

Doubts and Equilibria*

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Abstract

In real life strategic interactions decision-makers are likely to entertain doubts about the degree of optimality of their play. To capture this feature of real choice-making, we present here a model based on the doubts felt by an agent about how well is playing a game. The doubts are coupled with (and mutually reinforced by) imperfect discrimination capacity, which we model by means of similarity relations. These cognitive features, together with an adaptive learning process guiding agents' choice behavior leads to doubt-based selection dynamic systems. We introduce the concepts of Mixed Strategy Doubt Equilibria, Mixed Strategy Doubt-Full Equilibria and Mixed Strategy Doubtless Equilibria and show the theoretical and the empirical relevance of these concepts.

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1 Introduction

In real life situations, introspection before a choice is made is pervaded with doubts (or confidence) about how good will be the consequences associated to the each of the available alternatives. Even experienced decision-makers often face doubts in their domain of expertise. Doubts play an important role in decision making. Doubts could be a source of learning, but has attracted little attention in behavioural economics. The present work is an attempt at formalizing the feeling of doubts to become theoretically instrumental.

Doubt, in everyday language, is closely related to the notion of uncertainty. In the **Webster's Encyclopedic Unabridged Dictionary of the English Language**, the entry doubt (verb) is defined as *"to be uncertain in opinion"*. Doubt (noun), on the other hand, is *"a feeling of uncertainty about the truth, reality, or nature of something"*. The **Merriam-Webster Online Dictionary** defines doubt (noun) as *"uncertainty of belief or opinion that often interferes with decision-making"*.

To be more precise, let us relate doubts with the emotion of regret as studied in regret theory. Regret (or rejoice) arises by comparing the resolution of the chosen option with that of the forgone options. Thus, these negative or positive emotions arise from introspection with information feedback. It is assumed that regret can be anticipated and consequently it affects decision-making. Doubts, on the other hand, arise from a more basic assessment. They appear before a decision is taken, when the resolution of the alternatives is still unknown. Hence, in that respect, the feeling of doubts appear from introspection without information feedback. The decision maker does not know whether the decision he is about to make will be good or bad and consequently in his initial assessment doubts arise. We will assume that agents are doubt minimizers, and that they make the effort of distinguishing the alternatives by assigning levels of doubts to each of them. Needless to say that doubts are a subjective emotion, and, in general, the same option could induce different levels of doubts to different agents. Nevertheless, we shall make the following simplifying assumptions: all decisionmakers (i) measure the doubts associated to an option by the proportion of people who have chosen the same option and (ii) assign the same level of doubts to a given option, in other words, if there are n options there will be just n measures of doubts.

It is probably safe to say that strategic environments are more cognitive demanding and raise more doubts than the environment in which one has to choose, say, among simple lotteries (in part because strategic uncertainty is typically of a more ambiguous nature).

Thus, we think that embedding the notion of doubt in a strategic context is likely to be a fruitful enterprise. To this end, we relate doubts with strategic decisions in a dynamic model in which doubtful agents interact frequently. We characterize the long-run outcome of their choice behavior.

Let us think of a game continuously played by two player populations. Each population has a set of possible strategies from which to choose. We may imagine the following two separate, but very close, stages in the introspection process of a typical agent.

First stage: as a member of a population, I feel doubts about the exact consequences of choosing any of the strategies that are available to me, because they depend on the opponent's reactions, which, in most cases, are uncertain.

Second stage: how do I cope with that feeling? Does the environment provide me with information that I might use to construct an acceptable "measure" of my level of doubts?

The first stage depicts the situation in which a subject is directly feeling doubts as a consequence of the decision he is about to make. Let us talk about the second stage. When players make choices continuously, they obtain information about payoffs, but also about the fraction of fellow agents playing each strategy. If an agent observes that many others play a given strategy, it is natural for him to entertain less doubts about whether that strategy is a "good" option.¹ We shall assume that each individual has access to the information about the proportion of agents of his player population playing each strategy. Then each individual will use that information to build a "measure" of the doubts originated by his current strategy. This "measure" will be called the doubt function. Doubts about a strategy which decrease with the number of people using it are the most natural ones² and a great deal of the paper is dedicated to them.

Since the assumption about decreasing doubts is crucial for some of our results, let us discuss it in a bit more detail. Each agent has accumulated over time, as a participant in the game, a certain amount of playing experience, information and knowledge. Furthermore, people have a tendency to trust the intelligence or experience of others, and thus imitate them. Then, if the number of agents who play a strategy increase, it could be taken as the signal of an aggregate improvement in the experience of that strategy's users. This, on the one hand, increases its relative popularity in the eyes of those who are currently playing

¹Economists have understood for a long time that imitation of "common" behavior is a widespread human decision-making strategy (see e.g. Smallwood and Conlisk 1979 or Nelson and Winter 1982).

²Even if they are not the only possible type, as we shall see below.

it. Also, it will call the attention of those who are not using it. Hence, it seems natural to assume that the doubts and fears of an agent about a strategy will decrease with the popularity of the strategy.

An additional feature of doubts, which we discuss in this paper, is that *they give rise to thresholds and so they are closely related to imperfect discrimination capacity* (of real numbers, such as strategy frequencies and expected payoffs).³ *It is known that similarity judgments are part of observed decision procedures (see Tversky (1977), Rubinstein (1988) and Arieli et al. (2009)). Hence, it seems natural to assume that, with the purpose of increasing expected payoffs and decreasing doubts, the agent would build a procedure based on those similarity relations to decide whether to continue playing with the current strategy or switch to a different one (see also Uriarte (1999, 2007)).*

This adjusting behaviour, with which two opposing population of agents choose over time the strategies in a game, gives rise to, what we call, the *doubt-based selection dynamic model*. We explore the long run properties of this selection dynamics for constant-sum 2×2 games with a unique equilibrium in mixed strategies. The rest point of the *doubt-based selection dynamics* is called Mixed Strategy Doubt Equilibrium (MSDE). Some of the results of the paper depend on two limiting cases of doubtful behaviour: the *doubt-less* or *alert mode of play* and the *doubt-full* or *absent mode of play*. In the former case the agent has extremely low level of doubts, and in the latter case, the agent has extremely high level of doubts. In both cases, the agent is not sensitive to the strategy choices of his fellow agents.

We study the relationship between the MSDE and the Mixed Strategy Nash Equilibrium (MSNE) We show that an MSNE is, under some conditions, a rest point for the *doubt-based selection dynamic system*. More specifically, let us assume the situation in which all agents operate under the *doubt-full* or *absent* mode of play (i.e. agents have very high level of doubts). We show that the system converges to population frequencies close to the Mixed Strategy Nash Equilibrium when all agents are in the *doubt-full* mode of play. The following interpretation can be given to this result. Agents are aware that the proportions with which each strategy is being played over time are not truly random. Thus, they experience high levels of doubts out of a fear of being exploited by opponents. The high fear and the doubts together with the adaptive choices lead the system to the Mixed Strategy Nash Equilibrium.

