k^{-7/6+\delta/2}, \quad k \leq 1, \;\; t \in (0,T), \tag{5.3}$$
$$|a(t)| \leq L, \quad \text{for } \; t \in (0,T), \tag{5.4}$$

with L and T as in (5.1). The key idea to prove (5.1), (5.3), (5.4) is to use a standard continuity argument. More precisely, it turns out that the functions $f(t,k)$ solutions of problems (2.10)-(2.12) satisfy (5.1), (5.3), (5.4) in an interval of time $t \in [0, T(M,M')]$. This is proved in the next Lemma. In the rest of this subsection we extend the range of validity of these inequalities, to a time interval independent on M and M'. Since we are interested in the limit as M and M' approach to ∞, we will assume from now on that M and M' are larger than a positive fixed number.

Lemma 5.1 *For any $M > 0$ and $M' > 0$, there exists $T(M,M')$ such that the solution f of (2.10)-(2.12), with f_0 as in (2.3)-(2.5), obtained in Theorem 4.1, satisfies (5.1), (5.3), (5.4) with $L = 4\,B$, where B is as in (2.3)-(2.3), for $t \in [0, T(M,M')]$.*

Proof of Lemma 5.1. For $k > M'$ this is a consequence of the fact that $W_{M,M'}$ vanishes. For $k \leq M'$ the result is a consequence of (4.1) in Theorem 4.1. □

Our purpose is now to extend this estimates to a finite time T independent on M'. Let us denote from now on $T_{max}(M,M',L)$ the size of the largest interval of the form $[0,T]$ where (5.1), (5.3), (5.4) hold.

Lemma 5.2 *Let f be the solution of (2.10)-(2.12). There exists $T > 0$, $T = T(L)$ independent of M and M' such that*

$$f(t,k) \geq \frac{f_0(k)}{2}, \quad 1 \leq k \leq 2, \;\; t \in [0, \min\{T, T_{max}(M,M',L)\}), \tag{5.5}$$

Proof of Lemma 5.2. Notice that,

$$\frac{\partial f_1}{\partial t} \geq -f_1 \int_{D(k_1)} f_2(1+f_3+f_4) W_{M,M'}\, dk_3\, dk_4, \quad \text{for } \;\; 0 \leq t \leq T_{max}(M,M',L). \tag{5.6}$$

In order to derive a lower estimate for $\frac{\partial f}{\partial t}$ we need an upper estimate for the integral term on the right hand side of (5.6). To this end we first use that:

$$\int_{D(k_1)} f_2 W_{M,M'}\, dk_3\, dk_4 \leq \frac{1}{\sqrt{k_1}} \int_0^\infty \int_{-k_2-k_1}^{k_2+k_1} \sqrt{k_2}\, f_2\, d\xi\, dk_2 \tag{5.7}$$

where $\xi = k_4 - k_3$. Therefore

$$\begin{aligned}\int_{D(k_1)} f_2 W_{M,M'}\, dk_3\, dk_4 &\leq \frac{2\,L}{\sqrt{k_1}} \int_0^\infty \sqrt{k_2}\, \frac{e^{-Dk_2}}{k_2^{7/6}}\, (k_2 + k_1)\, dk_2 \\ &= C\, L\, (k_1^{-1/2} + \, k_1^{1/2}), \end{aligned} \tag{5.8}$$

where C is a positive constant independent of M, M' and L. On the other hand, a straightforward calculation, using (5.1), gives:

$$\int_{D(k_1)} f_2\, (f_3 + f_4) W_{M,M'}\, dk_3\, dk_4 \leq C\, L^2\, k_1^{-2/3}, \tag{5.9}$$

where C is a positive constant independent of M, M' and L. Combining (5.8), (5.9) we arrive at:

$$\int_{D(k_1)} f_2\, (1 + f_3 + f_4)\, W_{M,M'}\, dk_3\, dk_4 \leq C\, L\, (k_1^{-1/2} + \, k_1^{1/2}) + C\, L^2\, k^{-2/3} \tag{5.10}$$

for $0 \leq t \leq T(M, M')$. Therefore, by (5.6)

$$\frac{\partial f}{\partial t} \geq -C\, L\, (k_1^{-1/2} + \, k_1^{1/2}) - C\, L^2\, k^{-2/3}, \quad \text{for} \;\; 0 \leq t \leq T_{max}(M, M', L). \tag{5.11}$$

