SCALINGS FOR A BALLISTIC AGGREGATION EQUATION

August 13, 2010

Miguel Escobedo[1], Stéphane Mischler[2]

Abstract

We consider a mean field type equation for ballistic aggregation of particles whose density function depends both on the mass and impulsion of the particles. For the case of a constant aggregation rate we prove the existence of self-similar solutions and the convergence of more general solutions to them. We are able to estimate the large time decay of some moments of general solutions or to build some new classes of self-similar solutions for several classes of mass and/or impulsion dependent rates.

1 Introduction

The concern of this work is to establish quantitative estimates on the asymptotic behaviour of the solutions to some Smoluchowski like models for ballistic aggregation. By *ballistic aggregation*, also (improperly) called kinetic coalescence in previous works [6, 11], we mean aggregation phenomena taking place in a system of particles whose density function depends on mass and impulsion. It differs from the simplest aggregation mechanism introduced by Smoluchowski [22] in whose model the particles density function only depends on the mass.

In order to be more precise let us denote by $P = P_y$ with $y = (m, p)$ a particle of mass $m > 0$ and impulsion $p \in \mathbb{R}^d$. The space of particles states is then $Y = \mathbb{R}_+ \times \mathbb{R}^d$ and the velocity of the particle P_y is $v = p/m$. We assume that at a microscopic level (the level of particles) the rate of collision of two particles $P = P_y$ and $P' = P_{y'}$ is a given nonnegative function $a = a(y, y')$ and when these two particles collide they join to form one aggregated particle $P'' = P_{y''}$ in such a way that the mechanism conserves total mass and total impulsion. In other words, the microscopic mechanism reads

$$P_y + P_{y'} \overset{a(y,y')}{\longrightarrow} P_{y''},$$

[1]Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, E–48080 Bilbao, Spain. E-mail : `mtpesmam@lg.ehu.es`

[2]Ceremade - UMR 7534, Université Paris - Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France. E-mail : `mischler@ceremade.dauphine.fr`

with $y'' = (m'', p'')$ given by

$$m'' = m + m', \qquad p'' = p + p'.$$

It is worth mentioning that the above reaction dissipates kinetic energy since, denoting $\mathcal{E}^\sharp = m^\sharp\, |v^\sharp|^2/2$ the kinetic energy of particle $P^\sharp$, we have

$$\begin{aligned}\mathcal{E}'' - \mathcal{E} - \mathcal{E}' &= \frac{1}{2}\frac{|p+p'|^2}{m+m'} - \frac{1}{2}\frac{|p|^2}{m} - \frac{1}{2}\frac{|p'|^2}{m'} \\ &= -\frac{1}{2}\frac{m\,m'}{m+m'}\,|v-v'|^2 \le 0.\end{aligned}$$

At the mesoscopic (or statistical or mean field) level, the system is described at time $t \ge 0$ by the density function $f(t,y) \ge 0$ of particles with state $y \in Y$. For a given initial distribution f_{in}, the evolution of the density f is described by the Smoluchoswki/Boltzmann like equation:

$$\partial_t f = Q(f) \quad \text{in} \quad (0,+\infty) \times Y, \tag{1.1}$$
$$f(0) = f^{in} \quad \text{in} \quad Y. \tag{1.2}$$

The collision operator $Q(f)$ is given by $Q(f) = Q_1(f) - Q_2(f)$, where

$$Q_1(f)(y) = \frac{1}{2}\int_{\mathbb{R}^d}\int_0^m a(y', y-y')\; f(y')\; f(y-y')\; dm'dp', \tag{1.3}$$
$$Q_2(f)(y) = \int_{\mathbb{R}^d}\int_0^\infty a(y, y')\; f(y)\; f(y')\; dm'dp'. \tag{1.4}$$

The two following examples of functions a have been considered in relation with models in astrophysics [26, 1]:

$$a(y,y') = a_{HS}(y,y') := (m^{1/3} + m'^{1/3})^2\,|v-v'|, \tag{1.5}$$

$$a(y,y') = a_{NP}(y,y') := \frac{m+m'}{mm'}\,\frac{1}{|v-v'|^2}. \tag{1.6}$$

This model is seen as a simple test case or elementary analog of more realistic situations in fluid mechanics or astrophysics [4, 12]. We refer to the introduction of [20, 6, 11] for an elementary introduction to physics motivation of such a model. We also refer to [4, 12, 25, 26] and to the references quoted in [20, 6, 11] for a more detailed discussion on the physics of aggregation.

In the context described above it is very natural to impose on the initial data f_{in} to have finite number of particles and momentum. This condition reads:

$$0 \le f^{in} \in L^1\left(Y, (1+m+|p|)\, dydp\right). \tag{1.7}$$

Existence of solutions under that condition has been proved in [20, 6, 11]. It has also been proved that

$$f(t,\cdot) \to 0 \quad \text{in} \quad L^1(Y), \ \text{ as } t \to +\infty, \tag{1.8}$$

that is that the total number of particles tends to 0. This is a first result on the long time asymptotic behaviour of the solutions but still very partial.

A more detailed description of the asymptotic behaviour of the solutions may be obtained by considering scaling properties of the equation (1.1)-(1.4) and the corresponding self similar solutions. Suppose for example that given a solution $f(t,m,p)$ of (1.1)-(1.4), the function $f_r(t,m,p) = r^{-\lambda} f(rt, r^{-\mu} m, r^{-\nu} p)$ is still a solution for any $r > 0$ and for some exponents λ, μ, ν. A self similar solution is then a solution f such that $f = f_r$ for all $r > 0$. It is easy to check that such a function must be of the form $f(t,m,p) = t^\lambda f(1, t^\mu m, t^\nu p)$. These particular solutions may describe sometimes the long time asymptotic behaviour of the solutions of the equation for a suitable family of initial data. The existence of such self similar solutions may still be a delicate problem, see for example [8, 10] and the references therein for recent results in that direction for the Smoluchowski equation. We are very far from being able to treat the general case, when the aggregation kernel $a(y,y')$ actually depends on both mass and momentum of the two colliding particles, or even in the case where the aggregation kernel $a(y,y')$ only depends on the momentum of the two colliding particles. We may then be less ambitious and try only to partially improve on the convergence result (1.8). We may imagine to do so in several ways, listed below by order of accuracy. Let us just define before the moment $M_{\overline{\alpha}}(f)$ of order $\bar\alpha$ of a function f. It is done, depending on the model considered, as follows:

- when $f = f(y)$ with $y = m \in Y = (0,\infty)$ or $y = p \in Y = \mathbb{R}^d$, then $\bar\alpha = \alpha \in \mathbb{R}$ and
$$M_{\bar\alpha}(f) = M_\alpha(f) = \int_Y |y|^\alpha \, f \, dy; \tag{1.9}$$

- when $f = f(y)$ with $y = (m,p) \in Y = (0,\infty) \times \mathbb{R}^d$, then $\bar\alpha = (\alpha, \beta) \in \mathbb{R}^2$ and
$$M_{\bar\alpha}(f) = M_{\alpha,\beta}(f) = \int_Y m^\alpha \, |p|^\beta \, f \, dy. \tag{1.10}$$

The answers may then be:

- **Answer 1.** Upper bound on moment: $\exists\, \bar\alpha$, $\exists\, \nu$, $C \in (0,\infty)$ such that
$$M_{\bar\alpha}(f(t,.)) \le \frac{C}{t^\nu} \qquad \forall\, t \ge 1.$$

- **Answer 2.** Upper and lower bound on moments: $\exists\, \bar\alpha$, $\exists\, \nu_i = \nu_i(\bar\alpha)$, $C_i = C_i(\bar\alpha) \in (0,\infty)$, such that
$$\frac{C_1}{t^{\nu_1}} \le M_{\bar\alpha}(f(t,.)) \le \frac{C_2}{t^{\nu_2}} \qquad \forall\, t \ge 1.$$

- **Answer 3.** Existence of self-similar solution: there exists some profile function $\varphi_\infty : Y \to \mathbb{R}_+$, some exponents λ, μ, $\nu \in \mathbb{R}$ such such that the function
$$\varphi(t,m,p) := t^\lambda \, \varphi_\infty(t^\mu \, m, t^\nu \, p)$$
is a solution to equation (1.1), (1.3), (1.4).

- **Answer 4.** Self-similar behaviour: for any given solution f there exists a self-similar solution φ such that $f \sim \varphi$ as $t \to \infty$, in a sense to be specified.

Depending on the type of aggregation kernel $a(y, y')$ that we consider, we are able to prove one type of answer or another. The results obtained in this work, still very partials, may be classified as follows.

In Section 2 we consider the case of the kernel $a_{HS}(y, y')$ (which depends on both mass and momentum) and the only result that we are able to prove is an upper estimate on some moments (that is a result of type “Answer 1”).

In the remainder of the paper, we focus our attention on easier cases where the aggregation rate a only depends upon the impulsion or the mass, namely $a(y, y') = a(p, p')$, $a(y, y') = a(m, m')$ or even $a(y, y') \equiv 1$. In Section 3 we consider kernels a only depending on the momentum p and p'. A similar case has been considered in [24]. After integration of the particle density function $f(t, m, p)$ with respect to m, the resulting equation may be seen as describing a set of identical particles moving ballistically and such that when two particles moving with velocities v_1 and v_2 collide they form an aggregate particle moving with velocity $v = v_1 + v_2$. This simplified situation has been considered in [25]. We establish several moment estimates of type “Answer 2” when $a(p, p') = |p - p'|^\gamma$ and deduce that when $\gamma = 2$ equation (1.1), (1.3), (1.4) has no self-similar solutions of the form described above. This may suggest perhaps the non existence of self similar solutions neither in the case of the mass and impulsion hard spheres kernel.

Examples where $a \equiv 1$ or $a \equiv a(m, m')$ have already been treated in the literature as simplified models (see for example [19] and [12] for $a = a(m, m')$ and [13] for $a = 1$). When $a = a(m, m')$ it is possible to reduce the original equation (1.1), (1.3), (1.4) to the classical Smoluchowski equation for the zero order moment of the density functions. That type of aggregation rates is sometimes obtained assuming that the velocity v of the particles is determined by their mass m. We treat in Section 4 the case where the kernel depends only on the masses m and m' of the colliding particles and we exhibit a new class of self-similar solutions (that is an “Answer 3” type result). Finally, in Section 5 the case of constant kernel is treated, for which results of type “Answer 3” and “Answer 4” are established.

We end this introduction by some remarks and open questions. A common feature of these equations is that

$$M_{1,0}(t) \equiv M_{1,0}(0) \qquad \text{and} \qquad M_{0,0}(t) \to 0 \quad \text{as} \quad t \to \infty,$$

and when the cross-section a is homogeneous of order $\bar\gamma$ (which belongs to $\mathbb{R}$ or $\mathbb{R}^2$) it is likely that

$$M_{\bar\gamma}(t) \equiv \frac{1}{t} \quad \text{as} \quad t \to \infty, \tag{1.11}$$

a result which is also known to be true for the coagulation equation (see [10, 9, 8]) and for the inelastic Boltzmann equation (see [16] and the references therein). The identity (1.11) has been established when the aggregation rate depends only on the impulsion or on the mass. But only one side of such identity is proved in the case of the true hard spheres aggregation rate, that depends both on the mass and the impulsion of the particles. We ask then.

Open question 1. Is it true that the asymptotic identity behavior (1.11) holds for some true mass and impulsion depending aggregation rate?

An other interesting question should be to establish some asymptotic behavior of typical velocity or impulsion depending quantity. A way to express that in mathematical terms is the following:

Open question 2. Is it possible to exhibit some moment $M_{\bar{\alpha}}$ for which we may determinate the long time behavior of $M_{\bar{\alpha}}/M_0$ (even just saying that it converges as $t \to +\infty$)?

2 Mass and impulsion dependence case: a remark on the hard spheres model.

We start recalling an existence result for initial data f_{in} satisfying the symmetry property $f_{in}(p) = f_{in}(-p)$ for all $p \in \mathbb{R}^d$. The functions satisfying that property will be called even functions in all the remaining of this paper.

Theorem 2.1 *(cf. [11, Theorem 2.6, Theorem 2.8 and Lemma 3.3]) Assume that the aggregation rate a satisfies*

$$\begin{aligned} &0 \le a(y,y') = a(y',y) \le k_S(y)\, k_S(y'), \quad \forall\, y, y' \in Y, \\ &a(m,-p,m',-p') = a(m,p,m',p') \quad \forall\, (m,p),(m',p') \in Y, \\ &a(m,p,m',p') \le a(m,p,m',-p') \quad \forall\, (m,p),(m',p') \in Y \; s.t.\, \langle p,p' \rangle > 0, \end{aligned}$$

with $k_S(y) := 1 + m + |p| + |v|$. Then, for every non negative and even (in the p variable) initial condition $f_{in} \in L^1(Y; k_S^2(y)\, dy)$, there exists a unique solution of (1.1)-(1.4) $f \in C([0,T); L^1(Y; k_S(y)\, dy)) \cap L^\infty(0,T; L^1(Y; k_S^2(y)\, dy))$ $\forall\, T > 0$, satisfying furthermore:

$$\int_Y f(t,.)\, m\, dy \equiv \; Cst, \tag{2.1}$$

$$f(t,.) \text{ is even, so that } \int_Y f(t,.)\, p\, dy \equiv \; 0, \tag{2.2}$$

$$\int_Y f(t,.)\, |v|^k\, dy \le \int_Y f_{in}\, |v|^k\, dy, \quad \forall\, k > 0, \tag{2.3}$$

$$\int_Y f(t,.)\, |p|^2\, dy \le \int_Y f_{in}\, |p|^2\, dy, \tag{2.4}$$

$$\int_Y f\, m^\alpha\, dy \;\to\; 0 \quad \text{when} \quad t \to \infty, \quad \forall\, \alpha < 1. \tag{2.5}$$

Remark 2.2 *(i) It is worth mentioning that the hard spheres collision rate a_{HS} does satisfy the assumption of Theorem 2.1, but not the Manev rate a_{NP}.*

(ii) As a consequence of (2.1), (2.3), (2.4) and (2.5) we deduce that

$$M_{\alpha,\beta}(t) := \int_Y f(t,y)\, m^\alpha\, |p|^\beta\, dy \;\to\; 0 \quad as \quad t \to \infty \tag{2.6}$$

whenever (α,β) belongs to the region

$$\{\beta \in [0,2],\, \alpha < 1 - \beta/2\} \cup \{\beta \ge 2,\, \alpha < 2 - \beta\}.$$

In the case of the hard spheres model we are able to quantify the rate of decay of one of the moment functions of the solution. More precisely, we have the following result.