³The work of Kahneman and Tversky has plenty of examples about how the human cognitive system copes with such situations of limited capacity for discrimination. See, for instance, Tversky (1977), Kahneman and Tversky (1979), Kahneman (2003) and the references therein.

Once in equilibrium, payoffs are equalized across strategies but the doubt levels continue to be high and equal across strategies. Thus, we show the equilibrium is an asymptotically stable point for the dynamical system in the *doubt-full* mode of play. We also calculate the values of the doubt parameter that would stabilize the Mixed Strategy Nash Equilibrium of 2x2 games, and illustrate this finding with explicit calculations both for the “Penalty Kick Game” of Palacios-Huerta and Volij (2007)⁴ and the “Matching Pennies Game.”

A Mixed Strategy Nash Equilibrium could also be converted into a rest point of the doubt-based selection dynamics if agents have a very small level doubts (even if they still decrease in the frequency of play). This is the *doubt-less* or *alert* mode of play. Dynamics in this case are different as any perturbation, however small, sends the system away from the equilibrium. An interpretation for the result is that the extreme sensitivity to the “opinions” of others, leads play to a situation where players imitate, whenever doubtful, the current most fashionable action. This creates a tendency to diverge in population behavior. In addition, the doubt-less agents are quite satisfied with their current strategies and do not feel the need to experiment with new strategies to exploit the differences in payoffs and strategy proportions. Hence, a low level of imitation and strategy adjustment takes place, and the populations diverges very slowly to a situation where initially popular strategies dominate.

There are also quite interesting intermediate cases, with strictly decreasing doubts that are less extreme than the previous cases, in between the *doubt-full* and *doubt-less* modes of play. In this case, we find a kind of herding behavior. The equilibrium of the doubt-based dynamic system is not the Nash equilibrium and has the following feature: the most popular strategy has smaller (expected) payoffs. This is a *general* characteristic of equilibria with decreasing doubt functions. But in the *doubt-full* mode of play it is not so evident since the equilibrium is close to being Nash and we have asymptotic stability, whereas in the *doubt-less* mode, we get unstable dynamics. We believe that this feature of equilibria of *doubt-based selection dynamic system* is a relevant and robust testable implication for our model, and we provide some preliminary evidence to support it.

Finally, we should mention as well the case of *constant* doubts. This means that each agent’s hesitations and feelings of uncertainty are not affected by the fraction of fellow

⁴This interesting paper shows how professional football (soccer) players transfer the skills learnt in the field to the artificial setting of a laboratory and yet play close to the mixed strategy Nash Equilibrium.

agents from his population playing the same strategy. Thus, society does not have any direct influence on this type of agent. Then we show that the adjusting behavior would lead us to a doubt-based selection dynamics that is closely related to the replicator dynamics.⁵

How does this work relate with the existing literature? Given the nature of the paper, we think that it is in the realm of the experimental literature that we should look into. Particularly, the experiments where subjects are given information about the performance of the other participants. In Tang (2001), for instance, the participants in the experiment are given a precise information about the proportion of subjects playing each strategy as well as the average payoffs in the two player populations. This experiment contradicts one of our result, the one that says that the most popular strategy has smaller (expected) payoffs. To our defense, it should be said that it is not very realistic to provide such a precise information, -which, in fact, is only known by the experiment maker-, to subjects who are involved in the experiment. In fact, this kind of information would eliminate the "doubts" that the involved subjects might feel, a feature that plays a central role in the build-up of our model. *Our view is that experiments should be designed to capture decision schemes actually used by people. Thus, we think it is more realistic to assume doubtful agents who use similarity relations to compare expected payoffs and strategy frequencies separately and decide the strategy choice behaviour.*

In Binmore et al. (2001) a subject can compare his performance with the other subjects in the same population by seeing their median payoff. While holding the same critical stand as in the previous case, and taking into account that the information now is about payoffs, we note that the spiral trajectory converging to equilibrium that these authors observe in the experiment has a theoretical counterpart in the present paper with the *doubt-full* case, where the path to equilibrium is shown to be a spiral (sink).

Given the above limitations, we have looked for data coming from field experiment and provide a supportive piece of evidence for our (doubt) equilibrium condition.

To conclude, we think that this paper, by insisting on doubts related with imperfect perception, highlights the need of more evidence from fuzzier, that is, more realistic, experimental environments. Thus, our next step should be to test the theory presented in this paper.

⁵This result is yet another rationalization for the replicator dynamics. Other foundations for this dynamical system can be found in Binmore, Gale and Samuelson (1995), Weibull (1995), Cabrales (2000) and Schlag (1998), among others.

2 A model of doubt-based selection dynamics

2.1 Notation

Consider a noncooperative finite game G in normal form, with $K = \{1, 2, \dots, n\}$ denoting the set of players. For each player $k \in K$, let $S_k = \{1, 2, \dots, m_k\}$ be her finite set of pure strategies, for some integer $m_k \geq 2$.

Imagine that there exist n large populations, one for each of the n player positions in the game. Members of the n populations chosen at random -one member from each player population- are repeatedly matched to play the game. In what follows, we shall speak of *players* when referring to the game G and we shall speak of *agents* when referring to the members of the populations. Each agent is characterized by a pure strategy. From now on, we shall refer to the agent ki as a member of the player population $k \in K$ who plays pure strategy $i \in S_k$. Let $f_{ki}(t) \in F_{ki} = [0, 1]$ be the relative frequency of ki agents at time t , with $f(t)$ being the vector collecting such probabilities. Time index suppressed, $\pi_{ki}(f)$ will denote agent ki 's expected payoff given the population state f . Without loss of generality, we may assume that payoffs are strictly positive and do not exceed one; hence, $\pi_{ki}(f) \in \Pi_{ki} = (0, 1]$. Finally, $\bar{\pi}_k(f) = \sum_{i=1}^{m_k} f_{ki}(f) \pi_{ki}(f)$ is the average payoff in player population $k \in K$. To simplify notation, we shall denote $\pi_{ki}(f)$ as π_{ki} .

2.2 The doubt-based selection dynamics

The doubt function We assume the game is played by boundedly rational players who have doubts about how well they are playing. More precisely, every agent of each player population is endowed with a (primitive) function that we call the “doubt function”. This function, denoted d_{ki} , measures the doubts felt by agent ki about how good is his current strategy $i \in S_k$, available to player population $k \in K = \{1, 2, \dots, n\}$, as a response to the strategies that the remaining players are using. Agent ki relates his doubts to the proportion of individuals who are using his current strategy. We shall assume in this section that the agents are endowed with a strictly decreasing doubt function. That is, an agent's doubts about how well is playing gradually decrease when he observes (or is informed of) a larger number of agents from his player population playing the same strategy as the one he is currently using. The underlying logic of this assumption is a belief on the part of agents

about the collective “wisdom of crowds,” combined with the cognitive ease of trusting others relative to thinking through the decision problem.⁶ Formally,

The strictly decreasing doubt functions.