Integrating this equation for $k \in (1, 2)$ Lemma 5.2 follows. □

We now prove the following,

Lemma 5.3 *Suppose that f is a solution to (2.10)-(2.12) with initial data f_0. Then, there exists two positive constants $\rho = \rho(L)$ and $\kappa = \kappa(L)$, independent of M and M' such that*

$$\int_{D(k_1)} f_2\, (1 + f_3 + f_4)\, W_{M,M'}\, dk_3\, dk_4 \geq \frac{\kappa}{\sqrt{k_1}}\, \min\{M, k_1\}\, \chi\left(\frac{k_1}{M'}\right) \tag{5.12}$$

for $0 \leq t \leq T_{max}(M, M', L)$.

Proof of Lemma 5.3.

$$\begin{aligned}&\int_{D(k_1)} f_2\, (1 + f_3 + f_4)\, W_{M,M'}\, dk_3\, dk_4 \geq \chi\left(\frac{k_1}{M'}\right) \int_{D(k_1)} f_2\, W_{M,M'}\, dk_3\, dk_4 \\ &\geq \chi\left(\frac{k_1}{M'}\right) \int_0^{k_1/2} \int_{-k_1-k_2}^{k_1+k_2} W_{M,M'}\, d\xi\, f_2\, dk_2 \\ &\geq \frac{2}{\sqrt{k_1}} \chi\left(\frac{k_1}{M'}\right) \int_0^{k_1/2} \sqrt{k_2}\, f_2 \int_0^{k_1-k_2} \chi\left(\frac{\xi}{M}\right)\, d\xi\, dk_2 \\ &= \frac{1}{\sqrt{k_1}} \chi\left(\frac{k_1}{M'}\right)\, \min\{k_1, M\} \int_0^\infty \sqrt{k_2}\, f_2\, dk_2. \end{aligned} \tag{5.13}$$

using Lemma 5.2 we derive a uniform lower estimate for the last integral and the Lemma follows.

Lemma 5.4 *Suppose that f is a solution to (2.10)-(2.12) satisfying (5.1) with initial data f_0. Then, there exists a positive constant $\rho = \rho(L)$, independent on M and M' such that,*

$$\int_{D(k_1)} f_3\, f_4\, (1 + f_1 + f_2)\, W_{M,M'}\, dk_3\, dk_4 \le C\, \chi\left(\frac{k_1}{M'}\right) \frac{e^{-D\,k_1}}{k_1^{7/3}} \min\{k_1, M\} \tag{5.14}$$

for $k_1 \ge \rho$ and $t \le T_{max}(M, M')$.

Proof of Lemma 5.4. Estimate (5.1) implies that there exists $\rho = \rho(L) > 0$ such that $f_1 \le 1$ for $k_1 \ge \rho$. Then, for $k_1 \ge \rho$

$$\int_{D(k_1)} f_3\, f_4\, (1 + f_1 + f_2)\, W_{M,M'}\, dk_3\, dk_4 \le 2 \int_{D(k_1)} f_3\, f_4\, (1 + f_2)\, W_{M,M'}\, dk_3\, dk_4 \tag{5.15}$$

Using again (5.1) we may write

$$\int_{D(k_1)} f_3\, f_4\, (1 + f_2)\, W_{M,M'}\, dk_3\, dk_4 = C\, e^{-D\,k_1} \int_0^{\infty} dk_2 (1 + f_2)\, e^{-D\,k_2}\, J(k_1, k_2) \tag{5.16}$$

$$J(k_1, k_2) = (k_1 + k_2)^{-7/6} \int_0^{k_1+k_2} d\xi\, \frac{W_{M,M'}}{(k_1 + k_2 - \xi)^{7/6}} \tag{5.17}$$

where the change of variables $k_2 = k_3 + k_4 - k_1$, $\xi = k_4 - k_3$ as well as the fact that $k_1 + k_2 + \xi \ge k_1 + k_2$ for $\xi \ge 0$ have been used.