Lemma 2.3 *Assume that $a = a_{HS}$ and the assumption of Theorem 2.1 hold true. Then the solution f of (1.1)-(1.4) is such that $A^{-1} := M_{-1/3,1}(0) < \infty$ and*

$$M_{-1/3,1}(t) \le \frac{1}{A + t/4} \qquad \forall\, t \ge 0. \tag{2.7}$$

Proof of Lemma 2.3. Notice first that $M_{-1/3,1}(0) < \infty$ because

$$m^{-1/3}\,|p| = m^{2/3}\,|v| \le m^{4/3} + |v|^2 \le 2\,k_S^2.$$

Now, from the expression (1.1)-(1.2) of the collision kernel we have

$$\int_Y Q(f,f)\, m^{-1/3}\,|p|\,dy = \frac{1}{2}\int_Y\int_Y \Delta_{-1/3,\,1}\, f\, f'\,dydy',$$

with

$$\Delta_{-1/3,\,1} = [(m+m')^{-1/3}\,|p+p'| - m^{-1/3}\,|p| - (m')^{-1/3}\,|p'|]\,[r+r']^2\,|v-v'|.$$

On one hand $-\Delta_{-1/3,\,1} \ge 0$ because

$$(m+m')^{1/3}\left(\frac{|p|}{m^{1/3}} + \frac{|p'|}{(m')^{1/3}}\right) \ge |p| + |p'| \ge |p+p'|.$$

On the other hand, if we only take into account the values of v and v' where $v\cdot v' < 0$ and suppose that, for example, $|p| = \min(|p|,|p'|)$ we have

$$\begin{aligned}
-\Delta_{-1/3,\,1} &\ge \left(\frac{|p|}{m^{1/3}} + \left(\frac{|p'|}{(m')^{1/3}} - \frac{|p'|}{(m+m')^{1/3}}\right)\right)[r^2 + (r')^2]\,[|v| + |v'|] \\
&\ge \left(\frac{|p|}{m^{1/3}}\right)[(r')^2]\,[|v'|] = \frac{|p|}{m^{1/3}}\,\frac{|p'|}{(m')^{1/3}}.
\end{aligned}$$

Whence, using that f is even:

$$\begin{aligned}
\frac{d}{dt}\int_Y f\,\frac{|p|}{m^{1/3}}\,dy &\le -\frac{1}{2}\int_{Y^2,\,v\cdot v'<0} \frac{|p|}{m^{1/3}}\,\frac{|p'|}{(m')^{1/3}} f\,f'\,dydy' \\
&\le -\frac{1}{4}\left(\int_Y f\,\frac{|p|}{m^{1/3}}\,dy\right)^2,
\end{aligned}$$

from which (2.7) straightforwardly follows. □

Remark 2.4 *As it has already been noticed (cf. for example [2], [3], [23]), if the aggregation rate is of the form $a(y,y') = A(m,m')B(v,v')$ (as in a_{HS} or a_{NP}) and we assume that $f(t,m,p)$ is a solution of (1.1), (1.3), (1.4) of the form $F(t,m)\,\varphi(p\,m^{-\theta})$ for some function φ such that $\int \varphi(p)dp = 1$ and $\theta \in \mathbb{R}$, then $F(t,m)$ satisfies a Smoluchowski equation with a coagulation rate given by $A(m,m')C(m,m')$ with C depending on the function φ and on B:*

$$\begin{aligned}
\frac{\partial F}{\partial t}(t,m) = \frac{1}{2}\int_0^m & F(t,m-m')F(t,m')\,A(m-m',m')\,C(m-m',m')\,dm' - \\
& - \int_0^\infty F(t,m)F(t,m')\,A(m,m')\,C(m,m')\,dm'
\end{aligned} \tag{2.8}$$

$$C(m,m') = m^{-6\theta}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varphi\left(\frac{p}{m^\theta}\right)\varphi\left(\frac{p'}{m'^\theta}\right)B\left(\frac{p}{m},\frac{p'}{m'}\right)\,dp\,dp'. \tag{2.9}$$

Conversely, consider any φ for which the kernel $C(m,m')$ is well defined, and F a solution of the Smoluchowski equation (2.8) with coagulation rate $A(m,m')C(m,m')$, where $C(m,m')$ is given by (2.9). Then, the function $f(t,m,p)=\varphi(m\,p^{-\theta})\,F(t,m)$ satisfies a kind of averaged (in v) version of (1.1), (1.3), (1.4), namely:

$$\partial_t \int_{\mathbb{R}^3} f(t,m,p)\,dp = \int_{\mathbb{R}^3} (Q_1(f)-Q_2(f))(t,m,p)\,dp. \tag{2.10}$$

Of course all the available results on coagulation equation may be applied to (2.8). But the function f does not satisfies the equation (1.1), (1.3), (1.4) unless $B\equiv 1$ since it is not possible to find any function φ such that:

$$m^{-3\theta}\int_{\mathbb{R}^3}\varphi\left(\frac{p''}{m^\theta}\right)B\left(\frac{p''}{m},\frac{p'}{m'}\right)dp'' = B\left(\frac{p}{m},\frac{p'}{m'}\right).$$

Notice that, since no uniqueness result of the solutions to the Cauchy problem for (2.10) is known, we can not even know if the function $F(t,m)$ solving (2.8) (2.9) coincides with the "v-average" of the solution $f(t,m,p)$ of (1.1), (1.3), (1.4). The case $B\equiv 1$ is treated in Section 4.

3 The impulsion dependence case $a = a(p,p')$

We consider now the equation (1.1), (1.3), (1.4) with a collision kernel a independent of the mass of the colliding particles. We may then integrate the equation with respect to the mass and obtain that the function of t and p, $\int_0^\infty f(t,m,p)\,dm$, that we shall still denote f, satisfies the equation:

$$\partial_t f = Q(f,f) \quad \text{in} \quad (0,+\infty)\times\mathbb{R}^d, \tag{3.1}$$

$$f(0) = f_{in} \quad \text{in} \quad \mathbb{R}^d, \tag{3.2}$$

the collision operator $Q(f)$ is given by $Q(f,f)=Q_1(f,f)-Q_2(f,f)$, where

$$Q_1(f,f)(y) = \frac{1}{2}\int_{\mathbb{R}^d} a(p',p-p')\ f(p')\ f(p-p')\,dp', \tag{3.3}$$

$$Q_2(f,f)(p) = \int_{\mathbb{R}^d} a(p,p')\ f(p)\ f(p')\,dp'. \tag{3.4}$$

We focus on the cases

$$a(p,p') = |p-p'|^\gamma, \qquad \gamma\in[0,2], \quad d\in\mathbb{N}^*. \tag{3.5}$$

Before stating our main result we need some definitions and notations. We say that a function f on $\mathbb{R}^d$ is radially symmetric if

$$f(Rp) = f(p) \qquad \forall\, p\in\mathbb{R}^d,\ R\in SO(d)$$

where $SO(d)$ stands for the group of rotations on $\mathbb{R}^d$. For any weight function $k:\mathbb{R}^d\to\mathbb{R}_+$ we define the "moment of order k" of the non negative density measure$f\in M^1_{loc}(\mathbb{R}^d)$ by

$$M_k(f) := \int_{\mathbb{R}^d} k(p)\,f(dp),$$

and we define M^1_k as the set of Radon measures μ such that $M_k(|\mu|) < \infty$. For any $\alpha \in \mathbb{R}_+$ we use the shorthand notation

$$M_\alpha := \int_{\mathbb{R}} f(p)\, |p|^\alpha \, dp,$$

that is $M_\alpha = M_k(f)$ for $k(p) = |p|^\alpha$ and the shorthand notation $M^1_\alpha = M^1_\ell$ for $\ell(p) = 1 + |p|^\alpha$.

Theorem 3.1 *Consider the aggregation rate (3.5).*

(i) For any even initial datum $f_{in} \in M^1_{2\alpha}$, $\alpha \in \mathbb{N}\backslash\{0,1\}$, there exists a unique even solution $f \in C([0,T); M^1(\mathbb{R}^d) - weak) \cap L^\infty(0,T; M^1_{2\alpha}(\mathbb{R}^d))$ to equation (3.1)–(3.4). For any $\alpha \in [0,1]$ the function $t \mapsto M_\alpha(t)$ is decreasing and $f(t,.)$ is radially symmetric for any $t \ge 0$ if furthermore f_{in} is radially symmetric.

(ii) Moreover, the solution $f(t,.)$ satisfies

$$\frac{1}{M_\gamma(0)^{-1} + k_1\, t} \le M_\gamma(t) \le \frac{1}{M_\gamma(0)^{-1} + k_2\, t} \qquad \forall\, t \ge 0, \tag{3.6}$$

for some constants $k_i = k_i(\gamma, d) \in (0,\infty)$.

One of the main tools in order to establish that result is to consider the equations satisfied by the moments of the solution f. Using the classical argument for the coagulation equation, and one more change of variable $p' \to -p'$, it is easy to check that any even solution f to equation (3.1)–(3.4) satisfies (at least formally) the fundamental moment equation

$$\begin{aligned} \frac{d}{dt} M_\alpha &= \frac{1}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\, f'\, a(p,p')\, [|p+p'|^\alpha - |p|^\alpha - |p'|^\alpha]\, dp dp' \\ &= \frac{1}{4} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\, f'\, \big\{ a(p,p')\, \big[|p+p'|^\alpha - |p|^\alpha - |p'|^\alpha\big] \\ &\qquad\qquad + a(p,-p')\, \big[|p-p'|^\alpha - |p|^\alpha - |p'|^\alpha\big] \,\big\}\, dp dp'. \end{aligned} \tag{3.7}$$

We consider in this Section the case $\gamma \in (0,2)$ and $d \in \mathbb{N}^*$, the case $\gamma = 1$ and $d = 1$ and the case $\gamma = 2$ and $d \in \mathbb{N}^*$. The case $\gamma = 0$ and $d = 1$ is treated in Section 5.

3.1 Proof of the existence and uniqueness part in Theorem 3.1.

We prove in this subsection a uniqueness and existence result for a general class of aggregation rates by adapting some arguments from [14, 11], see also [18]. We then deduce the existence and uniqueness part in Theorem 3.1.

Lemma 3.2 *Consider a continuous aggregation rate $a : \mathbb{R}^{2d} \to \mathbb{R}_+$ which satisfies*

$$a(-p,-p') = a(p,p') \qquad \forall\, p, p' \in \mathbb{R}^d, \tag{3.8}$$

$$a(p,p') \le a(-p,p') \qquad \forall\, p, p' \in \mathbb{R}^d,\ p \cdot p' > 0, \tag{3.9}$$

and an even weight function $k : \mathbb{R}^d \to \mathbb{R}_+$. We define

$$\Delta_k(p,p') := a(p,p')\, [k(p'') + k(p') - k(p)], \quad \tilde{\Delta}_k(p,p') = \Delta_k(p,p') + \Delta_k(-p,p').$$

and assume that

$$a(p,p') \le C\,k(p)\,k(p') \quad and \quad \tilde\Delta_k(p,p') \le C\,k(p)\,k(p')^2. \tag{3.10}$$

Then, for any given even initial data $f_{in} \in M^1_k(\mathbb{R}^d)$ there exists at most one even solution $f \in C([0,T); M^1_k(\mathbb{R}^d)) \cap L^\infty(0,T; M^1_{k^2}(\mathbb{R}^d))$ to equations (3.1)–(3.4).

Remark 3.3 *(i) The same result holds without the assumption that the initial density function f_{in} is even when the second condition in (3.10) is replaced by*

$$\Delta_k(p,p') \le C\,k(p)\,k(p')^2.$$

The same kind of results was obtained in [14, 11] in the L^1 framework. The same result also holds for radially symmetric solutions when we assume that

$$a(Rp,Rp') = a(p,p') \quad \forall\, p,p' \in \mathbb{R}^d,\ R \in SO(d), \tag{3.11}$$

and the second condition in (3.10) is replaced by

$$\int_{R\in SO(d)} \Delta(p,R\,p')dR \le C\,k(p)\,k(p')^2.$$

(ii) The same kind of result holds for aggregation rate defined on Y^2 with $Y = (0,\infty)\times\mathbb{R}^d$ as it is the case when particles are identified by their mass and impulsion, see [11].

Proof of Lemma 3.2. *Step 1.* We claim that for any $g_{in} \in M^1_k$ there is a unique $g \in C([0,T); M^1_k - weak)$, $G \in L^1(0,T; M^1_k)$ and $b \in C((0,T)\times\mathbb{R}^d; \mathbb{R}_+)$ such that

$$\begin{aligned} &\partial_t g = G - b\,g \quad \text{in the sense of } \mathcal{D}'([0,T)\times\mathbb{R}^d),\\ &g(0 = g_{in} \quad \text{in } M^1_k. \end{aligned}$$

and that, the differential inequality

$$\frac{d}{dt}\|g\,k\|_{M^1} \le \|G\,k\|_{M^1} - \|b\,g\,k\|_{M^1} \tag{3.12}$$

holds in the sense of $\mathcal{D}'([0,T))$. First, it is clear using a classical duality argument that equation (3.12) has at most one solution. Suppose indeed, $g_1, g_2 \in C([0,T); M^1_k - weak)$ are two such solutions. For any $\bar t \in (0,T)$, and any $\overline{\varphi} \in C_0(\mathbb{R}^d)$ the function $\varphi(t,p) = \overline{\varphi}(p)\exp\int_{\bar t}^t b(s,p)\,ds$ satisfies $\varphi(t,\cdot) \in C_0(\mathbb{R}^d)$ for all $t > 0$, solves the dual homogeneous equation $\partial_t\varphi = b\,\varphi$ in $\mathbb{R}\times\mathbb{R}^d$ and $\varphi(\bar t) = \overline{\varphi}$. Let finally be $\psi_n \in C_0([0,T))$ such that $\psi_n(t) = 1$ on $t \in [0,\bar t]$, $\psi_n(s) \to \mathbf{1}_{[0,\bar t]}(s)$ for all $s \in [0,T)$ and $\psi_n' \to -\delta_{\bar t}$ as $n \to +\infty$. Then, on the one hand:

$$\begin{aligned} &(\partial_t(g_1 - g_2), \varphi\,\psi_n) = -\int_0^T \int_{\mathbb{R}^d} (\psi_n\varphi)_t\, d(g_2(s) - g_1(s))(p)\,ds\\ &= -\int_0^T \int_{\mathbb{R}^d} \varphi_t\,\psi_n\, d(g_2(s) - g_1(s))(p)\,ds - \int_0^T \int_{\mathbb{R}^d} \psi_{nt}\,\varphi\, d(g_2(s) - g_1(s))(p)\,ds\\ &= -\int_0^T \int_{\mathbb{R}^d} b\,\varphi\,\psi_n\, d(g_2(s) - g_1(s))(p)\,ds - \int_0^T \int_{\mathbb{R}^d} \psi_{nt}\,\varphi\, d(g_2(s) - g_1(s))(p)\,ds. \end{aligned}$$

And on the other hand,

$$(\partial_t(g_1-g_2),\varphi\,\psi_n) = -\,(b\,(g_1-g_2),\varphi\,\psi_n) = -\int_0^T\int_{\mathbb{R}^d} b\,\varphi\,\psi_n\,d(g_2(s)-g_1(s))(p).$$