Each agent ki is endowed with a differentiable doubt function d_{ki} in the set

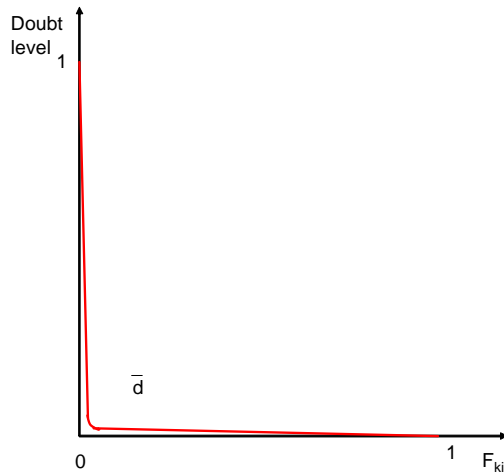
$$D = \left\{ \begin{array}{l} d_{ki} : F_{ki} \rightarrow [0, 1] : \text{with } \hat{f}_{ki} > \tilde{f}_{ki} \Rightarrow d_{ki}(\hat{f}_{ki}) < d_{ki}(\tilde{f}_{ki}) \\ \text{and } d_{ki}(0) = 1, d_{ki}(1) = 0 \end{array} \right\}$$

Given a proportion $f_{ki} \in F_{ki}$, -known by the ki agent-, and the $d_{ki} \in D$, $d_{ki}(f_{ki})$ measures the doubts (about how well is playing the game) felt by the agent ki when the proportion of agents in player population k playing strategy $i \in S_k$ at time t is f_{ki} .

Remark 1 We highlight the following two types of doubt functions in D , which are relevant for the results of section 3:

1. Function $d^\delta \in D$ which, for every $f_{ki} \in (0, 1)$, $d^\delta(f_{ki})$ is “arbitrarily close” to 0 (i.e., $d^\delta(f_{ki}) < \delta$ for all $f_{ki} > \delta > 0$, and δ an arbitrarily small number), as in figure 1.
2. Function $d^{1-\delta} \in D$ which, for every $f_{ki} \in (0, 1)$, $d^{1-\delta}(f_{ki})$ is “arbitrarily close” to 1 (i.e., $d^{1-\delta}(f_{ki}) > 1 - \delta$ for all $f_{ki} < 1 - \delta$, and $\delta > 0$ an arbitrarily small number), as in figure 2.

When $d_{ki} = d^\delta$, for an arbitrarily small $\delta > 0$, we say that the agent ki is in the doubt-less mode and when $d_{ki} = d^{1-\delta}$, for an arbitrarily small $\delta > 0$, we say that the agent is in the doubt-full mode.



⁶A similar argument is made in Smallwood and Conlisk (1979).

Figure 1: Doubt function of an agent in the doubt-less mode of play. His doubts are almost 0 in the interval $(0, 1)$.

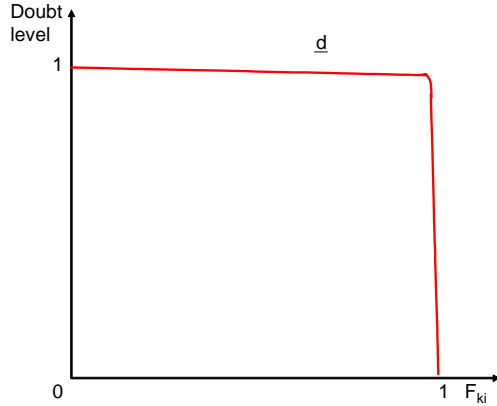


Figure 2: Doubt function of an agent in the *doubt-full* mode. His doubts are almost 1 in the interval $(0, 1)$.

An Index for dissatisfied agents Our adaptive agents are current users of some strategy and, very likely, past and future user of some others. Inside a player population its members are likely to share their experience and information about the game. This naturally leads to imitation processes which give rise to observational learning, herding and other forms of convergent behavior. Thus, it is reasonable to assume that adaptive agents have access to information about the relative popularity of each strategy available to them, as well as to their payoffs. We want to build a reasonable model about how these agents react to changes in those relevant variables. Let

$$\alpha_{ki} = \alpha_{ki}(\pi_{ki}, f_{ki}, \pi_{kj}, f_{kj}), i \neq j$$

denote the proportion of ki strategists who feel dissatisfied with strategy i at time t . We require the following reasonable properties (*ceteris paribus*) for α_{ki} :

1. The proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should decrease if expected payoffs to strategy $i \in S_k$, π_{ki} , increase.
2. If agents ki 's doubts decrease, because the popularity of strategy i , f_{ki} , has increased, the proportion of dissatisfied should decrease too.

3. The proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should increase if expected payoffs to strategy $j \in S_k$, π_{kj} , increase.
4. If the popularity of strategy $j \in S_k$, f_{kj} , increases, the proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should increase.

In the remainder we use a particular function which has these properties. In the Appendix we justify and microfound the choice of this function via a model of (correlated) similarities relations⁷

Let then

$$\alpha_{ki} = \frac{\lambda_{ki} - 1}{\sum_{i=1}^{m_k} \lambda_{ki}} = \frac{\lambda_{ki} - 1}{\lambda_k}$$

where the function $\lambda_{ki}(\pi_{ki}, f_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})}$, defined for $\pi_{ki} > d_{ki}(f_{ki})$, is used to build correlated similarities on the frequency space F_{ki} (a more detailed account of α_{ki} and λ_{ki} is given in Appendix A).

The Doubt-Based Selection Dynamics We assume that time is divided into discrete periods of length τ . In every period, $1 - \tau$ is the probability that the agent does retain his current strategy; thus, τ is the probability that each agent does not retain his current strategy. We make now the following assumption to build a selection dynamic model⁸

When an agent feels dissatisfied with his current strategy, she will choose a new strategy with a probability that is equal to the proportion of agents playing that strategy.

>From Assumption 2.2, $\tau \frac{(\lambda_{ki}-1)}{\lambda_k} f_{ki}$ will denote the proportion of ki strategists who will choose a new strategy (the *outflow*), and, since a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then $\tau \sum_{j=1}^{m_k} \frac{(\lambda_{kj}-1)}{\lambda_k} f_{kj} f_{ki} = \tau \frac{(\bar{\lambda}_k-1)}{\lambda_k} f_{ki}$ is the proportion of agents who will choose strategy i (the *inflow*), where $\bar{\lambda}_k = \sum_{j=1}^{m_k} \lambda_{kj} f_{kj}$.

Therefore,

$$f_{ki}(t + \tau) = f_{ki}(t) - \tau \frac{(\lambda_{ki} - 1)}{\lambda_k} f_{ki} + \tau \frac{(\bar{\lambda}_k - 1)}{\lambda_k} f_{ki}$$

As $\tau \rightarrow 0$, in the limit we get the *doubt-based* selection dynamic equation:

⁷For the definition of similarity relation, see Rubinstein (1988).