Consider first the case when $\rho < k_1 \le 2M$ and $k_2 \ge k_1$. Using the estimate $W_{M,M'} \le k_3^{1/2}\, k_1^{-1/2} = (k_1 + k_2 - \xi)^{1/2} k_1^{-1/2}$ we deduce that:

$$J(k_1, k_2) \le C\, k_1^{-1/2} (k_1 + k_2)^{-5/6} \le C\, k_1^{-4/3} \tag{5.18}$$

On the other hand, we use that $W_{M,M'} \le \min\{k_2^{1/2}\, k_1^{-1/2}, (k_1 + k_2 - \xi)^{1/2} k_1^{-1/2}\}$ holds if $\rho < k_1 \le 2M$ and $k_2 \le k_1$. Therefore, an explicit computation yields

$$J(k_1, k_2) \le C\, (k_1 + k_2)^{-7/6}\, k_2^{1/3}\, k_1^{-1/2} \le \frac{1}{k_1^{4/3}} \left(\frac{k_2}{k_1}\right)^{1/3}. \tag{5.19}$$

Where, in the derivation of this formula, we split the domain of integration in (5.17) in the intervals, $(0, k_1 - k_2)$ and $(k_1 - k_2, k_1 + k_3)$. In the original variables these regions are equivalent to $k_4 \le k_1$ and $k_4 \ge k_1$ respectively.

Suppose now that $k_1 \ge 2M$. In this case, a geometrical argument shows that, for the values of k_3, k_4 where $W_{M,M'} \ne 0$, the values of k_3, k_4 can be estimated from below by means of k_1.More precisely, there exists a positive constant κ, independent on k_1, k_3, k_4, M and M' such that, for $(k_3, k_4) \in D(k_1)$, and $W_{M,M'} \ne 0$, there holds $k_3 \ge \kappa k_1$ and $k_4 \ge \kappa k_1$. Using $W_{M,M'} \le \min(1, k_2^{1/2}\, k_1^{-1/2})$, it then follows that

$$J(k_1, k_2) \le C\, k_1^{-7/3}\, M\, \min\{1, k_2^{1/2}\, k_1^{-1/2}\} \le C\, k_1^{-7/3}\, M\, \min\{1, k_2^{1/3}\, k_1^{-1/3}\}. \tag{5.20}$$

By (5.18), (5.19) and (5.20) we obtain

$$J(k_1, k_2) \leq \frac{C}{k_1^{7/3}} \min\{k_1, M\} \min\left\{1, \left(\frac{k_2}{k_1}\right)^{1/3}\right\} \tag{5.21}$$

Plugging this in (5.16) (5.1), Lemma 5.4 follows. □

Combining now the two previous Lemmas we can obtain the following upper estimate for the solutions.

Lemma 5.5 *Suppose that f is a solution to (2.10)-(2.12) satisfying (5.1) in $0 \leq t \leq T_{max}(M, M')$ with initial data f_0 satisfying (2.3)-(2.5) . Then, there exists $\rho = \rho(L)$ independent on M and M' such that,*

$$f(t, k_1) \leq \frac{L}{2}\, k_1^{-7/6}\, e^{-D\, k_1}, \quad k_1 \geq \rho,\ 0 \leq t \leq T_{max}(M, M') \tag{5.22}$$

Proof of Lemma 5.5. Using in (2.10) the estimates (5.12) and (5.14) we obtain

$$\frac{\partial f}{\partial t} \leq \left(C\, k_1^{-7/3}\, e^{-D\, k_1} - \frac{\kappa}{\sqrt{k_1}}\, f\right) \min\{M, k_1\} \chi\left(\frac{k_1}{M'}\right) \tag{5.23}$$

By the maximum principle, we obtain,

$$f(k_1, t) \leq \max\left\{f_0(k_1), \frac{C}{\kappa}\, \frac{e^{-D\, k_1}}{k_1^{11/6}}\right\}. \tag{5.24}$$

Combining (5.1) and (5.24), Lemma 5.5 follows. □

As a final step, we prove that (5.22) also holds for $0 < k_1 \leq \rho$ as well as the improved estimates (5.3) (5.4). To this end we use the regularity estimates derived for the solutions of (2.10)-(2.12) in Section 3.