We deduce that, for all $n\geq 1$:

$$\int_0^T\int_{\mathbb{R}^d}\psi_{nt}\,\varphi\,d(g_2(s)-g_1(s))(p)\,ds = 0$$

and passing to the limit $n\to+\infty$, using that $g_i\in C([0,T),M^1_k-weak)$:

$$\int_{\mathbb{R}^d}\overline{\varphi}\,d(g_2(t)-g_1(t))(p) = 0.$$

In order to show the existence of a solution $g\in C([0,T);M^1_k-weak)$ of (3.12) we notice first that, for any $g_\varepsilon(0)\in C_{K_\varepsilon}:=\{u\in C(\mathbb{R}^d);\ \operatorname{supp}u\subset K_\varepsilon\}$, with $K_\varepsilon\subset\mathbb{R}^d$ a compact, and any $G_\varepsilon\in L^1(0,T;C_{K_\varepsilon})$ there exists a (unique) solution $g_\varepsilon\in C([0,T);C_{K_\varepsilon})$ to equation (3.12) which furthermore satisfies

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|g_\varepsilon|\,k\,dy &= \int_{\mathbb{R}^d}(G_\varepsilon-b\,g_\varepsilon)\,\mathrm{sign}g_\varepsilon\,k\,dy\\ &\leq \int_{\mathbb{R}^d}|G_\varepsilon|\,k\,dy-\int_{\mathbb{R}^d}|g_\varepsilon|\,b\,k\,dy. \end{aligned}\tag{3.13}$$

Here, $\mathrm{sign}g_\varepsilon=1$ if $g_\varepsilon>0$, $\mathrm{sign}g_\varepsilon=0$ if $g_\varepsilon=0$, $\mathrm{sign}g_\varepsilon=-1$ if $g_\varepsilon<0$. Finally, we can build (by a standard truncation and regularization by convolution process) the sequences (G_ε) and $g_\varepsilon(0)$ such that furthermore $G_\varepsilon\rightharpoonup G$, $g_\varepsilon(0)\rightharpoonup g(0)$ in the weak sense of measures in M^1_k, $\|G_\varepsilon(s)\|_{M^1_k}\leq\|G(s)\|_{M^1_k}$ for a.e. $s\in(0,T)$, $\|g_\varepsilon(0)\|_{M^1_k}\leq\|g(0)\|_{M^1_k}$. By the previous uniqueness argument we have $g_\varepsilon\rightharpoonup g$ in the weak sense of measure and we get (3.12) by passing to the limit in (3.13). This ends the proof of Step 1.

Step 2: End of the proof of Lemma 3.2.

Consider two even solutions $f_1,f_2\in C([0,T);M^1_k(\mathbb{R}^d))\cap L^\infty(0,T;M^1_{k^2}(\mathbb{R}^d))$ and let us denote $D=f_2-f_1$, $S=f_1+f_2$. By a standard algebraic computation D satisfies the following equation

$$\begin{aligned}\partial_t D &= \hat{Q}(f_2,f_2)-\hat{Q}(f_1,f_1)=\hat{Q}(D,S)\\ &= \hat{Q}_1(D,S)-S\,L(D)-L(S)\,D,\end{aligned}$$

where

$$\hat{Q}_i(\varphi,\psi)=\frac{1}{2}\,(Q_i(\varphi,\psi)-Q_i(\psi,\varphi)),\quad L(\varphi):=\int_{\mathbb{R}^d}a(p,p')\,\varphi(p')\,dp'.$$

Because of the assumption made on a and f we have $D\in C([0,T];M^1_k-weak)$, $G:=\hat{Q}_1(D,S)-S\,L(D)\in L^\infty(0,T;M^1_k)$ and $0\leq b:=L(S)\in C([0,T]\times\mathbb{R}^d)$ so that the first step implies

$$\begin{aligned}\frac{d}{dt}\|D\|_{M^1_k} &\leq \|(\hat{Q}_1(D,S)-S\,L(D))\,k\|_{M^1}-\|D\,k\,L(S)\|_{M^1}\\ &\leq \frac{1}{2}\iint a\,[k''+k']\,|D(dp)|\,S(dp')-\frac{1}{2}\iint a\,k\,|D(dp)|\,S(dp')\\ &\leq \frac{1}{4}\iint\tilde{A}\,|D(dp)|\,S(dp')\leq\frac{C}{4}\,\|S\|_{M^1_{k^2}}\,\|D\|_{M^1_k}.\end{aligned}$$

Uniqueness follows by using the Gronwall lemma. □

Lemma 3.4 *Consider a continuous aggregation rate $a : \mathbb{R}^{2d} \to \mathbb{R}_+$ which satisfies (3.8) (resp. (3.11)), (3.9) and such that, for some positive constant C:*

$$a(p,p') \le C\,(k(p) + k(p')) \qquad \forall\, p,\, p' \in \mathbb{R}^d, \tag{3.14}$$

for the weight function $k(p) = 1 + |p|^2$. Then, for any given even (resp. radially symmetric) initial datum $f_{in} \in M^1_{2\alpha}(\mathbb{R}^d)$ there exists at least one even (resp.radially symmetric) solution $f \in C([0,T); M^1(\mathbb{R}^d) - weak) \cap L^\infty(0,T; M^1_{2\alpha}(\mathbb{R}^d))$ to equation (3.1)–(3.4). This solution also satisfies that the map $t \mapsto M_\beta(t)$ is decreasing for any $\beta \in [0,1]$.

Remark 3.5 *It is likely that by adapting some arguments introduced in [17], see also [7, 14], for any even (resp. radially symmetric) initial datum $f_{in} \in L^1_{2\alpha}(\mathbb{R}^d)$ the approximating solution $f_n(t,.)$ built in the proof below is a Cauchy sequence in $C([0,T; L^1(\mathbb{R}^d))$ so that we may conclude $f \in C([0,T); L^1(\mathbb{R}^d)) \cap L^\infty(0,T; L^1_{2\alpha}(\mathbb{R}^d))$.*

Proof of Lemma 3.4. We define the sequence of bounded aggregation rates $a_n := a \wedge n$, for which classically fixed point argument (see for instance [11] which deals with some similar situation) implies the existence of a unique even (resp. radially symmetric) solution $f_n \in C([0,T); L^1_{2\alpha}(\mathbb{R}^d))$ to equation (3.1)–(3.4) associated with a_n for any initial datum $f_{in,n} \in L^1_{2\alpha+2}(\mathbb{R}^d)$, $\alpha \in \mathbb{N}$, $\alpha \ge 2$. Then, we have for any $\beta \in \mathbb{N}^*$, $\beta \le \alpha$

$$\begin{aligned} \frac{d}{dt}\int f_n\,(1+|p|^{2\beta}) &= \frac{1}{2}\int f_n\,f'_n\,a_n\,\Big[(|p|^2 + 2\,p\cdot p' + |p'|^2)^\beta - |p|^{2\beta} - |p'|^{2\beta} - 1\Big] \\ &= \int f_n\,f'_n\,a_n\,\Big[2\,\beta\,p\cdot p'\,|p|^{2(\beta-1)} - 1/2\Big] \\ &\quad + \sum \mu_{\beta_1,\beta_2,\beta_2} \int f_n\,f'_n\,a_n\,(p\cdot p')^{\beta_1}\,|p|^{2\,\beta_2}\,\,|p|^{2\,\beta_3}, \end{aligned}$$

where in the last sum the integers $\beta_1, \beta_2, \beta_3$ are such that $\beta_1 + \beta_2 + \beta_3 = \beta$ and must satisfy also: either $\beta_1 \ge 2$, or $\beta_2 \ge 1$ and $\beta_3 \ge 1$. This implies: $|p\cdot p'|^{\beta_1}\,|p|^{2\,\beta_2}\,|p|^{2\,\beta_3} \le |p|^{2\,\beta'}\,|p'|^{2\,(\beta-\beta')}$ with $1 \le \beta' \le \beta - 1$. Since we also have

$$\begin{aligned} &\int f_n\,f'_n\,a_n\,p\cdot p'\,|p|^{2(\beta-1)} = \\ &\quad = \int_{p\cdot p'>0} f_n\,f'_n\,(a(p,p')\wedge n - a(-p,p')\wedge n)\,p\cdot p'\,|p|^{2(\beta-1)} \le 0, \end{aligned}$$

we conclude with

$$\frac{d}{dt}\int f_n\,(1+|p|^{2\beta}) \le \sum_{1\le\beta'\le\beta-1} \mu_{\beta'} \int f_n\,f'_n\,a\,|p|^{2\,\beta'}\,|p'|^{2\,(\beta-\beta')}. \tag{3.15}$$

When $\beta = 1$ the set of admissible values of β' is empty, and we recover a result from [8]

$$\frac{d}{dt}\int f_n\,(1+|p|^2) \le 0,$$

so that

$$\sup_{[0,T]} \|f_n\|_{L^1_k} \le \|f_{in,n}\|_{L^1_k}. \tag{3.16}$$

When $\beta \geq 2$, gathering (3.14), (3.15) and (3.16), we easily conclude by a iterative argument that

$$\sup_{[0,T]} \|f_n\|_{L^1_{k^\beta}} \leq C_T(\beta, \|f_{in,n}\|_{L^1_{k^\beta}}). \tag{3.17}$$

Considering a sequence $(f_{in,n})$ such that $f_{in,n} \rightharpoonup f_{in}$ in the weak sense of measure and $\|f_{in,n}\|_{L^1_{k^\beta}}$ remains bounded, we easily pass to the limit in the equation satisfied by f_n thanks to (3.17). The fact that $t \mapsto M_\beta(t)$ is decreasing follows from the fact that $p \mapsto |p|^\beta$ is a sub-additive function when $\beta \in [0,1]$, so that $\Delta_\beta \leq 0$ and then $d/dt M_\beta(t) \leq 0$. $\square$

Proof of the existence and uniqueness part in Theorem 3.1. It is clear that $a(p,p') = |p-p'|^\gamma$ satisfies (3.8), (3.9), the first inequality in (3.10) and (3.14). Moreover, the second inequality in (3.10) holds since we have

$$\begin{aligned} \tilde\Delta_2(p,p') &= |p-p'|^\gamma \, (|p+p'|^2 + |p'|^2 - |p|^2 + 1) + |p+p'|^\gamma \, (|p-p'|^2 + |p'|^2 - |p|^2 + 1) \\ &= 2\,(|p-p'|^\gamma - |p+p'|^\gamma)\, p\cdot p' + (|p-p'|^\gamma + |p+p'|^\gamma)\,(2\,|p'|^2+1), \end{aligned}$$

where the first term in non positive and the second term is bounded by say $8\,(k')^2\,k$, using that $|p \pm p'|^\gamma \leq 2\,(|p|^\gamma + |p'|^\gamma)$. We conclude by using Lemma 3.2 and Lemma 3.4. $\square$

3.2 Proof of the rate decay part in Theorem 3.1 when $\gamma < 2$.

For an even initial datum $f_{in} \in M^1_4(\mathbb{R}^d)$ we consider the unique even solution $f \in C([0,T); M^1-weak) \cap L^\infty(0,T; M^1_4)$, $\forall\, T$, given by Theorem 3.1(i). It satisfies the moment equation

$$\frac{d}{dt} M_\gamma = \frac{1}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,\Delta_\gamma\,dpdp' = \frac{1}{4} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,\tilde\Delta_\gamma\,dpdp', \tag{3.18}$$

with

$$-\Delta_\gamma(p,p') = |p-p'|^\gamma\,[|p+p'|^\gamma - |p|^\gamma - |p'|^\gamma]$$

and

$$-\tilde\Delta_\gamma(p,p') = |p-p'|^\gamma\,\left[|p+p'|^\gamma - |p|^\gamma - |p'|^\gamma\right] + |p+p'|^\gamma\,\left[|p-p'|^\gamma - |p|^\gamma - |p'|^\gamma\right]. \tag{3.19}$$

We split the proof of Theorem 3.1(ii) in several steps.

Step 1. One the one hand, for any given $A > 0$ and any $p, p' \in \mathbb{R}^d$ such that $A^{-1}\,|p'| \leq |p| \leq A\,|p|$ we easily get

$$\begin{aligned} |\Delta_\gamma(p,p')| &\leq (|p|+|p'|)^\gamma \, \max\left[(|p|+|p'|)^\gamma, |p|^\gamma + |p'|^\gamma\right] \\ &\leq 2^4\,\max(|p|,|p'|)^{2\gamma} \leq 2^4\,A^\gamma\,(|p|\,|p'|)^\gamma. \end{aligned} \tag{3.20}$$

On the other hand, we define $M := \max(|p|,|p'|)$, $m := \min(|p|,|p'|)$, $x := m/M \in [0,1]$, $\varepsilon := \hat p \cdot \hat p' \in [-1,1]$ and we compute (in the first line we have assumed that $|p| = M$ which is not a restriction to the generality because of the symmetry of $\tilde\Delta_\gamma$)

$$\begin{aligned} -\tilde\Delta_\gamma(p,p') &= M^{2\gamma}\,\left\{|\hat p - x\,\hat p'|^\gamma\,\left[1 + x^\gamma - |\hat p + x\,\hat p'|^\gamma\right] + |\hat p + x\,\hat p'|^\gamma\,\left[1 + x^\gamma - |\hat p - x\,\hat p'|^\gamma\right]\right\} \\ &= M^{2\gamma}\,\Big\{(1+x^\gamma)\,[(1+2\,\varepsilon\,x + x^2)^{\gamma/2} + (1-2\,\varepsilon\,x+x^2)^{\gamma/2}] \\ &\qquad -2\,(1+2\,\varepsilon\,x+x^2)^{\gamma/2}\,(1-2\,\varepsilon\,x+x^2)^{\gamma/2}\Big\} \\ &= M^{2\gamma}\,\left\{2\,x^\gamma + \mathcal{O}(x^2)\right\} \leq 3\,M^{2\gamma}\,x^\gamma = 3\,(|p|\,|p'|)^\gamma \end{aligned} \tag{3.21}$$

uniformly on $\varepsilon \in [-1,1]$ and $x \le A_0^{-1}$ for $A_0 \ge 1$ large enough.
Using (3.20) and (3.21) we obtain

$$\frac{1}{4}\,\tilde\Delta_\gamma(p,p') \ge -k_1\,|p|^\gamma\,|p'|^\gamma \qquad \forall\, p,p' \in \mathbb{R}^d,$$

with $k_1 := \max(3/4, 2^3\,A_0^\gamma)/4$, and equation (3.18) then implies

$$\frac{d}{dt} M_\gamma \ge -k_1\, M_\gamma^2.$$

The first inequality in (3.6) follows straightforwardly by integrating this differential equation.