⁸For a justification see, for example, Binmore et al. (1995).

$$\dot{f}_{ki} = f_{ki} \left[\frac{\bar{\lambda}_k - \lambda_{ki}}{\lambda_k} \right] \quad (1)$$

To gain some intuition, let us now look at equation (1) in a less compact way. Let G be a two-population constant-sum game with $S_I = \{U, D\}$ and $S_{II} = \{L, R\}$ denoting player I and player II's strategy sets, respectively. Let x denote the probability of playing U , y the probability of playing L and $I = [(x^*, 1 - x^*), (y^*, 1 - y^*)]$ the Mixed Strategy Nash Equilibrium, with $x^* > 0$ and $y^* > 0$.

To avoid the use of four different doubt parameters, we shall assume that the four doubt functions are the same; that is, $d_{ki} = d \in D$ (where $k = I, II$ and $i = U, D, L, R$).

>From (1), the *doubt-based* selection dynamics for G is represented by the following system:

$$\dot{x} = \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)} (\pi_U d_D - \pi_D d_U) \equiv G_1(x, y) F_1(x, y) \quad (2)$$

$$\dot{y} = \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)} (\pi_L d_R - \pi_R d_L) \equiv G_2(x, y) F_2(x, y) \quad (3)$$

Clearly, a stationary point for the *doubt-based* system (2)-(3), with $x^* > 0$ and $y^* > 0$, requires $\pi_U d_D = \pi_D d_U$ and $\pi_L d_R = \pi_R d_L$. We call this point the Mixed Strategy Doubt Equilibrium (MSDE).

2.2.1 Mixed Strategy Nash Equilibrium (MSNE) and Mixed Strategy Doubt Equilibrium (MSDE)

We should distinguish between the Mixed Strategy Nash Equilibrium (MSNE) and the Mixed Strategy Doubt Equilibrium (MSDE) for the doubt-based dynamic system (1).

1. In a MSNE the requirement is that all strategies in the support of the equilibrium have equal payoffs; that is:

$$\pi_{ki}(f^*) = \pi_{kj}(f^*) \text{ for all } i, j \text{ with } f_i^* > 0 \text{ and } f_j^* > 0 \text{ and all } k.$$

2. From (1) we deduce that for a MSDE the requirement is (recall the assumption $d_{ki} = d \in D$):

$$\frac{\pi_{ki}(f^*)}{d(f_i^*)} = \frac{\pi_{kj}(f^*)}{d(f_j^*)} \text{ for all } i, j \text{ with } f_i^* > 0 \text{ and } f_j^* > 0 \text{ and all } k$$

Note that in this case, the expected payoffs to the strategies in the support of the equilibrium need not be equal, as it is required in the MSNE. We have the following result:

Proposition 1 *Suppose that agents are endowed with doubt functions $d_{ki} = d \in D$. Then for all k and all i, j , with $0 < f_{kj}^* < f_{ki}^* < 1$, since the doubt functions are strictly decreasing, $d(f_{ki}^*) < d(f_{kj}^*)$, and thus, in order to satisfy the Mixed Strategy Doubt Equilibrium condition, we must have $\pi_{ki}(f^*) < \pi_{kj}(f^*)$.*

Proof: Direct from the Mixed Strategy Doubt Equilibrium (MSDE) condition.

In words, the more frequent strategies in a MSDE should have lower expected payoffs. This situation is clearly distinct from a Nash equilibrium and is a general feature of the (decreasing) doubt-based dynamic system. In section 5 we shall explore the empirical justification of this result.

3 Doubt-based selection dynamics in constant sum games

In this section we shall explore the relationship between the MSNE and MSDE for different levels of doubts.

3.1 Relationship between a MSNE and a MSDE

Let us recall what game theorists say about a MSNE:

“The point of randomizing is to keep the other player(s) just indifferent between the strategies that the other player is randomizing among. One randomizes to keep one’s rivals guessing and not because of any direct benefit to oneself.” (Kreps 1990, p 408).

The doubt-based model is able to capture that state of players’ mutual guessing that characterizes a MSNE. Assume that we are dealing with 2×2 constant sum games having a unique mixed equilibrium with full support. Consider Player I; how would this player interpret different values of (his own probability) x , say 0.2 and 0.6? A rational Player I knows that Player II is randomizing to keep him indifferent between the strategies he is randomizing among. Therefore, in terms of our model of doubts, $x = 0.2$ and $x = 0.6$

would induce in the Player I's rational mind the same level of doubts as to which is the best probability distribution, because both of them get the same expected payoff. But, for the same reason, Player I's equilibrium strategy in the game will induce the same level of doubts as 0.2 or 0.6. In other words, Player I does not see, in a preference sense, any real difference between different probability distributions in the open unit interval $(0,1)$. As a consequence, he will have (nearly) equal level of doubts at any x in $(0,1)$. The same will happen to Player II.

The above suggests that we should ask first, *which are the level of doubts embedded in the players' mutual guessing that characterizes steady states very close to the MSNE*. This is answered in Proposition 2 below, where we show that, if all agents are playing in the *doubt-full mode*, any interior MSNE coincides with an MSDE ; that is, an MSNE is a Mixed Strategy Doubt-Full Equilibrium (MSDFE). We also show that any interior point of the simplex can be a Mixed Strategy Doubt-Less Equilibrium (MSDLE) for some kind of doubt-less behavior.

The second issue to deal with is the following: how is the MSNE reached? or, which is the equilibrating process that may lead to the MSNE? This will be answered in Propositions 4 and 5 below.

Let G be a two-population, two-strategy, constant-sum game with $I = [(x^*, 1 - x^*), (y^*, 1 - y^*)]$, $x^* > 0$, $y^* > 0$, denoting its MSNE.

Proposition 2

1. The (Euclidean) distance between an MSDE and MSNE converges to zero as δ goes to zero if every agent plays with a doubt function in the $D^{1-\delta}$ class; that is, if they play in a *doubt-full mode*. Hence, an MSNE $(x^*, y^*) \in (0, 1) \times (0, 1)$ is an MSDFE.
2. For any interior point of the simplex $A = [(x', 1 - x'), (y', 1 - y')]$ (i.e. with $0 < x' < 1$ and $0 < y' < 1$) there is a sequence of functions $d^\delta \in D^\delta$ such that the (Euclidean) distance between an MSDE and A converges to zero as δ goes to zero. That is, if they play in a *doubt-less mode*, any interior point of the simplex can be a MSDLE for some kind of doubt-less behavior.

Proof: See appendix C

A straightforward corollary of Proposition 2 is the following:

Corollary 3 *Let $C = [(1/2, 1/2), (1/2, 1/2)]$ and d any doubt function in D . Then, if C is the MSNE of G , it is also an MSDE (i.e. a rest point of the doubt-based dynamic system (2)-(3)).*

This means that if $C = [(x^*, 1 - x^*), (y^*, 1 - y^*)] = [(1/2, 1/2), (1/2, 1/2)]$ is the MSNE of G , then it is compatible (in the sense of Proposition 2) with agents playing in any of the two modes of play, *doubt-full* or *doubt-less*, as well as with agents endowed with any doubt function in D between those two extreme doubt functions.