Proposition 5.6 *Suppose that f is a solution to (2.10)-(2.12) satisfying (5.1), (5.3) and (5.4) in $0 \leq t \leq T_{max}(M, M')$ with initial data f_0 satisfying (2.3)-(2.5). There exists $T^* = T^*(A, B, \delta)$ such that, if $T_{max}(M, M') \leq T^*$*

$$f(t, k_1) \leq \frac{1}{2}\, L\, k_1^{-7/6}\, e^{-D\, k_1}, \quad 0 < k_1 \leq \rho \tag{5.25}$$

$$|f(t, k) - a(t)\, k^{-7/6}| \leq \frac{L}{2}\, k^{-7/6+\delta/2}, \quad k \leq 1, \tag{5.26}$$

$$|a(t)| \leq \frac{L}{2}, \tag{5.27}$$

for $0 \leq t \leq T_{max}(M, M')$.

Remark 5.7 *The key point in Proposition 5.6 is that T^* is independent on M, M'.*

Proof of the Proposition 5.6. Let us pick $M_0 > 0$ large enough but fixed ($M_0 = 4$ would work). We assume from now on that $M \geq M_0$, $M' \geq M_0$. The equation satisfied by f can be written as:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \int_{D(k_1)} W_{M_0,M_0}\, \tilde{q}(f)\, dk_3\, dk_4 + \int_{D(k_1)} (W_{M,M'} - W_{M_0,M_0})\; \tilde{q}(f)\, dk_3\, dk_4 \\ &+ \int_{D(k_1)} W_{M,M'}\; (f_3\, f_4 - f_1\, f_2)\; dk_3\, dk_4. \end{aligned} \tag{5.28}$$

Using (3.62), (3.63), (3.64) and (3.65) we can rewrite (5.28) as follows:

$$\frac{\partial f}{\partial t} = \lambda^2(t) \int_{D(k_1)} W_{M_0,M_0}\, \tilde{\ell}(f_0, g)\, dk_3\, dk_4 + \mathcal{S} \tag{5.29}$$

$$\begin{aligned}\mathcal{S} &= \lambda(t) \int_{D(k_1)} W_{M_0,M_0}\, \tilde{n}(f_0, g)\, dk_3\, dk_4 + \int_{D(k_1)} (W_{M,M'} - W_{M_0,M_0})\; \tilde{q}(f)\, dk_3\, dk_4 \\ &+ \int_{D(k_1)} W_{M,M'}\, r(f)\, dk_3\, dk_4 + \lambda^3(t) \int_{D(k_1)} W_{M_0,M_0}\, \tilde{q}(f_0)\, dk_3\, dk_4 \end{aligned} \tag{5.30}$$

$$= \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4. \tag{5.31}$$

In order to apply Theorem 3.1 we need to bound the source term $\mathcal{S}$ in (5.31). This is done in the following Lemma.

Lemma 5.8 *Suppose that f satisfies the asumptions in Proposition 5.6. Then,*

$$||\mathcal{S}(t)||_{3/2-\delta,\, \beta} \leq C(L), \quad 0 < t < T_{max}(M, M') \tag{5.32}$$

Proof of Lemma 5.8. The term $\mathcal{S}_1$ in (5.31) is estimated as the term $\mathcal{R}_2$ in the Lemma 4.4. In the third term $\mathcal{S}_3$, in order to obtain an estimate uniform with respect to M we use the exponential decay of f in (5.1) to bound the integral in the region where $k_3 \geq 1$ or $k_4 \geq 1$. To estimate the contribution in the region where $k_3 \leq 1, k_4 \leq 1$ we use the fact that $r(f)$ is quadratic with respect to f and therefore, its contribution is of lower order. Actually, the argument is exactly the same as the one which has been used in Lemma 4.4 to estimate the quadratic terms of $\mathcal{R}_2$. The main novelty arises in the estimate of $\mathcal{S}_2$. Notice that, the support of $W_{M,M'} - W_{M_0,M_0}$ is contained in the region where $|k_3 - k_4| \geq M_0$. On the other hand we write

$$|\mathcal{S}_2| \leq a_1(k_1)\, f_1 + a_2(k_1) \tag{5.33}$$

where,

$$a_1(k_1) = \int_{D(k_1)} |W_{M,M'} - W_{M_0,M_0}|\; (f_3\, f_4\, + f_2\, (f_3 + f_4))\; dk_3\, dk_4 \tag{5.34}$$

$$a_2(k_1) = \int_{D(k_1)} |W_{M,M'} - W_{M_0,M_0}|\; f_2\, f_3\, f_4\, dk_3\, dk_4 \tag{5.35}$$