Step 2. We still use the variables M, x, ε introduced in Step 1. We also define $r>0$ and $u\in[0,1]$ by setting $r^2 := |p|^2+|p'|^2$ and $u := 2\,p\cdot p'/r^2$, so that $|p\pm p'|^2 = r^2\,(1\pm u)$. Splitting the positive and the negative terms in identity (3.19), we have

$$\begin{aligned} -\tilde\Delta_\gamma(p,p') &= (|p|^\gamma+|p'|^\gamma)\,(|p-p'|^\gamma+|p+p'|^\gamma) - 2\,|p-p'|^\gamma\,|p+p'|^\gamma \\ &= r^{2\gamma}\left\{\frac{(|p|^2)^{\gamma/2}+(|p'|^2)^{\gamma/2}}{(|p|^2+|p'|^2)^{\gamma/2}}\,\Big[(1+u)^{\gamma/2}+(1-u)^{\gamma/2}\Big] - 2\,(1+u)^{\gamma/2}\,(1-u)^{\gamma/2}\right\}. \end{aligned}$$

Since $\gamma/2\in[0,1]$, the map $x\mapsto x^{\gamma/2}$ is sub-additive, and we obtain

$$\begin{aligned} -\tilde\Delta_\gamma(p,p') &\ge r^{2\gamma}\left\{\Big[(1+u)^{\gamma/2}+(1-u)^{\gamma/2}\Big] - 2\,(1+u)^{\gamma/2}\,(1-u)^{\gamma/2}\right\} \\ &\ge M^{2\gamma}\,(1+u)^{\gamma/2}\,(1-u)^{\gamma/2}\,\phi(u), \quad \phi(u) := \Big[(1-u)^{-\gamma/2}+(1+u)^{-\gamma/2}\Big]-2. \end{aligned}$$

We easily verify that ϕ is increasing on $[0,1]$ so that $\phi(u) > \phi(0) = 0$ for any $u\in[-1,1]$, $u\neq 0$. Coming back to the variables M, x and ε, that is $\phi(u)>0$ for any $p,p'\in\mathbb{R}^d$ such that the associated variables M, x and ε satisfy $M>0$, $x>0$ and $\varepsilon\neq 0$. Moreover, when $\varepsilon = 0$ (p and p' are orthogonal vectors) we also have

$$\begin{aligned} -\tilde\Delta_\gamma(p,p') &= 2(|p|^2+|p'|^2)^{\gamma/2}\,\Big[|p|^\gamma+|p'|^\gamma-(|p|^2+|p'|^2)^{\gamma/2}\Big] \\ &\ge 2\,M^{2\gamma}\,\Big[1+x^\gamma-(1+x^2)^{\gamma/2}\Big] > 0 \end{aligned}$$

for any $p,p'\in\mathbb{R}^d$ such that the associated variables M and x satisfy $M>0$, $x>0$, because the function $z\mapsto z^{\gamma/2}$ is strictly sub-additive on $\mathbb{R}_+$, that is $(z+z')^{\gamma/2} < z^{\gamma/2}+(z')^{\gamma/2}$ for any $z,z'>0$. From these two lower bounds on $-\tilde\Delta_\gamma$, we obtain

$$-\tilde\Delta_\gamma(p,p') \ge M^{2\,\gamma}\,\psi(x,\varepsilon) \tag{3.22}$$

with $\psi(x,\varepsilon)>0$ for any $x>0$ and $\varepsilon\in[-1,1]$.

Next, coming back to (3.21), we also deduce

$$-\tilde\Delta_\gamma(p,p') = M^{2\gamma}\,\left\{2\,x^\gamma + \mathcal{O}(x^2)\right\} \ge M^{2\gamma}\,x^\gamma \tag{3.23}$$

uniformly on $\varepsilon\in[-1,1]$ and $x\le A_0^{-1}$ for $A_0\ge 1$ large enough. We deduce from (3.22) and (3.23) that for some constant $k_2>0$ we have

$$\forall\, p,p'\in\mathbb{R}^d \qquad -\frac{1}{4}\tilde\Delta_\gamma \ge k_2\,M^{2\gamma}\,x^\gamma = k_2\,(|p|\,|p'|)^\gamma,$$

and equation (3.18) then implies

$$\frac{d}{dt}M_\gamma \le -k_2\, M_\gamma^2.$$

The second inequality in (3.6) is again obtained by integrating this differential equation.

3.3 The case $a(y,y') = |p-p'|$, $d=1$.

In the particular case $d=1$ and $\gamma=1$, it is possible to estimate more precisely the decay rate of the first moment M_1. It is also possible to estimate the decay rates of several other moments.

Lemma 3.6 *Assume $a(y,y') = |p-p'|$ and $d=1$. For any even initial data $f_{in} \in M^1_3(\mathbb{R})$ the unique solution $f \in C([0,T];M^1(\mathbb{R})) \cap L^\infty(0,T;M^1_3(\mathbb{R}))$ of (3.1)-(3.4) given by Theorem 3.1 satisfies for any $t \ge 0$*

$$\max\left(\frac{M_0(0)}{(1+M_1(0)\,t/2)^2}, \frac{2^{3/2}\,M_0(0)}{(2+3M_3^{1/3}(0)\,t)^{3/2}}\right) \;\le\; M_0(t) \;\le\; \frac{M_0(0)}{(1+M_1(0)t)^{1/2}} \tag{3.24}$$

$$\frac{1}{M_1(0)^{-1}+t} \;\le\; M_1(t) \;\le\; \frac{1}{M_1(0)^{-1}+t/2} \tag{3.25}$$

$$\frac{M_2(0)}{(1+M_1(0)\,t/2)^2} \;\le\; M_2(t) \;\le\; M_2(0) \tag{3.26}$$

$$\frac{M_3(0)}{(1+M_1(0)\,t/2)^2} \;\le\; M_3(t) \;\le\; M_3(0). \tag{3.27}$$

Remark 3.7 *The estimate (3.25) on $M_1(t)$ gives the exact value of the power of t at which the first moment decays for t large. That is not the case for the estimates on M_α, $\alpha = 0,2,3$ which are actually rather partial. They do not even allow to obtain the limit of any of the quotients of moments $M_\alpha(t)/M_1(t)$ for $\alpha = 0,2,3$ as $t \to \infty$. The value of such limits would indicate whether the solution $f(t)$ has a tendency to concentrate or to spread as t increases (see also below the discussion concerning the case $\gamma = 2$).*

Remark 3.8 *Let us perform the "Maxwellian approximation" as in [25], replacing the collision rate $a(p,p') = |p-p'|$ by the "root mean squared" velocity $V(t) := \sqrt{m_2(t)/m_0(t)}$, where here m_k denotes the k-th moment of the solution of that modified equation. We easily compute*

$$m_2(t) \equiv m_2(0) \quad \textit{and} \quad m_0(t) = \frac{1}{(\sqrt{m_2(0)}\,t/2 + m_0(0)^{-1/2})^2}.$$

It is worth emphasizing that $M_0(t)$ (the number of particles at time t for the equation with rate $a(p,p') = |p-p'|$) and $m_0(t)$ (the number of particles at time t for the equation with rate $a(p,p') = V(t)$) have definitely not the same long time behavior. As a conclusion, the "Maxwellian approximation" is not a good approximation here.

Proof of Lemma 3.6. Let us denote $M = \max(|p|,|p'|)$ and $m = \min(|p|,|p'|)$. We systematically exploit the differential equation

$$\frac{d}{dt}M_\alpha = \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,\Delta_\alpha\,dpdp' \tag{3.28}$$

with

$$\Delta_\alpha := [M-m]\,[(M+m)^\alpha - M^\alpha - m^\alpha] + [M+m]\,[(M-m)^\alpha - M^\alpha - m^\alpha].$$

Step 1: We suppose $\alpha = 1$. In that case

$$\Delta_1 = -2\,(M+m)\,m,$$

from where we deduce

$$\frac{d}{dt}M_1(t) = -\frac{M_1^2(t)}{2} - \frac{B_1(t)}{2}, \qquad B_1(t) := \int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,\{\min(|p|,|p'|)\}^2\,dpdp'.$$

Since $0 \le \{\min(|p|,|p'|)\}^2 \le |p|\,|p'|$, we have $0 \le B_1(t) \le M_1^2(t)$ and therefore

$$-M_1^2(t) \le \frac{d}{dt}M_1(t) \le -\frac{M_1^2(t)}{2}, \tag{3.29}$$

from where (3.25) follows.

Step 2: We suppose $\alpha = 0$. Then, since

$$\Delta_0 = -2\,M,$$

we have

$$\frac{d}{dt}M_0(t) = -\frac{B_0(t)}{2}, \qquad B_0(t) := \int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,\max(|p|,|p'|)\,dpdp'. \tag{3.30}$$

Using $|p| \le \max(|p|,|p'|) \le |p| + |p'|$, we deduce $M_0\,M_1 \le B_0 \le 2\,M_0\,M_1$ and then

$$-M_0\,M_1 \le \frac{d}{dt}M_0 \le -\frac{1}{2}\,M_0\,M_1. \tag{3.31}$$

By the previous estimate (3.25) on $M_1(t)$ we get

$$-\frac{M_0(t)}{M_1^{-1}(0)+t/2} \le \frac{d}{dt}M_0(t) \le -\frac{M_0(t)}{2(M_1^{-1}(0)+t)},$$

and we obtain the first lower estimate as well as the upper bound in (3.24).

Step 3: The case $\alpha = 2$. We deduce from

$$\Delta_2 = -4\,m\,M^2$$

that:

$$\frac{d}{dt}M_2(t) = -B_2(t), \qquad B_2(t) := \int_{\mathbb{R}}\int_{\mathbb{R}} f\,f'\,\min(|p|,|p'|)\,|p|\,|p'|\,dpdp'.$$

Using that $0 \le \min(|p|,|p'|)\,|p|\,|p'| \le |p|^2\,|p'|$ together with (3.25), we obtain

$$-M_2\,\frac{1}{M_1(0)^{-1}+t/2} \le -M_2\,M_1 \le \frac{d}{dt}M_2(t) \le 0,$$

and (3.26) follows.

Step 4: The case $\alpha = 3$. From the following estimates on Δ_3

$$0 \geq \Delta_3 = -2\, M\, m^3 - 2\, m^4 \geq -4\, M\, m^3 \geq -4\, |p|^3\, |p'|,$$

we deduce

$$0 \geq \frac{d}{dt} M_3(t) \geq -M_1\, M_3,$$

which again implies (3.27).

Step 5: Suppose $\alpha = 0$ again. We complete now the lower estimate of M_0 in (3.24). To this end we write for any $\varepsilon > 0$

$$\begin{aligned} \frac{d}{dt} M_0 &= -\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} f\, f'\, |p' - p|\, dp dp' \\ &\geq -\frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} f\, f'\, (\varepsilon + \frac{1}{\varepsilon}\, |p - p'|^2)\, dp dp' \\ &\geq -\frac{\varepsilon}{4}\, M_0^2 - \frac{2}{\varepsilon}\, M_0\, M_2. \end{aligned}$$

By interpolation we have $M_2(t) \leq M_0^{1/3}(t)\, M_3^{2/3}(t)$. Since, by (3.27), $M_3(t) \leq M_3(0)$ for all $t > 0$ we deduce $M_2(t) \leq M_0^{1/3}(t)\, M_3^{2/3}(0)$. Therefore

$$\frac{d}{dt} M_0(t) \geq -\frac{\varepsilon}{4}\, M_0^2 - \frac{2}{\varepsilon}\, M_0^{4/3}\, M_3^{2/3}(0)$$

We now chose $\varepsilon \equiv \varepsilon(t) > 0$ such that $\varepsilon\, M_0^2 = \frac{1}{\varepsilon}\, M_0^{4/3}\, M_3^{2/3}(0)$, or equivalently $\varepsilon = M_0^{-1/3}\, M_3^{1/3}(0)$. With that choice of $\varepsilon(t)$ the equation reads

$$\frac{d}{dt} M_0(t) \geq -\frac{9}{4}\, M_3^{1/3}(0)\, M_0^{5/3},$$

and the second lower estimate in (3.24) follows. □

Remark 3.9 *In the last step, we may also argue as follows. Combining the estimate* $\max(|p|, |p'|) \geq (|p|\, |p'|)^{1/2}$*, (3.27), the differential equation (3.30) and the interpolation estimate* $M_1^{5/2} \leq M_{1/2}^2\, M_3^{1/2}$ *we obtain:*

$$\frac{d}{dt} M_0 \leq -\frac{1}{dt}\, M_3^{1/2}(0)\, M_1^{5/2}(t).$$

Using (3.25) we recover the second lower estimate in (3.24).

3.4 The case $a = |p - p'|^2$

When $\gamma = 2$ and $d \in \mathbb{N}^*$, the family of moment equations may be closed, for all the "even" moments $M_{2\alpha}$, $\alpha \in \mathbb{N}$. This allows to prove a non existence result of self similar solutions in that case. We first obtain in the next lemma the exact expressions of the even moments of order less that or equal to four of the solutions to the equation (3.1)-(3.4).

Lemma 3.10 *Assume $a(y,y') = |p-p'|^2$ and $d \in \mathbb{N}^*$. Then, there exist numerical constants $k_d \in (0,\infty)$, $k_1 := 2$, such that for any radially symmetric initial datum $f_{in} \in M^1_6(\mathbb{R}^d)$ the unique radially symmetric solution $f \in C([0,T]; M^1(\mathbb{R}^d)) \cap L^\infty(0,T; M^1_6(\mathbb{R}^d))$ of (3.1)-(3.4) given by Theorem 3.1 satisfies for any $t \geq 0$*

$$M_0(t) = \frac{M_0(0)}{(M_2(0)^{-1} + 2\,k_d\,t)^{1/(2k_d)}} \tag{3.32}$$

$$M_2(t) = \frac{1}{M_2(0)^{-1} + 2\,k_d\,t} \tag{3.33}$$

$$M_4(t) = M_4(0)\,(M_2(0)^{-1} + 2\,k_d\,t)^{\frac{1}{k_d}-2}. \tag{3.34}$$

Proof of Lemma 3.10. We proceed in several steps.
Step 1: If $\alpha = 2$. Using the fact that f is radially symmetric (so that the odd moments of f vanish) and the notations $p = r\,\sigma$, $r = |p|$, $p' = r'\,\sigma'$, $r' = |p'|$, the fundamental moment identity (3.7) implies

$$\begin{aligned} \frac{d}{dt}M_2 &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,[|p|^2 - 2\,p\cdot p' + |p'|^2]\,(2\,p\cdot p')\,dpdp' \\ &= -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,[p\cdot p']^2\,dpdp' \\ &= -2\int_0^\infty\int_0^\infty f(r)\,f(r')\,r^{d+1}\,(r')^{d+1}\,drdr' \times \int_{S^{d-1}}\int_{S^{d-1}}[\sigma\cdot\sigma']^2\,d\sigma d\sigma' \\ &= -2\,k_d\,M_2^2, \end{aligned}$$

with

$$\begin{aligned} k_d &:= \omega_d^{-2}\left(\int_{S^{d-1}}\int_{S^{d-1}}[\sigma\cdot\sigma']^2\,d\sigma d\sigma'\right) \\ &= \omega_d^{-1}\int_{S^{d-1}}\sigma_1^2\,d\sigma. \end{aligned}$$

We compute $k_1 = 1$, $k_2 = 1/2$. The expression (3.33) immediately follows by integrating that ODE.
Step 2: If $\alpha = 0$. In that case the fundamental moment identity (3.7) and the fact that f is radially symmetric imply

$$\begin{aligned} \frac{d}{dt}M_0 &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,[|p|^2 - 2\,p\cdot p' + |p'|^2]\,(-1)\,dpdp' \\ &= -M_2\,M_0. \end{aligned}$$