Even though every $d_{ki} \in D$ is strictly decreasing, the exact values of the mixed strategy equilibrium, (x^*, y^*) , with x^* and $y^* > 0$, do not matter since every ki agent is endowed either with a doubt function in the absent mode or in the alert mode. In particular, this means that Proposition 2 does not impose any restriction on the equilibrium values that f_{ki} might take nor it does relate those probability values with their corresponding expected payoff values, π_{ki} , $k = I, II$ and $i = 1, 2$, in a particular manner.

3.2 Learning to Play a Mixed Strategy Nash Equilibrium (MSNE)

We have seen that an MSNE and an MSDE satisfy different equilibrium properties and therefore, in general, they do not coincide. However, from Proposition 2, we know that an MSNE could be converted into an MSDE when all agents are either in the *doubt-full* mode or in the *doubt-less* mode. In other words, an MSNE could be converted into a rest point of the doubt-based dynamic system (2)-(3). Hence, now we are ready to answer the question: how do the boundedly rational player populations learn to coordinate in the MSNE? Proposition 4, below, shows that an introspective element, such as doubts, could be crucial for learning to play optimally.

We know that a fully rational player must avoid being guessed by the opponents and that to achieve this he will behave in such a way so as to create a random sequence of choices. This suggests that a *doubt-less* mode of playing -that implies almost no strategy switching behavior- would be far from being an adjusting process leading to the Nash equilibrium, and indeed for most MSNE that is the case as proposition 2 shows. It seems that, in an equilibrating process, what makes more sense is that players should behave in the *doubt-full* mode. In our deterministic dynamic model, permanent *doubt-full* agents will have a tendency to keep trying new strategies and, thus, generating not a truly random sequences of choices, but individual processes of trial-and-error adjustments which could find their way

to the MSNE. In Proposition 4 below we show that this is the case: if every agent behaves as if he were constantly with a high level of doubts, the agents' adjusting behavior would lead them to the MSNE and endow the equilibrium with a strong stability property. Proposition 5 shows that the *doubt-less* mode of play has just the opposite consequence.

Proposition 4 *Let G be a two-population, two-strategy, constant-sum game with $I^* \equiv [(x^*, 1 - x^*), (y^*, 1 - y^*)]$, $x^* > 0$ and $y^* > 0$, denoting its MSNE. Then a point close to I^* is asymptotically stable for the doubt-based dynamic system (2)-(3) if every agent plays in the doubt-full or absent mode of play.*

Proof: See appendix C

Proposition 5 *Let G be a two-population, two-strategy, constant-sum game with $I^* \equiv [(x^*, 1 - x^*), (y^*, 1 - y^*)]$, $x^* > 0$ and $y^* > 0$, denoting its MSNE. If every agent is in the doubt-less or alert mode of play and the initial conditions of the doubt-based dynamic system (2)-(3) are different from $[(1/2, 1/2), (1/2, 1/2)]$, the MSDLE is a source.*

Proof: See appendix C

One may then ask about why the modes of play of Proposition 4 and 5 would arise. Needless to say, doubts are a subjective feeling and hence it is difficult to ascertain the precise reason why they may arise in each particular case. Proposition 4 suggests that the origin of high level of doubts lies in the fact that every agent seems to be aware that the proportion with which each available strategy is being played and the sequence that the agents, as a player population, are producing is not random. Thus, the high levels of doubts felt by every member of each player population would arise from the fear of being guessed and exploited by the opponent. As a consequence, since agents are very unhappy with their current strategies (measured by a very high valued α_{ki}) a high proportion of agents will experiment with new strategies in the next period. The fear and the doubts of the agents will continue to be high and, joint with the choices that exploit the variations both in the payoffs and in the strategy proportions, the adjusting behavior would lead the system to the Mixed Strategy Nash Equilibrium. Once in the equilibrium, payoffs are equalized across strategies and the doubt levels continue to be very high and equal across strategies too. Thus, the doubt-full mode of play endow the MSNE with strong stability properties

Proposition 5 suggests that agents seem to be too confident and satisfied with the pure strategies they are currently playing (they have a very low valued α_{ki}). With almost no doubts, they would just produce small strategy choice changes, not taking care of the randomness of their sequences. Thus, imitation is almost non-existent and the resulting dynamics is not sensitive enough to payoff and strategy proportion changes, however small. These features would explain why the dynamics do not converge to equilibrium from any initial point in the state space different from the equilibrium itself.

4 Examples

In the numerical examples, we shall assume, without loss of generality, that the doubt function takes the following form: $d_{ki}(f_{ki}) = (1 - f_{ki})^\alpha$. Assuming that $\alpha \in (0, \infty)$, we would obtain a large enough subclass of doubt functions in the set D . The convex combinations of elements in this class belong to D as well. Note, in particular, that this class contains the two extreme types of doubt functions introduced in Remark 1: when α is very small, near zero, the doubt parameter characterizing agent ki , denoted as $\delta = \frac{1}{\alpha}$, is very high for any $f_{ki} \in (0, 1)$. Then the function will have a graph looking like the one of figure 2, and we shall say that the agent is in the absent or *doubt-full mode of play*. When α is very high, the graph of d_{ki} is close to the axes, as in figure 1, and so the doubt parameter, $\Omega = \frac{1}{\alpha}$, is very small, for any $f_{ki} \in (0, 1)$. This is the agent in the alert or *doubt-less mode of play*.

As in Binmore et al. (1995), we approach equation

$$\dot{f}_{ki} = f_{ki} \left[\frac{\bar{\lambda}_k - \lambda_{ki}}{\lambda_k} \right] \quad (4)$$

by means of the equation

$$f_{ki}(t + \tau) - f_{ki}(t) = \tau f_{ki} \left[\frac{\bar{\lambda}_k(t) - \lambda_{ki}(t)}{\lambda_k(t)} \right] \quad (5)$$

where the step size $\tau = 0.01$. We shall consider, like Binmore et al.(1995), that the system has converged on a point when the first 15 decimals are unchanging.

When doing the numerical examples, we can avoid the problems caused by the degenerate similarity relation to the dynamic system (2)-(3), when $\pi_{ki} \leq d_{ki}(f_{ki})$, - because the agents are

in the *doubt-full mode*-, by adding a constant $c > 1$ to both the numerator and denominator of λ_{ki} . Then the system will be:

$$\begin{aligned}\dot{x} &= \frac{x(1-x)}{(\pi_U + c)(\pi_D + c - d_D) + (\pi_D + c)(\pi_U + c - d_U)} [(\pi_U + c)d_D - (\pi_D + c)d_U] \\ \dot{y} &= \frac{y(1-y)}{(\pi_L + c)(\pi_R + c - d_R) + (\pi_R + c)(\pi_L + c - d_L)} [(\pi_L + c)d - (\pi_R + c)d_L]\end{aligned}$$

Example **The Penalty Kick Game**

Palacios-Huerta (2003) found that the equilibrium theory predictions are observed in the professional players' behavior: (i) their choices follow a random process and (ii) that the probability that a goal will be scored must be the same across each player's strategies and equal to the equilibrium scoring probability (that is, in the Mixed Strategy Nash Equilibrium each player is indifferent among the available strategies). Palacios-Huerta and Volij (2007) extend this result by observing that professional players are capable of transferring their skills from the field to the laboratory, a completely unknown setting for them, and yet behave in a way that is significantly near the Nash equilibrium.