Since the integration in these two formulas takes place in the region where $|k_3 - k_4| > M_0$, the function a_1 and a_2 in (5.34), (5.35) can be bounded by a constant independent on M and M' due to the exponential decay of f. Moreover, both functions a_1 a_2 decay exponentially fast as $k_1 \to +\infty$, due to the exponential decay of the function f. □

End of the Proof of Proposition 5.6. The basic idea is to apply once more Theorem 3.1. Notice that Theorem 3.1 is written using the time variable τ, insted of t. However, (2.21) as well as the fact that $\frac{1}{2} \leq \lambda(t) \leq 2$, imply that the result of Theorem 3.1 can also be applied in the t variable as it has been made in the Sections 3 and 4. Therefore, Theorem 3.1 combined with Lemma 5.8 yields,

$$|f(k,t)| \leq \frac{B}{k^{7/6}} e^{-D\,k} + C\,\frac{T^{3\delta}}{k^{7/6}}, \quad 0 < k < \rho,\ 0 \leq t \leq T_{max}(M, M'), \tag{5.36}$$

where C depends on M_0 but is independent on M and M'. Formula (5.25) follows choosing T small enough but independent on M, M'. Similarly,

$$|a(t)| \leq B + C\,T^{3\delta/2} \quad 0 \leq t \leq 1, \tag{5.37}$$

$$|f(t,k) - a(t)\,k^{-7/6}| \leq (B + CT^{-3\delta/2} k^{-7/6+\delta/2}).$$

Henceforth, (5.26), (5.27) follow choosing again T small enough but independent on M, M'. This concludes the proof of the Proposition 5.6. □

Lemma 5.9 *Suppose that f is a solution to (2.10)-(2.12) satisfying (5.1), (5.3) and (5.4) in $0 \leq t \leq T_{max}(M, M')$ with initial data f_0 satisfying (2.3)-(2.5). Then, the solution f constructed in Theorem 4.1 can be extended to a time interval $[0, T)$ where $T = (A, B, D, \delta)$ is independent of M and M'.*

Proof of the Lemma 5.9. Let us denote as $T_{exist}(M, M')$ the maximal existence time of the solutions constructed in Theorem 4.1. If for some $M > M_0$, $M' > M_0$ we have $T_{exist}(M, M') \geq T^*$ the Lemma will follow. Let us suppose that, on the contrary, for some $M > M_0$ and $M' > M_0$, we have $T_{exist}(M, M') < T^*$. By definition, $T_{max}(M, M') \leq T_{exist}(M, M')$. Moreover, we claim that $T_{max}(M, M') = T_{exist}(M, M')$. Indeed, if $T_{max}(M, M') < T_{exist}(M, M')$, Lemma 5.5 and Proposition 5.6 yield a contradiction since estimates (5.1), (5.3) and (5.4), could be extended beyond $T_{max}(M, M')$. Therefore, as long as the solution of (2.10)-(2.12) exists these estimates hold. The constants arising in the contractivity argument that gives the Theorem 4.1 are then independent on M M' (cf. Remark 4.9). We deduce that there exists a lower bound for $T_{exist}(M, M')$ independent of M and M' and the result follows. □

5.2 Taking the limit $M \to +\infty$, $M' \to +\infty$.

Proposition 5.10 *Suppose that $f = f_{M,M'}$ are the solutions of (2.10)-(2.12) constructed in Theorem 4.1, and defined in the interval of time T independent of M and M'. Then $\lim_{M,M'\to\infty} f_{M,M'}(t,k) = \overline{f}(t,k)$ uniformly on compact sets of $\mathbb{R}^+ \times [0,\overline{T})$. The function $\overline{f}$ is such that $\overline{f} \in \mathbf{Y}_{7/6,\beta}$, $\partial_t \overline{f} \in \mathbf{Y}_{3/2,\beta}$, it solves (2.1), (2.2) for $0 \le t \le T$ and moreover satisfies (5.1), (5.3), (5.4).*