Integrating that ODE with the help of (3.33) we get (3.32).
Step 3: If $\alpha = 4$. When $\alpha = 4$, the fundamental moment identity (3.7) and the fact that f is radially symmetric imply

$$\begin{aligned} \frac{d}{dt}M_4 &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,[|p|^2 - 2\,p\cdot p' + |p'|^2]\,[4\,(p\cdot p')^2 + 8\,|p|^2\,(p\cdot p') + 2\,|p|^2\,|p'|^2]\,dpdp' \\ &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,\left\{[2\,|p|^2]\,[4\,(p\cdot p')^2 + 2\,|p|^2\,|p'|^2] - 16\,|p|^2\,(p\cdot p')^2\right\}\,dpdp' \\ &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f\,f'\,\left\{|p|^4\,|p'|^2 - 2\,|p|^2\,(p\cdot p')^2\right\}\,dpdp' \\ &= (2 - 4\,k_d)\,M_2\,M_4. \end{aligned}$$

Integrating that ODE with the help of (3.33) we get (3.34). □

Remark 3.11 *It is straightforward to check that if* $\Phi(t,m,p)$ *solves (1.1), (1.3), (1.4) with* $a(y,y')\equiv|p-p'|^\gamma$*, so does*

$$\Phi_\rho(t,m,p)=\rho^{-\lambda}f(\rho\, t,\rho^{-\delta}\, m,\rho^{-\nu}\, p) \tag{3.35}$$

for all $\rho>0$*, whenever the exponents satisfy:*

$$\lambda=\nu(\gamma+1)+\delta-1. \tag{3.36}$$

This may suggest the existence of self similar solutions of (1.1), (1.3), (1.4) with $a(y,y')\equiv|p-p'|^\gamma$ *of the form :*

$$\Phi(t,m,p)=t^\lambda\Theta\left(t^\delta\, m,t^\nu\, p\right) \tag{3.37}$$

for some function Θ*. Therefore the function*

$$\begin{aligned} g(t,p) &= \int_{\mathbb{R}^d} t^\lambda\Theta\left(t^\delta\, m,t^\nu\, p\right)dm=t^{\lambda-\delta}\int_{\mathbb{R}^d}\Theta\left(m,t^\nu\, p\right)dm\\ &=: t^\mu G\left(t^\nu\, p\right) \end{aligned} \tag{3.38}$$

with $\mu=\lambda-\delta=\nu(\gamma+1)-1$ *would be a self similar solution of (3.1)-(3.5) and its moments of order* $\alpha\in\mathbb{R}$*, would then be:*

$$M_\alpha(g(t,.))=M_\alpha(G)\, t^{\mu-(d+\alpha)\,\nu}, \tag{3.39}$$

On the other hand, by Lemma 3.10, when $\gamma=2$ *the solutions* f *of equation (3.1)-(3.4) with initial data* $f_{in}\in M_6^1(\mathbb{R})$ *satisfy*

$$M_0(f(t,.))\sim C_0'\, t^{-\frac{1}{2k_d}},\quad M_2(f(t,.))\sim C_2'\, t^{-1},\quad M_4(f(t,.))\sim C_4'\, t^{\frac{1}{k_d}-2}. \tag{3.40}$$

as $t\to+\infty$ *for some positive constants* C_0'*,* C_2' *and* C_4'*. It is easy to check that to have both (3.39) and (3.40) requires :*

$$(3-d)\,\nu-1=-\frac{1}{k_d};\ \ (1-d)\,\nu-1=-1;\ \ -(1+d)\,\nu-1=\frac{1}{k_d}-2$$

that is impossible. We deduce that when $\gamma=2$*, the equation (3.1)-(3.5) has no self similar solution of the form (3.38) with self-similar profile* $G\in M_6^1(\mathbb{R})$*. Therefore the equation (1.1), (1.3), (1.4) with* $a(y,y')\equiv|p-p'|^2$ *has no self similar solution of the form (3.37) with* Θ *such that* $M_{1,0}(\Theta)<\infty$ *and* $M_{0,6}(\Theta)<\infty$*.*

Remark 3.12 *Consider again any solution* f *of (3.1)-(3.4) with initial data* $f_{in}\in M_6^1$ *and suppose, only for the sake of simplicity, that we are in the case* $d=1$*. Then the moments* $M_\gamma(f(t))$ *for* $0\le\gamma\le 2\alpha$ *satisfy:*

$$\frac{d}{dt}M_{2\alpha}=\sum_{\beta=1}^{\alpha-1}\binom{2\alpha}{2\beta}\, M_{2\beta}\, M_{2(\alpha+1-\beta)}-\sum_{\beta=0}^{\alpha-1}\binom{2\alpha}{2\beta+1}\, M_{2\beta+2}\, M_{2(\alpha-\beta)}.$$

In particular:

$$\frac{d}{dt} M_6 = 3\, M_2\, M_6 - 5\, M_4^2.$$

When $M_2(0) = 1/2$ *(only for the sake of simplicity again), the solution is*

$$M_6(t) = \left(M_6(0) - 2\, M_4(0)^2 + \frac{2\, M_4(0)^2}{(1+t)^{5/4}} \right) (1+t)^{3/2} \qquad \forall\, t \geq 0,$$

with, by Holder's inequality, $M_6(0) - 2\, M_4(0)^2 > 0$. *It then follows that the moments of* f *satisfy the following*

$$M_0(t) \sim \kappa_0\, t^{-1/2}, \quad \frac{M_2(t)}{M_0(t)} \sim \kappa_1\, t^{-1/2}, \quad \frac{M_6(t)}{M_0(t)} \sim \kappa_2\, t^{3/2}, \tag{3.41}$$

as $t \to +\infty$, *for some positive constants* $\kappa_i, i = 0, 1, 2$. *We notice that the "mean second moment" tends to* 0 *as* $t \to +\infty$. *The behavior of the mean second moment* M_2/M_0 *is then comparable with the behaviour of the energy (second moment) of the solutions to the inelastic Boltzmann equation that decreases as* $t \to +\infty$. *The opposite will be true for a model considered in section 4. On the other hand, (3.41) shows that the behaviour of the mean sixth moment* M_6/M_0 *is similar to that of the high moments of the solutions of the Smoluchoski equation that increase as* $t \to +\infty$.

4 The mass dependence case $a = a(m, m')$

Consider now the problem (1.1)-(1.4) where the kernel $a(y, y')$ only depends on the masses of the particles, namely

$$a(y, y') = a(m, m'), \tag{4.1}$$

and introduce the associated Smoluchowski equation

$$\frac{\partial F}{\partial t}(t, m) = \frac{1}{2} \int_0^m F(t, m-m') F(t, m')\, a(m-m', m')\, dm' - \int_0^\infty F(t, m) F(t, m')\, a(m, m')\, dm'. \tag{4.2}$$

For any function $\psi \in L^1(\mathbb{R}^3)$ we define the Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ by

$$\hat{\psi}(\eta) = (\mathcal{F}\psi)(\eta) = \int_{\mathbb{R}^3} \psi(p)\, e^{-i\, p\cdot\eta}\, dp, \qquad (\mathcal{F}^{-1}\psi)(p) = (2\,\pi)^{-3} \int_{\mathbb{R}^3} \psi(\eta)\, e^{i\, p\cdot\eta}\, dp.$$

Theorem 4.1 *For any continuous function* a *on* $\mathbb{R}^3$, *homogeneous of degree* θ^{-1}, $\theta \in (0, \infty)$, *and such that* $\varphi := \mathcal{F}^{-1}(e^{-a(\cdot)}) \geq 0$, *and for any solution* $F \equiv F(t, m)$ *to the coagulation equation (4.2) with coagulation kernel* $a(m, m')$, *the function* $f(t, m, p)$ *defined by*

$$f(t, m, p) = m^{-3\,\theta}\, F(t, m)\, \varphi\left(\frac{p}{m^\theta}\right), \tag{4.3}$$

is a solution of the equation (1.1), (1.3), (1.4) for the same aggregation kernel.

Remark 4.2 *The functions defined on $\mathbb{R}^d$ that are the Fourier transform of non nonnegative measures are, by a theorem of S. Bochner (see for example [21]), the so called positive definite functions. Examples of such functions are $e^{-\beta |p|^r}$ for all $\beta > 0$ and $r \in (0, 2]$. We deduce that $a(p) = |p|^{1/\theta}$, with $\theta \geq 1/2$ are admissible examples in Theorem 4.1. For $\theta = 1/2$ the velocity distribution function is then of Maxwellian type. In the case $\theta = 1$ the velocity distribution function is of the type Lynden-Bell obtained in [15].* `Another example is` $a(p) = |p_1| + |p_2| + |p_3|$ `for which` $\theta = 1$.

Remark 4.3 *Notice that the initial data of the solutions (4.3) in Theorem 4.1 are of the form $f(0, m, p) = m^{-3\theta} F_{in}(m) \varphi\left(p/m^\theta\right)$ for some F_{in}. Theorem 4.1 is not therefore a general existence result of solutions to the Cauchy problem associated to (1.1), (1.3), (1.4).*

Proof of Theorem 4.1. We have to check that the function $f(t, m, p)$ defined by (4.3) solves (1.1), (1.3), (1.4). We start with writing

$$\begin{aligned}
\frac{\partial f}{\partial t} &= m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) \frac{\partial F}{\partial t} \\
&= m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) \left[\frac{1}{2} \int_0^m F(t, m - m') F(t, m')\, a(m - m', m')\, dm' \right. \\
&\qquad \left. - \int_0^\infty F(t, m) F(t, m')\, a(m, m')\, dm' \right]. \qquad (4.4)
\end{aligned}$$

On the one hand, using that

$$\int_{\mathbb{R}^3} \varphi(p)\, dp = \mathcal{F}(\varphi)(0) = e^{-a(0)} = 1,$$

the last term in (4.4) gives

$$\begin{aligned}
& m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) \int_0^\infty F(t, m) F(t, m')\, a(m, m')\, dm' = \\
&= m^{-3\theta} \varphi\left(\frac{p}{m^\theta}\right) F(t, m) \int_0^\infty a(m, m')\, F(t, m') \int_{\mathbb{R}^3} (m')^{-3\theta} \varphi\left(\frac{p'}{m'^\theta}\right) dp' \\
&= f(t, m, p) \int_0^\infty \int_{\mathbb{R}^3} a(m, m')\, f(t, m', p')\, dp'. \qquad (4.5)
\end{aligned}$$

On the other hand, let us define the function

$$g(m, p) = m^{-3\theta} \varphi(p/m^\theta).$$

Using the definition of φ and the homogeneity of a, it satisfies for any $0 < m' < m$

$$\begin{aligned}
\hat{g}(m, \eta) &= \hat{\varphi}(m^\theta \eta) = \exp(-a(m^\theta \eta)) = \exp(-m\, a(\eta)) \\
&= \exp(-m'\, a(\eta)) \exp(-(m - m')\, a(\eta)) \\
&= \hat{g}(m', \eta)\, \hat{g}(m - m', \eta),
\end{aligned}$$

or coming back to the origin function

$$g(m, p) = \int_{\mathbb{R}^3} g(m', p')\, g(m - m', p - p')\, dp'.$$

Using that identity in the first (gain) term in (4.4), we get

$$
\begin{aligned}
& m^{-3\theta}\varphi\left(\frac{p}{m^\theta}\right)\int_0^m F(t,m-m')F(t,m')\,a(m-m',m')\,dm' = \\
& = g(m,p)\int_0^m F(t,m-m')F(t,m')\,a(m-m',m')\,dm' \\
& = \int_{\mathbb{R}^3}\int_0^m F(t,m-m')g(m-m',p-p')\,F(t,m')g(m',p')\,a(m-m',m')\,dm'dp' \\
& = \int_{\mathbb{R}^3}\int_0^m f(t,m-m',p-p')\,f(t,m',p')\,a(m-m',m')\,dm'dp'. \qquad (4.6)
\end{aligned}
$$

We conclude that f satisfies (1.1), (1.3), (1.4) by gathering (4.4), (4.5) and (4.6). □

Remark 4.4 *Solutions of the form $n(m,t)\,\varphi(m,v)$ or $n(t,m)\,e^{-m|v|^2}$ (which corresponds to $\theta = 1/2$) have been considered in previous references as for example [2], [3], [23]. Sometimes this form is obtained from physical arguments, sometimes it is postulated as a simplifying ansatz.*

The previous Theorem is useful in order to prove the existence of self similar solutions for some kernels $a(m,m')$ as it is seen in the following corollary.

Corollary 4.5 *Suppose that a and θ are as in Theorem 4.1. Assume further that F is a self similar solution of the coagulation equation with coagulation kernel $a(m,m')$. Then the function f defined by (4.3) is a self similar solution of (1.1), (1.3), (1.4).*

Proof of Corollary 4.5. The hypothesis on F means that for some functions Φ, $\nu(t)$ and $\mu(t)$ it may be written as:

$$F(t,m) = \nu(t)\Phi(\mu(t)\,m).$$

Therefore f is a self-similar function since it may be written as

$$
\begin{aligned}
f(t,m,p) &= m^{-3\theta}\,\nu(t)\Phi(\mu(t)\,m)\,\varphi\left(\frac{p}{m^\theta}\right) \\
&= \nu(t)\mu(t)^{3\theta}\,(\mu(t)\,m)^{-3\theta}\;\Phi(\mu(t)\,m)\varphi\left(\frac{\mu(t)^\theta\,p}{(\mu(t)m)^\theta}\right) \\
&= \nu(t)\mu(t)^{3\theta}\,\Psi\left(\mu(t)\,m,\mu(t)^\theta\,p\right)
\end{aligned}
$$

with $\Psi(M,P) = M^{-3\theta}\,\Phi(M)\,\varphi\left(P/M^\theta\right)$. □

Remark 4.6 *The existence of self similar solutions of (1.1), (1.3), (1.4) corresponding to the case $\theta = 1/2$ of Corollary 4.5 had already been proved in [11].*

Remark 4.7 *Self similar solutions of the coagulation equation are well known to exist for the cases $a(m,m') = 1$, $a(m,m') = m + m'$ and $a(m,m') = m\,m'$. Their existence has been proved in [9] and [10], for several other kernels with homogenetity $\lambda < 1$. In that case they are of the form:*

$$F(t,m) = t^{-\frac{2}{1-\lambda}}\Phi\left(\frac{m}{t^{\frac{1}{1-\lambda}}}\right). \qquad (4.7)$$

We deduce that, under the assumption of the Corollary 4.5, and for these kernels $a(m,m')$ with homogeneity $\lambda < 1$,

$$f(t,m,p) = t^{-\frac{2}{1-\lambda}} m^{-3\theta}\,\Phi\left(\frac{m}{t^{\frac{1}{1-\lambda}}}\right)\varphi\left(\frac{p}{m^{\theta}}\right). \tag{4.8}$$

is a self similar solutions to equation (1.1), (1.3), (1.4). A straightforward calculation yields

$$P_k(t) = \int_{\mathbb{R}^d}\int_0^\infty |p|^k\,f(t,m,p)\,dm\,dp = t^{-\frac{1-k\theta}{1-\lambda}}\int_{\mathbb{R}^d}|P|^k\varphi(P)dP\int_0^\infty M^{k\theta}\,\Phi(M)\,dM. \tag{4.9}$$

As a consequence, we have $P_0 \to 0$, $P_1 \to 0$ and more generally $P_k \to 0$ whenever $k < \theta^{-1}$ but $P_k/P_0 \to \infty$ for any $k > 0$ and $P_k \to \infty$ whenever $k > \theta^{-1}$. The interpretation in terms of the model is that the total number of particles in the gas and the total impulsion of the gas decrease and tend to zero, while, for instance, the mean second moment P_2/P_0 tends to infinity as t tends to infinity. This behavior is quite similar to that of the solutions to the Smoluchowski equation (where the mean impulsion moment $P_k/P_0 \to \infty$ for any $k > 0$) and is completely different to that discussed in Remark 3.11.