Palacios-Huerta and Volij (2007), from a sample of 2,717 penalty kicks collected from European first division football (soccer) leagues during the period 1995-2004, built the following two player (Player I: the kicker and Player II: goal keeper) two strategy (Left, Right) game.

	(y) L	R
(x)L	0.60, 0.40	0.95 , 0.05
R	0.90, 0.10	0.70, 0.30

where $\pi_I(i, j)$ denotes the kicker's probability of scoring when he chooses i and the goalkeeper chooses j , for $i, j \in \{L, R\}$. The Mixed Strategy Nash Equilibrium of this game is: $x^* = 0.36364, y^* = 0.45455$.

Football matches are continuously played and players' game is based on the study of the opponents in the field and watching their play on TV and videotapes, so that their behavior in the penalty kicks is collected and analyzed. Thus, there is a history of play of each player and, hence, an interactive learning process. Thus, a natural issue is to investigate the type of dynamic process that may lead to the result found by Palacios-Huerta (2003). The *doubt-based* model seems to be a suitable model for this task.

The *doubt-based* selection dynamic system (2)-(3) corresponding to this game is the following:

$$\begin{aligned}\dot{x} &= \frac{x(1-x)((0.95-0.35y)x^\alpha - (0.2y+0.7)(1-x)^\alpha)}{2(0.95-0.35y)(0.2y+0.7) - (0.95-0.35y)x^\alpha - (0.2y+0.7)(1-x)^\alpha} \\ \dot{y} &= \frac{y(1-y)((0.1+0.3x)y^\alpha - (0.3-0.25x)(1-y)^\alpha)}{2(0.1+0.3x)(0.3-0.25x) - (0.1+0.3x)y^\alpha - (0.3-0.25x)(1-y)^\alpha}\end{aligned}$$

The vector field defining (2)-(3) is

$$F(x, y) = \left(\frac{x(1-x)((0.95-0.35y)x^\alpha - (0.2y+0.7)(1-x)^\alpha)}{2(0.95-0.35y)(0.2y+0.7) - (0.95-0.35y)x^\alpha - (0.2y+0.7)(1-x)^\alpha}, \frac{y(1-y)((0.1+0.3x)y^\alpha - (0.3-0.25x)(1-y)^\alpha)}{2(0.1+0.3x)(0.3-0.25x) - (0.1+0.3x)y^\alpha - (0.3-0.25x)(1-y)^\alpha} \right)$$

We compute first the derivative $DF(x, y)$ and then evaluate $DF(x, y)$ at $(0.363\ 64, 0.454\ 55)$ to get the following Jacobian matrix:

$$DF(0.363\ 64, 0.454\ 55) = \begin{bmatrix} \frac{\alpha}{1.581\ 8 - 2 \times 0.363\ 64^\alpha} & \frac{0.146\ 29(-0.2 \times 0.636\ 36^\alpha - 0.35 \times 0.363\ 64^\alpha)}{0.790\ 91 - 0.363\ 64^\alpha} \\ 0.592\ 88 \frac{0.25 \times 0.545\ 45^\alpha + 0.3 \times 0.454\ 55^\alpha}{0.209\ 09 - 0.454\ 55^\alpha} & \frac{\alpha}{0.418\ 18 - 2 \times 0.454\ 55^\alpha} \end{bmatrix}$$

It is easy to see that for values of $\alpha \in (0, 0.231\ 88)$, all the eigenvalues of $DF(0.363\ 64, 0.454\ 55)$ have negative real parts and the associated determinants are all positive. Thus, the equilibrium $(0.363\ 64, 0.454\ 55)$ is a *spiral sink*, for those values of α , and, therefore, it is asymptotically stable. This means that the doubt functions of professional football (soccer) players are in a set which includes one having a graph looking, approximately, like the one of figure 4. The latter would correspond to the player whose performance shows fewer level of doubts, $\delta = 1/0.231\ 88 = 4.312\ 6$, for any frequency level in $(0, 1)$.

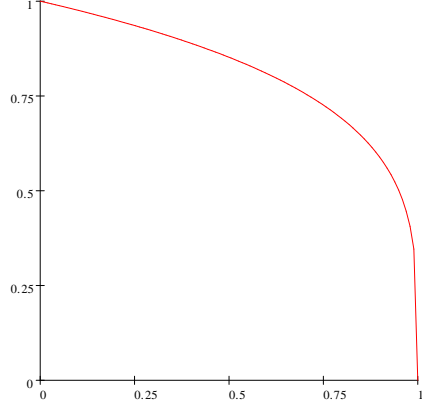


Figure 4. The graph of the doubt function $d(f_{ki}) = (1 - f_{ki})^{0.231}$. The horizontal axis measures the proportion f_{ki} of agents in population k playing the pure strategy i . The vertical axis measures the doubt level associated to each f_{ki} .

Example 1 *The Matching Pennies Game*

	(y) L	R
(x) U	1, 0.5	0.5, 1
D	0.5, 1	1, 0.5

The Mixed Strategy Nash equilibrium of this game is $(1/2, 1/2)$, and the doubt-based system (2)-(3) corresponding to the game is the following:

$$\begin{aligned} \dot{x} &= \frac{x(1-x)(0.5(1+y)x^\alpha - (1-0.5y)(1-x)^\alpha)}{2(0.5 + 0.25y - 0.25y^2) - 0.5(1+y)x^\alpha - (1-0.5y)(1-x)^\alpha} \\ \dot{y} &= \frac{y(1-y)((1.0 - 0.5x)y^\alpha - (0.5x + 0.5)(1-y)^\alpha)}{2(0.5 + 0.25x - 0.25x^2) - (1.0 - 0.5x)y^\alpha - (0.5x + 0.5)(1-y)^\alpha} \end{aligned}$$

We show now the conditions that makes $(1/2, 1/2)$ asymptotically stable in the above system. More specifically, we show that $(1/2, 1/2)$ is a *spiral sink*.