Proof of Proposition 5.10. The idea is to prove that the family $\{f_M\}_{M>M_0}$ satisfies the Cauchy condition with the norm $|||f|||_{7/6-\delta/2,\beta}$. Let us write $f = f_{M,M'}$ and $\tilde{f} = f_{\widetilde{M},\widetilde{M}'}$. It is convenient to use in all this argument the time variable τ instead of t. Notice that the definition on τ in terms of t in (2.21) is different for the solutions f and $\tilde{f}$. We also define $g = f - \lambda(\tau)\, f_0$ and $\tilde{g} = \tilde{f} - \tilde{\lambda}(\tau)\, f_0$ where $\lambda = \lambda_{M,M'}$ and $\tilde{\lambda} = \tilde{\lambda}_{\widetilde{M},\widetilde{M}'}$. Notice that both functions g and $\tilde{g}$ solve problem (2.22) and (2.23). Then

$$\begin{aligned}\frac{\partial(g-\tilde{g})}{\partial\tau} &= \int\int_{D(k)} W_{M,M'}\, \tilde{\ell}(f_0, g-\tilde{g})\; dk_3\, dk_4 + (\lambda_\tau - \tilde{\lambda}_\tau)\, f_0 \\ &\quad + \mathcal{S}_1 + \mathcal{S}_2, \end{aligned} \tag{5.38}$$

where,

$$\begin{aligned}\mathcal{S}_1 &= \int\int_{D(k)} \left(W_{M,M'} - W_{\widetilde{M},\widetilde{M}'}\right)\, \tilde{\ell}(f_0,\tilde{g})\, dk_3\, dk_4 \\ \mathcal{S}_2 &= \int\int_{D(k)} \left(\frac{W_{M,M'}}{\lambda}\, s(f_0,g) - \frac{W_{\widetilde{M},\widetilde{M}'}}{\tilde{\lambda}}\, s(f_0,\tilde{g})\right)\; dk_3\, dk_4 \\ &\quad + \left(\frac{\mathcal{R}_1+\mathcal{R}_2}{\lambda^2} - \frac{\widetilde{\mathcal{R}}_1+\widetilde{\mathcal{R}}_2}{\widetilde{\lambda}^2}\right)\end{aligned}$$

where $\mathcal{R}_i$ and $\widetilde{\mathcal{R}}_i$, $i = 1,2$ are defined by means of (2.19), (2.20) using the functions g and $\tilde{g}$ respectively.

Lemma 5.11 *Let us denote $m \equiv \min(M, M', \widetilde{M}, \widetilde{M}')$. Then for some positive constant $C = C(A,B,D,\delta)$,*

$$||\mathcal{S}_1(t)||_{3/2-\delta/2,\beta} \le C\, e^{-\frac{D\,M}{2}}, \tag{5.39}$$

$$||\mathcal{S}_2(t)||_{3/2-\delta/2,\beta} \le C\, e^{-\frac{D\,M}{2}} + |||g-\tilde{g}|||_{7/6-\delta/2,\beta} + ||\lambda - \tilde{\lambda}||_{L^\infty(0,T)}. \tag{5.40}$$

Proof of Lemma 5.11. We assume without any loss of generality that $\widetilde{M} \ge M$. The estimate (5.39) is a consequence of the exponential decay of the functions g, $\tilde{g}$ and the fact that the support of $W_{M,M'} - W_{\widetilde{M},\widetilde{M}'}$ is contained in the region where $k_3 > M/2$, $k_4 > M/2$. To estimate $\mathcal{S}_2$ we decompose it in the sum of different terms containing the differences $W_{M,M'} - W_{\widetilde{M},\widetilde{M}'}$, $g - \tilde{g}$ and $\lambda - \widetilde{\lambda}$, by means of the usual triangular argument. □

Lemma 5.12 *Under the asumptions of Proposition 5.10*

$$|\lambda(\tau)-\widetilde{\lambda}(\tau)|+|\lambda_\tau(\tau)-\widetilde{\lambda}_\tau(\tau)|\le C|||g-\tilde g|||_{7/6-\delta/2,\beta},\quad 0\le\tau\le T. \tag{5.41}$$