5 The constant case $a = 1$

We consider in this Section the aggregation kernel $a = 1$. Equation (1.1), (1.3) (1.4) reads then:

$$\begin{aligned}\partial_t f(t,m,p) = \frac{1}{2}\int_{\mathbb{R}^d}\int_0^m f(t,m-m',p-p')\,f(t,m',p')dm'\,dp'\\ -f(t,m,p)\int_{\mathbb{R}^d}\int_0^\infty f(t,m',p')\,dm'\,dp'.\end{aligned} \tag{5.1}$$

The first result is on the existence of self similar solutions.

Theorem 5.1 *Let $\Phi \in C^1(\mathbb{R})$ such that:*

$$\lim_{\zeta\to 0,\,\xi\to 0}\zeta\Phi\left(\frac{\xi^2}{\zeta}\right) = 0 \tag{5.2}$$

and suppose that

$$g(y,x) = \mathcal{F}_\xi^{-1}\mathcal{L}_\zeta^{-1}\left(\frac{2}{2\,\zeta\Phi\left(\frac{\xi^2}{\zeta}\right)+1}\right) \tag{5.3}$$

satisfies $g \in L^1(\mathbb{R}^+\times\mathbb{R})$. Then, for all real positive numbers β_i, $i = 1,\cdots,d$ the function

$$t^{-\frac{d+4}{2}} g\left(\frac{m}{t},\beta_1\frac{p_1}{\sqrt{t}}\cdots,\beta_d\frac{p_d}{\sqrt{t}}\right). \tag{5.4}$$

is a self similar `weak` *solution to (1.1), (1.3), (1.4) with $a = 1$. If moreover $g \in C^1(\mathbb{R}^+\times\mathbb{R})$ then it is a classical solution.*

Remark 5.2 *As it will be seen in the Remark 5.3 below, it is easy to obtain different self similar solutions of the equation (5.1) using Theorem 5.1. Notice nevertheless that these self similar solutions are not necessarily non negative.*

Proof of Theorem 5.1.

The formal argument leading to the expression (5.3) and condition (5.2) is the following. If we look for a self similar solution of (5.1) of the form (5.4) the function g must then solve:

$$-\frac{d+2}{2}\,g - y\partial_y g - \frac{1}{2}x\cdot\nabla_x g = \frac{1}{2}\int_{\mathbb{R}}\int_0^y g(y-y',x-x')\,g(y',x')dy'\,dx' - \\ -g\int_{\mathbb{R}}\int_0^\infty g(y',x')dy'\,dx'. \qquad (5.5)$$

We integrate this equation with respect to x and y and obtain

$$\int_{\mathbb{R}}\int_0^\infty g(y',x')dy'\,dx' = 2. \qquad (5.6)$$

We now Fourier transform with respect to x and Laplace transform with respect to y:

$$\zeta\partial_\zeta\widehat{g} + \frac{1}{2}\xi\cdot\nabla_\xi\widehat{g} = \frac{1}{2}\widehat{g}^2 - \widehat{g}. \qquad (5.7)$$

We divide by $\widehat{g}^2$ and define $G = 1/\widehat{g}$:

$$\zeta\partial_\zeta G + \frac{1}{2}\xi\cdot\nabla_\xi G = G - \frac{1}{2}. \qquad (5.8)$$

The function G may then be any function of the form:

$$G(\zeta,\xi) = \zeta\Phi\left(\frac{|\xi|^2}{\zeta}\right) + \frac{1}{2} \qquad (5.9)$$

for any arbitrary, derivable function Φ. Therefore we should have:

$$\widehat{g}(\zeta,\xi) = \frac{2}{2\,\zeta\Phi\left(\frac{|\xi|^2}{\zeta}\right)+1}, \qquad (5.10)$$

with, due to (5.6):

$$\lim_{\zeta\to 0,\,\xi\to 0}\frac{2}{2\,\zeta\Phi\left(\frac{|\xi|^2}{\zeta}\right)+1} = 2 \Longleftrightarrow \lim_{\zeta\to 0,\,\xi\to 0}\zeta\Phi\left(\frac{|\xi|^2}{\zeta}\right) = 0.$$

It is then straightforward to check that, given any function $\Phi\in C^1(\mathbb{R})$ satisfying (5.2) the function G defined by (5.9) is such that G^{-1} satisfies (5.7). Therefore, if (5.3) defines a function $g\in L^1(\mathbb{R}^+\times\mathbb{R})$, the function g satisfies (5.5) in the weak sense of distributions in $\mathbb{R}^+\times\mathbb{R}$ and the function (5.4) is a weak solution of (5.1). □

Remark 5.3 *If* $\Phi(z) = z + 1$,

$$\hat{g}(\zeta, \xi) = \frac{2}{2\,\zeta\left(\frac{|\xi|^2}{\zeta} + 1\right) + 1} = \frac{2}{2\,(|\xi|^2 + \zeta) + 1}$$

$$= \mathcal{L}\left(\mathcal{F}\left(\frac{e^{-\frac{y}{2}}\,e^{-\frac{|x|^2}{4y}}}{\sqrt{2}\sqrt{y}}\right)\right).$$

and then,

$$f(t, m, p) = t^{-(d+4)/2}\frac{1}{\sqrt{2}\sqrt{m}}e^{-\frac{m}{2\,t}}\,e^{-\frac{1}{4m}\sum_{i=1}^{d}\beta_i^2\,p_i^2} \tag{5.11}$$

is a self similar solution of (5.1). This is, up to a constant, the profile of the self similar solution that appears in Theorem 5.4 below. It is possible to obtain other self similar solutions of (5.1). Some of them are explicit others are not. If, for example, $\Phi \equiv 1$ *then* $g(y, x) = e^{-y^2}\delta_{x=0}$. *Another explicit example is for* $\Phi(z) = z$ *which gives* $g(y, x) = \sqrt{\pi}\delta_{y=0}e^{-\frac{|x|}{\sqrt{2}}}$. *Notice that in all these three examples the solution g is non negative. On the other hand, if we take* $\Phi(z) = \sqrt{z}$, *the inverse Laplace transform, let us call it* $h(y, \xi)$, *is still explicit:*

$$h(y, \xi) = \mathcal{L}_\zeta^{-1}\left(\frac{2}{2\sqrt{\zeta\,\xi^2}} + 1\right) = \frac{\frac{\sqrt{\xi^2}}{\sqrt{\pi}\,\sqrt{y}} - e^{\frac{y}{\xi^2}}\,Erfc\left(\sqrt{\frac{y}{\xi^2}}\right)}{4\xi^2}. \tag{5.12}$$

It remains to check that $h(y, \cdot)$ *has an inverse Fourier transform with respect to the variable* ξ. *It is easily checked that, for all* $y > 0$ *fixed:*

$$h(y, \xi) = \mathcal{O}\left(\frac{\xi}{y^{3/2}}\right), \quad as\ \xi \to 0$$

$$h(y, \xi) = \frac{\frac{1}{\sqrt{\pi}}\sqrt{\frac{\xi^2}{y}} - 1}{4\xi^2} + \mathcal{O}\left(\frac{y}{\xi^2}\right) \quad as\ |\xi| \to +\infty.$$

This function is then in $L^2(\mathbb{R})$ *with respect to the* ξ *variable and has then an inverse Fourier transform with respect to* ξ *which is* $g(y, x)$:

$$g(y, x) = \mathcal{F}_\xi^{-1}(h(y, \cdot))(x).$$

Moreover, for all $y > 0$, $g(y, \cdot) \in L^2(\mathbb{R})$ *and the convolution of* $g(y, \cdot)$ *with itself is well defined*

$$\mathcal{F}\left(g(y - y', \cdot) * g(y', \cdot)\right)(\xi) = h(y - y', \xi)\,h(y', \xi)$$

and

$$\int_0^y \mathcal{F}\left(g(y - y', \cdot) * g(y', \cdot)\right)(\xi)dy = \int_0^y h(y - y', \xi)\,h(y', \xi)\,dy.$$

Therefore,

$$\int_{\mathbb{R}} \left|\mathcal{F}\left(g(y - y', \cdot) * g(y', \cdot)\right)(\xi)\right| d\xi \;\leq\; \int_0^y \int_{\mathbb{R}} \left|h(y - y', \xi)\,h(y', \xi)\right| d\xi\,dy = \sum_{k=1}^{6} I_k,$$

with

$$
\begin{aligned}
I_1 &:= \int_0^{y/2} \int_{|\xi| \le {y'}^{1/2} \le (y-y')^{1/2}} \left| h(y-y', \xi)\, h(y', \xi) \right| d\xi\, dy', \\
I_2 &:= \int_0^{y/2} \int_{{y'}^{1/2} \le |\xi| \le (y-y')^{1/2}} \left| h(y-y', \xi)\, h(y', \xi) \right| d\xi\, dy', \\
I_3 &:= \int_0^{y/2} \int_{{y'}^{1/2} \le (y-y')^{1/2} \le |\xi|} \left| h(y-y', \xi)\, h(y', \xi) \right| d\xi\, dy', \\
I_4 &:= \int_{y/2}^{y} \int_{|\xi| \le (y-y')^{1/2} \le {y'}^{1/2}} \left| h(y-y', \xi)\, h(y', \xi) \right| d\xi\, dy', \\
I_5 &:= \int_{y/2}^{y} \int_{(y-y')^{1/2} \le |\xi| \le {y'}^{1/2}} \left| h(y-y', \xi)\, h(y', \xi) \right| d\xi\, dy', \\
I_6 &:= \int_{y/2}^{y} \int_{(y-y')^{1/2} \le {y'}^{1/2} \le |\xi|} \left| h(y-y', \xi)\, h(y', \xi) \right| d\xi\, dy'.
\end{aligned}
$$

We must verify that each term is finite. Indeed, we have

$$
\begin{aligned}
I_1 &\le C \int_0^{y/2} \int_{|\xi| \le {y'}^{1/2} \le (y-y')^{1/2}} \frac{\xi^2}{{y'}^{3/2} (y-y')^{3/2}} d\xi\, dy' \\
&\le C \int_0^{y/2} \frac{min\{{y'}^{3/2},\, (y-y')^{3/2}\}}{{y'}^{3/2}(y-y')^{3/2}} dy' < \infty; \\
I_2 &\le C \int_0^{y/2} \int_{{y'}^{1/2} \le |\xi| \le (y-y')^{1/2}} \frac{|\xi|}{(y-y')^{3/2}} \left(\frac{1}{\sqrt{y'}\,|\xi|} + \frac{1}{\xi^2} + \mathcal{O}\left(\frac{y'}{\xi^2} \right) \right) d\xi\, dy' \\
&\le \frac{C}{y^{3/2}} \int_0^{y/2} \int_{{y'}^{1/2} \le |\xi| \le (y-y')^{1/2}} \left(\frac{1}{\sqrt{y'}} + \frac{1}{|\xi|} + 1 \right) d\xi dy \\
&\le \frac{C}{y^{3/2}} \int_0^{y/2} \int_{{y'}^{1/2} \le |\xi| \le (y-y')^{1/2}} \left(\frac{2}{\sqrt{y'}} + 1 \right) d\xi dy < \infty; \\
I_3 &\le C \int_0^{y/2} \int_{{y'}^{1/2} \le (y-y')^{1/2} \le |\xi|} \left(\frac{1}{\sqrt{y'}\,|\xi|} + \frac{1}{\xi^2} + \mathcal{O}\left(\frac{y'}{\xi^2} \right) \right) \times \\
&\quad \times \left(\frac{1}{\sqrt{y-y'}\,|\xi|} + \frac{1}{\xi^2} + \mathcal{O}\left(\frac{y-y'}{\xi^2} \right) \right) d\xi\, dy' \\
&\le C \int_0^{y/2} \int_{{y'}^{1/2} \le (y-y')^{1/2} \le |\xi|} \left(\frac{1}{\sqrt{y'}\sqrt{y-y'}\,|\xi|^2} + \frac{1}{|\xi|^3} \left(\frac{1}{\sqrt{y-y'}} + \frac{1}{\sqrt{y'}} \right) + \right. \\
&\quad \left. + \frac{1}{\xi^4} + \mathcal{O}\left(\frac{y'}{\sqrt{y-y'}|\xi|^3} \right) + \mathcal{O}\left(\frac{y-y'}{\sqrt{y'}|\xi|^3} \right) + \mathcal{O}\left(\frac{y+y^2}{|\xi|^4} \right) \right) d\xi dy \\
&\le \frac{C}{\sqrt{y}} \int_0^{y/2} \frac{1}{\sqrt{y'}} \int_{{y'}^{1/2} \le (y-y')^{1/2} \le |\xi|} \frac{d\xi}{|\xi|^2}\, dy + \\
&\quad + C \int_0^{y/2} \left(\frac{1}{\sqrt{y}} + \frac{1}{\sqrt{y'}} \right) \int_{{y'}^{1/2} \le (y-y')^{1/2} \le |\xi|} \frac{d\xi}{|\xi|^3}\, dy + \\
&\quad + C \int_0^{y/2} \int_{{y'}^{1/2} \le (y-y')^{1/2} \le |\xi|} \frac{d\xi}{|\xi|^4}\, dy
\end{aligned}
$$

$$\leq \frac{C}{\sqrt{y}}\int_0^{y/2}\frac{dy}{\sqrt{y'}\sqrt{(y-y')}}+C\int_0^{y/2}\left(\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{y'}}\right)\frac{dy}{y-y'}+C\int_0^{y/2}\frac{dy}{(y-y')^2}.$$

Similar estimates show that the integrals I_4, I_5 and I_6 converge. The function $\int_0^y h(y-y',\xi)\,h(y',\xi)\,dy$ is then in $L^1(\mathbb{R})$ and has then an inverse Fourier transform which is

$$\int_0^y (g(y-y',\cdot)*g(y',\cdot))(\xi)\,dy.$$

In the second result of this Section we obtain solutions of (5.1) for a particular class of initial data and we describe their long time asymptotic behaviour in terms of self similar solutions (cf. Remark 5.5).