The vector field defining (8)-(9) is

$$F(x, y) = \left(\frac{x(1-x)(0.5(1+y)x^\alpha - (1-0.5y)(1-x)^\alpha)}{2(0.5 + 0.25y - 0.25y^2) - 0.5(1+y)x^\alpha - (1-0.5y)(1-x)^\alpha}, \frac{y(1-y)((1.0 - 0.5x)y^\alpha - (0.5x + 0.5)(1-y)^\alpha)}{2(0.5 + 0.25x - 0.25x^2) - (1.0 - 0.5x)y^\alpha - (0.5x + 0.5)(1-y)^\alpha} \right)$$

We compute first the derivative $DF(x, y)$ and then evaluate $DF(x, y)$ at $(1/2, 1/2)$ to get the following matrix:

$$DF(1/2, 1/2) = \begin{bmatrix} \frac{\alpha}{1.5-2 \times 0.5^\alpha} & \frac{0.16667}{0.75-0.5^\alpha} 0.5^\alpha \\ -\frac{0.16667}{0.75-0.5^\alpha} 0.5^\alpha & \frac{\alpha}{1.5-2 \times 0.5^\alpha} \end{bmatrix}$$

We see that the elements $j_{ij}(\alpha)$ of the Jacobian matrix are three functions whose signs depend on the value of the parameter α . Furthermore, these functions are all multiplied by $\frac{1}{0.75-0.5^\alpha}$, and $0.75 - 0.5^\alpha = 0$ when $\alpha = 0.41504$. Then it is easy to see that only for values of α in $(0, 0.41504)$ all the eigenvalues of the matrix $DF(1/2, 1/2)$ have negative real parts. As in the previous example, the equilibrium $(1/2, 1/2)$ is a *spiral sink*. ■

5 Testable implications of doubt-based selection dynamics

Recall that in a Mixed Strategy Doubt Equilibrium (MSDE), the requirement is that for all i, j with $f_{ki}^* > 0$ and $f_{kj}^* > 0$,

$$\frac{\pi_{ki}(f^*)}{d(f_{ki}^*)} = \frac{\pi_{kj}(f^*)}{d(f_{kj}^*)}$$

Notice that this condition applies as well as to a pure decision problem than to a non-trivial game situation. So a supportive piece of evidence for our equilibrium condition could come from consumer choice situations. Suppose that several brands of a product are sold (say automobiles). For a particular category of product (a family sedan, a pickup truck), sufficiently narrowly defined *so that no horizontal or vertical differentiation of quality is possible*, the presence of multiple brands suggests according to standard theory that the consumer should be (close to) indifferent between them (*in our language* $\pi_{ki}(f^*) = \pi_{kj}(f^*)$). Our model, on the other hand, suggests that the quality is lower for brands with higher sales/market share. In our words, when $f_{ki}^* > f_{kj}^*$ we should observe $\pi_{ki}(f^*) < \pi_{kj}(f^*)$. Table 1, compiles statistics of mechanical troubles of cars compiled by the German Automobile Club for 2002 (measured by the number of calls for towing-and-repairing to the Club per thousand vehicles of that kind sold that year), as well as sales in February 2007. It is interesting to note that for the three best kinds of car in all categories, there is a significant correlation between sales of a model and mechanical troubles (a correlation coefficient of

0.65).⁹ This is, of course, far from a proof of our result. The overall correlation coefficient is of rather uncertain sign,¹⁰ but we suspect this is not a stable situation and the “worst” cars will eventually exit the market. But it is strongly suggestive and it points to an interesting testable implication from our model.

Our conclusions could also be tested in the experimental laboratory. However, subjects in experiments usually do not have information about the proportion of people using each strategy. For example, the only experiment from those surveyed in chapter 3 of Camerer (2003) in which agents are given that information is the one carried out by Tang (2001). In that experiment, participants are given a precise information about the proportion of subjects playing each strategy as well as the average payoffs in the two player populations, and, contrary to our predictions, the most frequently played strategies have a higher ex-post average payoff¹¹. We suspect, though, that the highly precise (and, we would argue, unnatural) form of the feedback given to subjects,- see Tang’s description of the experimental procedure -, eliminates the “doubt” considerations that are important in the build-up of our model.

In real choice situations, an agent does make calculations about what would be the expected payoffs to his current strategy and spend time and effort in gathering information about its popularity. It would be unrealistic to assume that the agents get the correct numbers. But, either with accurate or inaccurate numbers, we think it is more realistic to assume that agents use similarity relations to compare the expected payoffs and the strategy frequencies separately. It is known that similarity judgments are part of observed decision procedures (see Tversky (1977), Rubinstein (1988) and Arieli et al. (2009)). Our view is that experiments should be designed to capture the decision schemes that are actually used by subjects. Doubts and the derived similarity relations allow to introduce a procedure used by agents to build a preference relation between vectors of expected payoffs-strategy frequencies. This preference relation will tell the agent the degree of satisfaction with his current strategy and thus determine his adaptive behaviour.

We believe that more evidence, and hopefully, from “fuzzier” (more realistic) environments would be useful to confront some predictions made in this work. Hence, a translation

⁹The same computation by category gives a number in excess of 0.75 for each one.

¹⁰And there are, of course, lots of omitted important variables

¹¹Curiously enough, in the experiment of Tang (2001) only about a fourth of the subjects participating in that experiment used repeatedly this information on frequencies of play.

Table 1: Quality and sales by category. Source: ADAC 2002

Model	Mechanical problems	Sales
Small cars		
Fiat Punto	34,5	1176
Renault Clio	32,7	1506
Seat Ibiza/ Cordoba	28,3	1399
Opel Corsa	17,9	4983
VW Polo	16,2	4437
Ford Fiesta	15,9	2906
Medium-sized cars		
Renault Mégane	46,2	1508
Ford Escort	27,8	2933
Opel Astra	16,7	5207
VW Golf/Vento/Bora	16,2	11072
Audi A3/S3	15,1	4052
Toyota Corolla	9,8	2236
Large cars		
Volvo S40/V40	27,7	811
BMW 3	17,8	6043
Mercedes C	17,5	2848
VW Passat	16,1	7500
Audi A4/S4	14,0	5267
Mazda 626	10,2	754
Toyota Carina/Avensis	7,6	1025

of our theoretical model into an experimental design should be our next task.

6 Constant doubt-based selection dynamics

The individual choice model that we are going to use in this section is derived from a choice procedure introduced by Aizpurúa, Ichiishi, Nieto and Uriarte (1993), (referred to as AINU from now on), in the space of simple lotteries. We consider now the case when the level of doubts felt is constant, for any value of $f_{ki} \in F_{ki}$. This means that society has no influence upon the doubt level of the agents. Formally,

The Constant Doubt Function. For all $k \in K$, $i \in S_k$ and $f_{ki} \in F_{ki}$, the function $d_{ki} : F_{ki} \rightarrow [0, 1]$ is constant; i.e.

$$d_{ki}(f_{ki}) = \epsilon_k \in (0, 1)$$

We assume that the constant level of doubts ϵ_k felt by agent ki induces *threshold levels* in both expected payoffs and strategy frequencies and that these threshold levels are described by means of similarity relations.

As in the previous case, it is by means of Assumption 6 about the doubt function that we may define a similarity relation on $\Pi_{ki} = (0, 1]$ and correlated similarity relations on $F_{ki} = [0, 1]$. Suppose that (π_{ki}, f_{ki}) is the vector of expected payoff-strategy proportion attached to strategy i at time t .