Proof of Lemma 5.12. This result is a consequence of the estimates obtained for the derivatives of the solution of the integral equation (4.26) (cf. (4.46)). On the other hand, using the equation (4.47)

$$\begin{aligned}|\frac{d}{d\tau}\lambda(\tau)-\frac{d}{d\tau}\tilde\lambda(\tau)|\le\frac{1}{A}\int_0^\tau\left|\frac{\partial a}{\partial\tilde\tau}(\tau,\overline{\tau})\right|\left|(\lambda_\tau-\tilde\lambda_\tau)(\overline{\tau})\right|d\overline{\tau}+\\+\frac{1}{A}\int_0^\tau\left|\left(\frac{\partial a}{\partial\tilde\tau}-\frac{\partial a}{\partial\tilde\tau}\right)(\tau,\overline{\tau})\right||\tilde\lambda_\tau|\,d\tilde\tau+|b_\tau(\tau)-\tilde b_\tau(\tau)|\end{aligned} \tag{5.42}$$

The first term in the right hand side of (5.42) is estimated, using (3.90) , by $CT^{3\delta}||\lambda_\tau-\tilde\lambda_\tau||_{L^\infty(0,T)}$. The second one is estimated, applying Theorem 3.1 to (3.85), (3.86), by $CT^{3\delta}||\lambda-\tilde\lambda||_{L^\infty(0,T)}$. Finally, the last one can be estimated, applying once more Theorem 3.1 to (4.23), (4.24), by $CT^{3\delta/2}||\lambda-\tilde\lambda||_{L^\infty(0,T)}+C|||g-\tilde g|||_{7/6-\delta/2,\beta}$. A similar argument with the equation (4.26) shows that

$$||\lambda-\widetilde{\lambda}||_{L^\infty(0,T)}\le CT^{3\delta/2}||\lambda-\widetilde{\lambda}||_{L^\infty(0,T)}+C|||g-\tilde g|||_{7/6-\delta/2,\beta} \tag{5.43}$$

Combining these estimates the Lemma follows for T sufficiently small. □

End of the Proof of Proposition 5.10. Combining Theorem 3.1 with Lemmas 5.11 and 5.12 we obtain

$$\begin{aligned}||\lambda-\widetilde{\lambda}||_{L^\infty(0,T)}+|||g-\tilde g|||_{7/6-\delta/2,\beta}\ \le\ & CT^{3\delta/2}\left(||\lambda-\widetilde{\lambda}||_{L^\infty(0,T)}+|||g-\tilde g|||_{7/6-\delta/2,\beta}\right)\\ &+Ce^{-\frac{DM}{2}}\end{aligned}$$

whence, for T sufficiently small,

$$||\lambda-\widetilde{\lambda}||_{L^\infty(0,T)}+|||g-\tilde g|||_{7/6-\delta/2,\beta}\le Ce^{-\frac{DM}{2}}$$

and the Proposition follows. This shows the existence of $\overline{f}$ as defined in the statement of the Proposition. Notice that, we deduce from (2.10) and (2.12):

$$f_{M,M'}(t,k)=f_0(k)+\int_0^t Q_{M,M'}(f_{M,M'})(s,k)ds.$$

Taking the limit $M,M'\to+\infty$ we deduce that,

$$\overline{f}(t,k)=f_0(k)+\int_0^t Q(\overline{f})(s,k)ds. \tag{5.44}$$

Since the second term in the right hand side of (5.44) is a differentiable function of time, we deduce $\partial_t\overline{f}=Q(\overline{f})\in\mathbf{Y}_{3/2,\beta}$.