Theorem 5.4 *Suppose that the initial data $f_{in}\in L^1(Y;k_S^2(y)\,dy)$ is even, non negative and such that*

$$\int_0^\infty\int_{\mathbb{R}^d} e^{s_0 m} f_{in}(m,p)\,dp\,dm<+\infty \tag{5.13}$$

for some $s_0>0$. Suppose also that the function F_0 defined as:

$$F_0(\zeta,\xi)=\frac{1}{\sqrt{2\pi}}\int_0^\infty\int_{\mathbb{R}} e^{-ip\xi}e^{-m\zeta}f_{in}(m,p)\,dp\,dm \tag{5.14}$$

satisfies:

$$\int_{\mathbb{R}^d}\int_{-\infty}^\infty |F_0(u+iv,\,\xi)|\,dv\,d\xi<\infty \tag{5.15}$$

when $u>s_0$. Then the function

$$f(t,m,p)=\mathcal{F}^{-1}\left(\mathcal{L}^{-1}F\right)(t,m,p) \tag{5.16}$$

$$F(t,\zeta,\xi)=\frac{H_0^2}{(H_0+(t/2))^2\left(\frac{1}{F_0(\zeta,\xi)}-\frac{H_0\,t/2}{H_0+(t/2)}\right)}, \tag{5.17}$$

with $H_0:=M_{0,0}(0)^{-1}$ as defined in (2.6), is such that $f\in C^1((0,\infty);L^1(\mathbb{R}^+\times\mathbb{R}^d))$ and satisfies (1.1), (1.2), (1.3) (1.4) with $a(y,y')\equiv 1$ for all $t>0$ and almost every $y\in\mathbb{R}^+\times\mathbb{R}^d$.

Furthermore, f satisfies

$$t^{(d+4)/2}f(t,t\,m,\sqrt{t}\,p)\ \rightharpoonup\varphi_\infty(m,p):=C_1\frac{e^{-C_2m}}{\sqrt{m}}\prod_{i=1}^d\sqrt{C_{3,i}}\,e^{-C_{3,i}^2\frac{|p_i|^2}{m}}, \tag{5.18}$$

$$=C_1\frac{e^{-C_2m}}{\sqrt{m}}\left(\prod_{i=1}^d\sqrt{C_{3,i}}\right)e^{-\sum_{i=1}^d C_{3,i}^2\frac{|p_i|^2}{m}} \tag{5.19}$$

in the weak sense of measures $\sigma(M^1(Y),C_c(Y))$, as $t\to+\infty$, where

$$C_1=\frac{4}{(2\pi)^{d/2}M_{1,0}(f_{in})},\qquad C_2=\frac{2}{M_{1,0}(f_{in})}, \tag{5.20}$$

$$C_{3,i}=\sqrt{\frac{2M_{1,0}(f_{in})}{M_{0,0}(f_{in})N_i(f_{in})}},\quad N_i(f_{in})=\int_0^\infty\int_{\mathbb{R}^d}p_i^2f_{in}(m,p)\,dp\,dm. \tag{5.21}$$

Remark 5.5 *Theorem 5.4 shows that the solutions of (5.1), for the set of initial data satisfying the hypothesis, behave asymptotically when $t \to +\infty$ as one of the self similar solutions obtained in Theorem 5.1. Notice indeed that (5.18) means that the solution $f(t, m, p)$ behaves, in a weak sense, as the self similar solution:*

$$t^{-(d+2)/2}\varphi_\infty \left(\frac{m}{t}, \beta_1 \frac{p_1}{\sqrt{t}}, \cdots, \beta_d \frac{p_d}{\sqrt{t}}\right)$$

as $t \to +\infty$. Notice also that this self similar solutions is essentially the same as (5.11) in Remark 5.3. It is also noteworthy that, among all the possible self similar solutions, the solution f converges to one that is regular and non negative.

Proof of Theorem 5.4. By Theorem 5.4, there exists a unique solution f of the equation (5.1) in $C([0, T); L^1(Y; k_S(y)\,dy)) \cap L^\infty(0, T; L^1(Y; k_S^2(y)\,dy))$ for all $T > 0$. Moreover, if the initial data is non negative so is the solution for all time. It turns out that the equation (5.1) may be explicitly solved using Fourier transform with respect to $p \in \mathbb{R}$ and Laplace transform with respect to $m > 0$. Let us then consider such a transform defined as:

$$F(t, \zeta, \xi) = \frac{1}{(2\pi)^{d/2}} \int_0^\infty \int_{\mathbb{R}^d} e^{-m\,\zeta}\, e^{-i\,p\,\xi} f(t, m, p)\, dp\, dm. \tag{5.22}$$

Due to the properties of the solution f we may then apply the transform (5.22) to both sides of the equation (5.1) and obtain the following Bernouilli equation:

$$\partial_t F(t, \zeta, \xi) = \frac{1}{2} F^2(t, \zeta, \xi) - M_0(t)\, F(t, \zeta, \xi) \tag{5.23}$$

$$M_0(t) = F(t, 0, 0). \tag{5.24}$$

We first notice, taking $\zeta = \xi = 0$ in (5.23), that $M_0(t)$ satisfies $\frac{d}{dt} M_0(t) = -\frac{1}{2} M_0^2(t)$ from where

$$M_0(t) = \frac{1}{H_0 + t/2}. \tag{5.25}$$

The unique solution of (5.23) is the function $F(t, \zeta, \xi)$ given by (5.17). On the one hand, the function $t \mapsto (H_0\, t/2)/(H_0 + t/2)$ is strictly increasing with limit in infinity equal to H_0, so that for any $\delta \in (0, 1)$ there exists $T \in (0, \infty)$

$$\forall\, t \in [0, T] \qquad \frac{H_0\, t/2}{H_0 + t/2} \le H_0\, (1 - \delta), \tag{5.26}$$

and on the other hand

$$|F(0, \zeta, \xi)| \le \int_0^\infty \int_{\mathbb{R}} f(0, m, p)\, dm dp = H_0^{-1}. \tag{5.27}$$

Gathering (5.26) and (5.27) the fraction in the right hand side of (5.17) is well defined for all $t > 0$. More precisely for any $t \in [0, T]$ there is $\delta = \delta(t) > 0$ such that

$$\begin{aligned} \left| \frac{1}{F(0, \zeta, \xi)} - \frac{H_0\, t/2}{H_0 + (t/2)} \right| &\ge \left| \frac{1}{F(0, \zeta, \xi)} \right| - \frac{H_0\, t/2}{H_0 + (t/2)} \\ &\ge |F(0, \zeta, \xi)|^{-1} - H_0\, (1 - \delta) \ge \delta\, |F(0, \zeta, \xi)|^{-1}. \end{aligned} \tag{5.28}$$

By the hypothesis on f_{in}, for any fixed $\xi \in \mathbb{R}$ the function $F_0(\cdot,\,\xi)$ is analytic in the half plane $Re(\zeta) > -s_0$, it tends towards zero when in the half plane $Re(\zeta) \geq -s_0/2$, ζ tends two-dimensionally towards infinity. It immediately follows from (5.17) and (5.56) that $F(t,\cdot,\xi)$ satisfies the same properties for all $t > 0$ and $\xi \in \mathbb{R}^d$ as well as

$$\int_{\mathbb{R}^d} \int_{-\infty}^{\infty} |F(t, u + iv,\, \xi)|\, dv\, d\xi < \infty \tag{5.29}$$

for all $u > s_0$ for all $t > 0$. We deduce that for any $t > 0$ and almost every $\xi \in \mathbb{R}^d$

$$\int_{-\infty}^{\infty} |F(t, u + iv,\, \xi)|\, dv < \infty \tag{5.30}$$

and therefore the function $F(t,\cdot,\xi)$ is the Laplace transform of the function:

$$\mathcal{L}^{-1}(F(t,\cdot,\xi))(m) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{m\,\zeta}\, F(t,\zeta,\xi)\, d\zeta$$

the integral being independent of $x > s_0$. Notice that we have, for any $x > 0$:

$$\frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \int_{\mathbb{R}^d} |e^{m\,\zeta}\, F(t,\zeta,\xi)|\, d\zeta\, d\xi = \frac{e^{x\,m}}{2\pi} \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} |F(t, x + iv, \xi)|\, dv\, d\xi < \infty.$$

The function $\xi \mapsto \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{m\,\zeta}\, F(t,\zeta,\xi)\, d\zeta$ is then integrable and we have:

$$\begin{aligned} f(t,m,p) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ip\xi} \mathcal{L}^{-1}(F(t,\cdot,\xi))(m) d\xi \\ &= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \int_{\mathbb{R}^d} e^{ip\xi}\, e^{m\,\zeta}\, F(t,\zeta,\xi)\, d\zeta\, d\xi. \end{aligned} \tag{5.31}$$

In order to study the behaviour of $f(t,m,p)$ as $t \to \infty$ it is a classical argument to consider the rescaled function φ associated to f by the relation

$$\varphi(t, M, P) := t^{(d+4)/2}\, f(t, t\,M, \sqrt{t}\,P), \tag{5.32}$$

so that

$$f(t,m,p) = t^{-(d+4)/2}\, \varphi\left(t, \frac{m}{t}, \frac{p}{\sqrt{t}}\right). \tag{5.33}$$

Taking the Fourier and Laplace transform in both side yields

$$F(t,\zeta,\xi) \;=\; t^{-1}\Phi(t, t\,\zeta, \sqrt{t}\,\xi) \tag{5.34}$$

with

$$\Phi(t,\zeta,\xi) \;=\; \frac{tH_0^2}{(H_0 + (t/2))^2 \left(\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})} - \frac{H_0\, t/2}{H_0 + (t/2)}\right)}. \tag{5.35}$$

Since we are interested in the long time behaviour of $\Phi(\cdot,\zeta,\xi)$ for all ζ and ξ fixed , we may write:

$$\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})}-\frac{H_0\,t/2}{H_0+(t/2)} \quad = \quad \frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})}-H_0+\frac{H_0^2}{H_0+(t/2)}$$

and consider the auxiliary function

$$\begin{aligned}\Psi(t,\zeta,\xi) &= \frac{tH_0^2}{((t/2))^2\left(\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})}-H_0+\frac{H_0^2}{(t/2)}\right)} \\ &= \frac{4H_0^2}{t\left(\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})}-H_0+\frac{2\,H_0^2}{t}\right)}. \end{aligned} \tag{5.36}$$

We perform the following expansion up to the order $o(1/t)$:

$$\begin{aligned}\frac{1}{F(0,\frac{\zeta}{t},\frac{\xi}{\sqrt{t}})}-H_0 &= \frac{\zeta}{t}\frac{\partial F^{-1}}{\partial\zeta}(0,0,0)+\frac{\xi}{\sqrt{t}}\cdot\nabla_\xi F^{-1}(0,0,0)+ \\ &+\frac{1}{2\,t}\sum_{i,j=1}^{d}\xi_i\,\xi_j\frac{\partial^2 F^{-1}}{\partial\xi_i\,\partial\xi_j}(0,0,0)+o\left(\frac{1}{t}\right). \end{aligned} \tag{5.37}$$

Since by hypothesis f is even with respect to p, we have

$$\frac{\partial F}{\partial\xi_k}(0,0,0)=-i\int_0^\infty\int_{\mathbb{R}^d}f_{in}(m,p)\,p_k\,dp\,dm=0$$

and then:

$$\frac{\partial F^{-1}}{\partial\xi_k}(0,0,0)=-\frac{1}{F(0,0,0)^2}\frac{\partial F}{\partial\xi_k}(0,0,0)=0. \tag{5.38}$$

We also have

$$\frac{\partial^2 F}{\partial\xi_i\partial\xi_j}(0,0,0)=-\int_0^\infty\int_{\mathbb{R}^d}p_i\,p_j\,f(0,m,p)dpdm,$$

which with the help of (5.38) implies

$$\begin{aligned}\frac{\partial^2 F^{-1}}{\partial\xi_i\partial\xi_j}(0,0,0) &= -F^{-2}(0,0,0)\frac{\partial^2 F}{\partial\xi_i\partial\xi_j}(0,0,0) \\ &= H^2(0)\int_0^\infty\int_{\mathbb{R}^d}p_i\,p_j\,f(0,m,p)dpdm \quad =:2\,\mathcal{B}_{i,j}. \end{aligned} \tag{5.39}$$

Since $f_{in}(m,p)$ is even with respect to the p variable we have $B_{i,j}=0$ whenever $i\neq j$. Similarly, we compute

$$\frac{\partial F}{\partial\zeta}(0,0,0) \quad = \quad -\int_0^\infty\int_{\mathbb{R}^d}m\,f(0,m,p)dpdm,$$

which implies

$$\frac{\partial F^{-1}}{\partial \zeta}(0,0,0) = -\frac{1}{F^2(0,0,0)}\frac{\partial F}{\partial \zeta}(0,0,0) =: \mathcal{A}. \tag{5.40}$$

Thanks to (5.37), (5.38), (5.39) and (5.40), we deduce that (5.36) reads now:

$$\Psi(t,\zeta,\xi) = \frac{4H^2(0)}{\left(\zeta\,\mathcal{A} + \sum_i \mathcal{B}_{i,}\,\xi_i^2 + 2H^2(0) + o(1)\right)}$$

from where

$$\lim_{t\to+\infty} \Phi(t,\zeta,\xi) = \lim_{t\to+\infty} \Psi(t,\zeta,\xi) = \frac{4H^2(0)}{\mathcal{A}\,\zeta + \sum_i \mathcal{B}_i\,\xi_i^2 + 2H^2(0)} =: \Psi_\infty(\zeta,\xi). \tag{5.41}$$

In order to come back to the original variables, we recall that from standard integral calculus for any $\mathcal{C},\mathcal{D} > 0$

$$\frac{1}{(2\pi)^{1/2}}\int_0^\infty \int_{\mathbb{R}} e^{-m\,\zeta}\, e^{-i\,p\,\xi}\, \frac{e^{-\mathcal{C}\,m} e^{-\frac{|p|^2}{2\,\mathcal{D}\,m}}}{\sqrt{\mathcal{D}\,m}}\, dp\, dm = \frac{1}{\zeta + \mathcal{D}\,\xi^2 + \mathcal{C}},$$

from where choosing $\mathcal{C} := 2\,H_0^2/\mathcal{A}$ and $\mathcal{D}_i := \mathcal{B}_i/\mathcal{A}$, we obtain

$$\begin{aligned}(\mathcal{F}^{-1}\mathcal{L}^{-1})\,(\Psi_\infty) &= \frac{4\,H_0^2}{(2\pi m)^{1/2}\,\mathcal{A}} e^{-\mathcal{C}\,m} \prod_{i=1}^d \frac{e^{-\frac{|p_i|^2}{2\,\mathcal{D}_i\,m}}}{\sqrt{\mathcal{D}_i}} \\ &= \frac{4\,H_0^2 \mathcal{A}^{\frac{d}{2}-1}}{(2\pi m)^{1/2}} e^{-\mathcal{C}\,m} \prod_{i=1}^d \frac{e^{-\frac{\mathcal{A}}{2m}\left|\frac{p_i}{\sqrt{\mathcal{B}_i}}\right|^2}}{\sqrt{\mathcal{B}_i}} = \varphi_\infty(m,p)\end{aligned}$$

as defined in (5.18). Finally, (5.65) implies that $\varphi(t,.) \rightharpoonup \varphi_\infty$ in the weak sense of measures, which is nothing but (5.18). ☐

It is very easy to obtain a simplified version of Theorem 5.4 for the equation (3.1)-(3.4) with $a(p,p') = 1$:

Theorem 5.6 *Suppose that the initial data $f_{in} \in M^1_{2\alpha}(\mathbb{R}^d)$, $\alpha \in \mathbb{N}\backslash\{0,1\}$ is non negative and satisfies $f_{in}(p) = f_{in}(-p)$ and let F_0 be its Fourier transform:*

$$F_0(\xi) = \mathcal{F}\,(f)\,(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^d} e^{-i\,p\,\xi}\, f_{in}(p)\, dp. \tag{5.42}$$

Suppose moreover that

$$\int_{\mathbb{R}^d} |F_0(\xi)|\, d\xi < +\infty \tag{5.43}$$

Then the function

$$f(t,p) = \mathcal{F}^{-1}\,(F)\,(t,p) \tag{5.44}$$

$$F(t,\xi) = \frac{H_0^2}{(H_0 + (t/2))^2\left(\frac{1}{F_0(\xi)} - \frac{H_0\,t/2}{H_0 + (t/2)}\right)}, \tag{5.45}$$

with $H_0^{-1} := F_0(0)$, *is such that* $f \in C([0,T); M^1(\mathbb{R}^d) - weak) \cap L^\infty(0,T; M^1_{2\alpha}(\mathbb{R}^d))$, $f \in C([0,+\infty); L^\infty(\mathbb{R}^d)) \cap C([0,+\infty) \times \mathbb{R}^d)$ *and satisfies (3.1)-(3.4) with* $a(y,y') \equiv 1$ *for all* $t > 0$ *and almost every* $y \in \mathbb{R}^+ \times \mathbb{R}^d$.