6 End of the Proof of Theorem 2.1.

6.1 Uniqueness of Solutions.

Proposition 6.1 *Suppose that f_0 satisfies (2.3)-(2.5). Then, there exists a unique solution of (2.1), (2.2) satisfying (5.1), (5.3) and (5.4).*

Proof of Proposition 6.1. The proof is basically the same as that of Proposition 5.10. Indeed, if f and $\tilde{f}$ are two solutions of (2.1), (2.2) satisfying (5.1), (5.3) and (5.4) then they are of the form $f = \lambda(\tau)\, f_0 + g$, $\tilde{f} = \widetilde{\lambda}(\tau)\, f_0 + \widetilde{g}$ with g and $\widetilde{g}$ in the space $\mathbf{Y}_{7/6-\delta/2,\beta}$. Arguing exactly as in the proof of (5.44) we obtain

$$||\lambda - \widetilde{\lambda}||_{L^\infty(0,T)} + |||g - \tilde{g}|||_{7/6-\delta/2,\beta} \;\; \leq \;\; CT^{3\delta/2}\left(||\lambda - \widetilde{\lambda}||_{L^\infty(0,T)} + |||g - \tilde{g}|||_{7/6-\delta/2,\beta}\right).$$

that yields the desired uniqueness for T small enough. □

6.2 End of the Proof of Theorem 2.1.

The proof of Theorem 2.1 is just a consequence of Proposition 5.10 and Proposition 6.1

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References

[1] R. Baier, T. Stockkamp *Kinetic equations for Bose Einstein condensates from the 2PI effective action*, hep-ph/0412310.

[2] A. M. Balk, *On the Kolmogorov-Zakharov spectra of weak turbulence*, Physica D **139** (2000), 137–157.

[3] A. M. Balk, V. E. Zakharov, *Stability of Weak-Turbulence Kolmogorov Spectra* in Nonlinear Waves and Weak Turbulence, V. E. Zakharov Ed., A. M. S. Translations Series 2, Vol. 182 (1998), 1-81.

[4] C. Cercignani, *The Boltzmann equation and its applications*; Applied Math. Sciences **67**. Springer Verlag, New York 1988.

[5] M. Escobedo, S. Mischler, J. J. L. Velazquez, *On the fundamental Solution of a Linearized Uehling Uhlenbeck Equation*, Preprint.

[6] M. Escobedo, M. A. Herrero, J. J. L. Velazquez, *A Nonlinear Fokker-Planck equation modelling the approach to thermal equilibrium in a homogeneous plasma*, Trans. AMS, **350** (1998), 3837–3901.

[7] C. Josserand and Y. Pomeau, *Nonlinear aspects of the theory of Bose-Einstein condensates*, Nonlinearity **14** (2001), R25-R62.

[8] A. V. Kats, V. M. Kantorovich, *Symmetry properties of the collision integral and non-isotropic stationary solutions in weak turbulence theory*, JETP **37** (1973), 80–85.

[9] A. V. Kats, V. M. Kantorovich, *Anisotropic turbulent distributions for waves with a non-decay dispersion law*, JETP **38** (1974), 102–107.

[10] R. Lacaze, P. Lallemand, Y. Pomeau, S. Rica, *Dynamical formation of a Bose-Einstein condensate*, Physica D **152-153** (2001), 779–786.

[11] F. Leyvraz, *Scaling Theory and Exactly Solved Models in the Kinetics of Irreversible Aggregation* Phys. Reports **383** (2003), 95–212.

[12] X. G. Lu, *A modified Boltzmann equation for Bose-Einstein particles: isotropic solutions and long time behavior*, J. Stat. Phys. **98** (2000), 1335–1394.

[13] X. G. Lu, *On isotropic distributional solutions to the Boltzmann equation for Bose-Einstein particles*, J. Stat. Phys. **116** (2004), 1597–1649.

[14] D.V. Semikov, I.I. Tkachev, *Kinetics of Bose condensation*, Phys. Rev. Lett. **74** (1995), 3093-3097.

[15] D.V. Semikov, I.I. Tkachev, *Condensation of Bosons in the kinetic regime*, Phys. Rev. D **55**, 2 (1997), 489-502.

[16] E. A. Uehling, G. E. Uhlenbeck, *Transport phenomena in Einstein-Bose and Fermi-Dirac gases*, Physical Review **43** (1933), 552-561.

[17] V. E. Zakharov, *Collapse of Langmuir Waves* , Sov. JETP **35** (1972), 908–914.

[18] V. E. Zakharov, Kolmogorov spectra in weak turbulence problems, *Basic Plasma Physics* (A. A. Galeev, R. Sudan, eds.), vol. 2, North Holland, Amsterdam, 1984.