Furthermore, f *satisfies*

$$t^{d/2} f(t, \sqrt{t}\, p) \rightharpoonup \varphi_\infty(p) \tag{5.46}$$

in the weak sense of measures $\sigma(M^1(Y), C_c(Y))$, *as* $t \to +\infty$ *where*

$$\varphi_\infty(p) = \mathcal{F}^{-1}(\Psi_\infty)(p) \tag{5.47}$$

$$\Psi_\infty(\xi) = \frac{4H^2(0)}{\sum_{i=1}^d \mathcal{B}_i\, \xi_i^2 + 2H^2(0)} \tag{5.48}$$

$$\mathcal{B}_i = \int_{\mathbb{R}^d} p_i^2\, f_{in}(p)\, dp. \tag{5.49}$$

In particular, when $d = 1$*:*

$$\varphi_\infty(p) = 2\sqrt{\frac{2\,\pi}{M_2(f_{in})}}\, e^{-C_1\, |p|}. \tag{5.50}$$

Proof of Theorem 5.6. The equation (3.1)-(3.4) may be explicitly solved using Fourier transform with respect to $p \in \mathbb{R}$. By Theorem 3.1 we already know the existence and uniqueness of a solution $f \in C([0,T); M^1(\mathbb{R}^d) - weak) \cap L^\infty(0,T; M^1_{2\alpha}(\mathbb{R}^d))$. Applying Fourier transform to both sides of the equation (3.1)-(3.4) we obtain the following Bernouilli equation:

$$\partial_t F(t,\xi) = \frac{1}{2} F^2(t,\xi) - M_0(t)\, F(t,\xi) \tag{5.51}$$

$$M_0(t) = F(t,0). \tag{5.52}$$

We first notice, taking $\xi = 0$ in (5.51), that $M_0(t)$ satisfies $\frac{d}{dt} M_0(t) = -\frac{1}{2} M_0^2(t)$ from where

$$M_0(t) = \frac{1}{H_0 + t/2}. \tag{5.53}$$

It is then straightforward to solve (5.51) and obtain the function $F(t,\xi)$ given by (5.45). On the one hand, the function $t \mapsto (H_0\, t/2)/(H_0 + t/2)$ is strictly increasing with limit in infinity equal to H_0, so that for any $\delta \in (0,1)$ there exists $T \in (0,\infty)$

$$\forall\, t \in [0,T] \qquad \frac{H_0\, t/2}{H_0 + t/2} \le H_0\, (1-\delta), \tag{5.54}$$

and on the other hand

$$|F(0,\xi)| \le \int_{\mathbb{R}^d} f(0,p)\, dp = H_0^{-1}. \tag{5.55}$$

Gathering (5.54) and (5.55) the fraction in the right hand side of (5.45) is well defined for all $t > 0$. More precisely for any $t \in [0,T]$ there is $\delta = \delta(t) > 0$ such that

$$\begin{aligned} \left| \frac{1}{F(0,\zeta,\xi)} - \frac{H_0\, t/2}{H_0 + (t/2)} \right| \;&\ge\; \left| \frac{1}{F(0,\zeta,\xi)} \right| - \frac{H_0\, t/2}{H_0 + (t/2)} \\ &\ge\; |F(0,\zeta,\xi)|^{-1} - H_0\,(1-\delta) \ge \delta\, |F(0,\zeta,\xi)|^{-1}. \end{aligned} \tag{5.56}$$

Therefore:

$$|F(t,\xi)| \le \frac{1}{\delta}\,|F_0(\xi)|$$

and (5.44) follows.

In order to study the behaviour of $f(t,p)$ as $t\to\infty$ it is a classical argument to consider the rescaled function φ associated to f by the relation

$$\varphi(t,P) := t^{d/2}\, f(t,\sqrt{t}\,P), \tag{5.57}$$

so that

$$f(t,p) = t^{-d/2}\,\varphi\left(t,\frac{p}{\sqrt{t}}\right). \tag{5.58}$$

Taking the Fourier and Laplace transform in both side yields

$$F(t,\xi) \;=\; t^{-1}\Phi(t,\sqrt{t}\,\xi) \tag{5.59}$$

with

$$\Phi(t,\xi) \;=\; \frac{tH_0^2}{(H_0+(t/2))^2\left(\frac{1}{F(0,\frac{\xi}{\sqrt{t}})}-\frac{H_0\,t/2}{H_0+(t/2)}\right)}. \tag{5.60}$$

Since we are interested in the long time behaviour of $\Phi(\cdot,\xi)$ for all ξ fixed , we may write:

$$\frac{1}{F(0,\frac{\xi}{\sqrt{t}})}-\frac{H_0\,t/2}{H_0+(t/2)} \;=\; \frac{1}{F(0,\frac{\xi}{\sqrt{t}})}-H_0+\frac{H_0^2}{H_0+(t/2)}$$

and consider the auxiliary function

$$\begin{aligned}\Psi(t,\xi) \;&=\; \frac{tH_0^2}{((t/2))^2\left(\frac{1}{F(0,\frac{\xi}{\sqrt{t}})}-H_0+\frac{H_0^2}{(t/2)}\right)}\\ &=\; \frac{4H_0^2}{t\left(\frac{1}{F(0,\frac{\xi}{\sqrt{t}})}-H_0+\frac{2\,H_0^2}{t}\right)}.\end{aligned} \tag{5.61}$$

We perform the following expansion up to the order $o(1/t)$:

$$\frac{1}{F(0,\frac{\xi}{\sqrt{t}})}-H_0 = \frac{\xi}{\sqrt{t}}\cdot\nabla_\xi F^{-1}(0,0)+\frac{1}{2\,t}\sum_{i,j=1}^{d}\xi_i\,\xi_j\,\frac{\partial^2 F^{-1}}{\partial\xi_i\,\partial\xi_j}(0,0)+o\left(\frac{1}{t}\right). \tag{5.62}$$

Since by hypothesis f_{in} is even with respect to p, we have

$$\frac{\partial F}{\partial\xi}(0,0) = -i\int_{\mathbb{R}^d} f_{in}(p)\,p\,dp = 0$$

and then:

$$\frac{\partial F^{-1}}{\partial \xi}(0,0) = -\frac{1}{F(0,0)^2}\frac{\partial F}{\partial \xi}(0,0) = 0. \tag{5.63}$$

We also have

$$\frac{\partial^2 F^{-1}}{\partial \xi_i\, \partial \xi_j}(0,0) = -\int_{\mathbb{R}^d} p_i\, p_j\, f(0,p) dp dm,$$

which with the help of (5.63) implies

$$\begin{aligned} \frac{\partial^2 F^{-1}}{\partial \xi_i \partial \xi_j}(0,0) &= -F^{-2}(0,0)\frac{\partial^2 F}{\partial \xi_i \partial \xi_j}(0,0) \\ &= H^2(0)\int_{\mathbb{R}^d} p_i\, p_j\, f(0,p) dp \quad =: 2\,\mathcal{B}_{i,j}. \end{aligned} \tag{5.64}$$

Since $f_{in}(m,p)$ is even with respect to the p variable we have $B_{i,j} = 0$ whenever $i \neq j$. We denote $\mathcal{B}_{i,i} = \mathcal{B}_i$. Thanks to (5.62), (5.63) and (5.64), we deduce that (5.61) reads now:

$$\Psi(t,\xi) = \frac{4H^2(0)}{\left(\sum_{i=1}^d \mathcal{B}_i\, \xi_i^2\, + 2H^2(0) + o(1)\right)}$$

from where

$$\lim_{t\to+\infty} \Phi(t,\xi) = \lim_{t\to+\infty} \Psi(t,\xi) = \frac{4H^2(0)}{\sum_{i=1}^d \mathcal{B}_i\, \xi_i^2 + 2H^2(0)} =: \Psi_\infty(\xi). \tag{5.65}$$

and (5.65) implies that $\varphi(t,.) \rightharpoonup \varphi_\infty = \mathcal{F}^{-1}(\Psi_\infty)$ in the weak sense of measures, which is nothing but (5.46).

When $d = 1$, we recall from standard integral calculus that for any $C > 0$

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\,p\,\xi}\frac{d\xi}{1 + C\,\xi^2} = \sqrt{\frac{\pi}{2\,C}}\; e^{-\left|\frac{p}{\sqrt{C}}\right|},$$

from where for $C = \mathcal{B}/2H_0^2$ we obtain

$$\varphi_\infty(p) = \mathcal{F}^{-1}\left(\Psi_\infty\right)(p) = 2\sqrt{\frac{\pi}{\mathcal{B}}}\, H_0\, e^{-\left|\frac{\sqrt{2}\,H_0}{\sqrt{\mathcal{B}}}\,p\right|}$$

and (5.50) follows. □

Acknowledgments

M.E. thanks support by Grant MTM2008-03541 from the Spanish Ministerio de Ciencia e Innovación and IT-305-07 from the Basque Government.

References

[1] Bobylev, A.V. , Illner, R.: Collision integrals for attractive potentials. J. Statist. Phys. 95, 633–649 (1999)

[2] Brilliantov, N. V., Bodrova, A. S., Krapivsky, P. L.: A model of ballistic aggregation and fragmentation. J. Stat. Mech. P06011 (2009)

[3] Brilliantov, N. V., Spahn, F.: Dust coagulation in equilibrium molecular gas. Maths. & Comp. in Simulations 72, 93–97 (2006)

[4] Carnevale, G. F., Pomeau, Y., Young, W. R.: Statistics of Ballistic Agglomeration. Phys. Rev. Lett. 64, 2913-2916 (1990)

[5] Doesch, G.: Introduction to the Theory and Application of the Laplace Transformation. Springer-Verlag Berlin Heidelberg New York (1974)

[6] Escobedo, M., Laurençot, Ph., Mischler, S.: On a kinetic equation for coalescing particles. Comm. in Maths. Phys. 246, 237–267 (2004)

[7] Escobedo, M., Mischler, S.: On a Quantum Boltzmann equation for a gas of photons. J. Math. Pures Appl. 80, 471-515 (2001)

[8] Escobedo, M., Mischler, S.: On self-similarity for the coagulation equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 23, 331–362 (2006)

[9] Escobedo, M., Mischler, S., Ricard, M. R.: On self-similarity and stationary problem for fragmentation and coagulation models. Ann. Inst. H. Poincaré Anal. Non Linéaire 22, 99-125 (2005)

[10] Fournier, N., Laurençot, Ph.: Existence of self-similar solutions to Smoluchowski's coagulation equation. Comm. Math. Phys. 256, 589-609 (2005)

[11] Fournier, N., Mischler, S.: A Boltzmann equation for elastic, inelastic, and coalescing collisions. J. Math. Pures Appl. 84, 1173-1234 (2005)

[12] Jiang, Y., Leyvraz, F.: Scaling theory for ballistic aggregation. J. Phys. A: Math. Gen. 26, L179-L186 (1993)

[13] Krapivsky, P. L., Ben-Naim, E.: Aggregation with multiple conservation caws. Phys. Rev. E. 53, 291–298 (1996)

[14] Laurençot, Ph., Mischler, S.: On coalescence equations and related models. In: Modeling and computational methods for kinetic equations, Model. Simul. Sci. Eng. Technol, pp. 321–356. Birkhuser Boston, Boston, MA, (2004)

[15] Lynden-Bell, D.: Statistical mechanics of violent relaxation in stellar systems. Mon. Not. Astr. Soc. 136, 101–121 (1967)

[16] Mischler, S., Mouhot, C.: Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres. Comm. Math. Phys. 3, 287 (2009)

[17] Mischler, S., Wennberg, B.: On the Spatially Homogeneous Boltzmann equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 16, 467-501 (1999)

[18] Norris, J.R.: Smoluchowski's coagulation equation: uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent. Ann. Appl. Probab. 9, 78–109 (1999)

[19] Morfill, G., Röser, S., Tscharnuter, W., Völk, H.: The dynamics of dust in a collapsing protostellar cloud amd its possible role in planet formation. The Moon and the Planets 19, 211-220 (1978)

[20] Roquejoffre, J.M., Villedieu, Ph.: A kinetic model for droplet coalescence in dense sprays. Math. Models Methods Appl. Sci. 11, 867–882 (2001)

[21] Rudin, W.: Functional Analysis. McGraw and Hill, NewYork (1991)

[22] Smoluchowski, M.: Versuch einer mathematischen Theorie der Koagulationskinetik kolloider Lösungen. Zeitschrift f. Physik. Chemie 92, 129–168 (1917)

[23] Spahn, F., Albers, N., Sremčević, M., Thornton, C.: Kinetic description of coagulation and fragmentation in dilute granular particle ensembles. Europhys. Lett. 67, 545–551 (2004)

[24] Trizac, E., Hansen, J. P.: Dynamic scaling behavior of ballistic coalescence. Phys. Rev. Lett. 74, 4114–4117 (1995)

[25] Trizac, E., Krapivsky, P.L.: Correlations in ballistic processes. Phys. Rev. Lett. (2003) doi: 10.1103/PhysRevLett.91.218302

[26] Wetherill, G.: The Formation and Evolution of Planetary Systems. Cambridge University Press, Cambridge (1988)