The similarity relation on Π_{ki} , denoted $S\Pi_{ki}$, is assumed to be of the difference type and it is defined as follows

$$\pi_{ki} S\Pi_{ki} \pi'_{ki} \Leftrightarrow |\pi_{ki} - \pi'_{ki}| \leq \epsilon_k$$

On F_{ki} , we define now the correlated similarity relations as follows. First, for all $\pi_{ki} > \epsilon_k > 0$ we build the function $\phi_{ki} : \Pi_{ki} \rightarrow (1, \infty]$ as follows,

$$\phi_{ki}(\pi_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - \epsilon_k} > 1$$

Then, we can establish the following similarity relation (of the ratio-type) between f_{ki} and other frequencies in F_{ki} , such as f'_{ki} , given π_{ki} .

$$f_{ki} S F_{ki}(\pi_{ki}) f'_{ki} \Leftrightarrow \frac{1}{\phi_{ki}(\pi_{ki})} \leq \frac{f_{ki}}{f'_{ki}} \leq \phi_{ki}(\pi_{ki})$$

We call $SF_{ki}(\pi_{ki})$ a **correlated** similarity relation because the similarity on F_{ki} depends on the level of expected payoff π_{ki} at period t . For values of $\pi_{ki} \leq \epsilon_k$ the function ϕ_{ki} is not defined and we assume that in that case that $SF_{ki}(\pi_{ki})$ is the degenerate similarity relation (see Rubinstein (1988)).

Remark 2 *The threshold level in the frequency space is inversely related to expected payoffs: $\frac{\partial \phi_{ki}(\pi_{ki})}{\partial \pi_{ki}} < 0$. This means that as the expected payoffs at stake increases, the discrimination on the frequency space F_{ki} increases.*

Every agent in a given player position is able to observe the relative frequency of every strategy available to that position. When an agent feels dissatisfied with his current strategy, he will choose a new strategy with a probability that is equal to the proportion of agents playing that strategy.

We proceed as in the previous case (for simplicity we shall write ϕ_{ki} instead of $\phi_{ki}(\pi_{ki})$). Let the ratio

$$\frac{\phi_{ki} - 1}{\sum_{i=1}^{m_k} \phi_{ki}} = \frac{\phi_{ki} - 1}{\phi_k}$$

denote the proportion of ki strategists who feel dissatisfied with strategy i . Note that, everything equal, this function increases with ϕ_{ki} . Hence, an increase in ϕ_{ki} , due to a decrease in the expected payoffs π_{ki} , will increase the proportion of dissatisfied ki strategists.

As before, $\tau \frac{(\phi_{ki}-1)}{\phi_k} f_{ki}$ denotes the proportion of ki strategists who will choose a new strategy at time t (the *outflow*). Since a particular strategy is chosen with a probability that is equal to the proportion of agents playing that strategy, then $\tau \sum_{j=1}^{m_k} \frac{(\phi_{kj}-1)}{\phi_k} f_{kj} f_{ki} = \tau \frac{(\bar{\phi}_k-1)}{\phi_k} f_{ki}$ denotes the proportion of agents who choose strategy i ; i.e. the *inflow* (where $\bar{\phi}_k = \sum_{j=1}^{m_k} \phi_{kj} f_{kj}$ is the average perception in player population k at time t).

Therefore

$$f_{ki}(t + \tau) = f_{ki}(t) - \tau \frac{[\phi_{ki} - 1]}{\phi_k} f_{ki} + \tau \frac{[\bar{\phi}_k - 1]}{\phi_k} f_{ki}. \quad (6)$$

Proposition 6 *As $\tau \rightarrow 0$, equation (6) becomes*

$$\dot{f}_{ki} = f_{ki} \left[\frac{\bar{\phi}_k - \phi_{ki}}{\phi_k} \right] \quad (7)$$

1. If for all player position $k \in K = \{1, 2, \dots, n\}$, the strategy set S_k consists of two elements, i.e. if $m_k = 2$ then, equation (7) is just the standard Replicator Dynamics (RD) multiplied by a positive function (i.e. is aggregate monotonic).
2. If $m_k > 2$, then we obtain a selection dynamics that approximates the RD, but preserves only the positive sign of the RD (i.e. is weakly payoff positive).

Proof: See appendix C

7 Concluding remarks

In 2×2 games with Mixed Strategy Nash Equilibria, the introduction of agents with doubts coupled with (and mutually reinforced by) imperfect discrimination capacity, permits a departure from the long-run behavior of traditional selection dynamic systems. For instance, if we assume that the feeling of doubts is sensitive to the popularity of a pure strategy we then obtain doubt-based selection dynamics that are not payoff monotonic. The main feature of the doubt-based system is that its equilibrium does not require expected payoffs to be equalized across strategies. Nevertheless, the curvature of the decreasing doubt functions has strong implications on the long run behavior of the system. If agents play in the *doubt-full mode*,- i.e. agents are endowed with an extremely concave doubt function-, a Mixed Strategy Nash Equilibrium is a Mixed Strategy Doubt-Full Equilibrium and it is shown to be asymptotically stable. But stability is lost when agents are in the *doubt-less mode*, i.e. when they are endowed with an extremely convex doubt function. An interesting case occurs when the doubt functions are between those two extreme cases. Then an agent's doubts are heavily influenced by the relative popularity of each of the pure strategies available to his player role. This leads to a kind of conformity behavior and to a mixed strategy doubt equilibrium in which the most popular strategies receive lower expected payoffs. We present some preliminary data whose qualitative features are along the lines of this theoretical result.

The present behaviorally-based theoretical work could be applicable to experimental research. To do this, experiments should be designed to capture two features of bounded rationality that, in our view, are embedded in our daily choice-making behavior: doubts and imperfect perception. For example, feedback on the popularity of different strategies would be important (most experiments on games with mixed strategy equilibria reported

in Camerer 2003 did not provide subjects with such information), as well as less sharply defined payoffs. We think that evidence from environments capturing those features would open fruitful lines of experimental research.

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Appendices

A Doubts and Imperfect Discrimination Modeled by Similarity relations

In the present model, doubts are closely related to imperfect discrimination capacity (of real numbers, such as strategy frequencies and expected payoffs). An environment shaped by uncertainty and doubts about the correctness of the choices made is effort demanding for the cognitive system of decision-makers. One way subjects cope with the ambiguous nature of this situation is by simplifying its complexity; for instance, by grouping numbers in intervals of similarity. Inside those intervals, whose size depend on threshold levels that change continuously, values - of, say, expected payoffs and strategy frequencies -, are not distinguished. We, thus, model subjects' imperfect discrimination by means of similarity relations¹². Let (π_{ki}, f_{ki}) be the vector of expected payoff-proportion of agents of player population k attached to strategy $i \in S_k$ at time t .

1. d_{ki} defines on the space of expected payoffs, Π_{ki} correlated similarities of the difference-type as follows: given f_{ki} , the similarity interval of π_{ki} is:

$$[\pi_{ki} - d_{ki}(f_{ki}), \pi_{ki} + d_{ki}(f_{ki})]$$

Thus, given f_{ki} , the doubt level, $d_{ki}(f_{ki})$, determines the threshold level on Π_{ki} . Payoffs inside the similarity interval are not discriminated by the agent. By Assumption 2.2, if f_{ki} increases, the threshold, $d_{ki}(f_{ki})$, decreases and so the similarity interval of π_{ki} shrinks. This means that when f_{ki} increases, the discrimination capacity on the space of expected payoffs to strategy i , Π_{ki} , increases (probably because the accumulated experience with strategy i has increased due to the increased number of agents from population k currently playing strategy i). This increase in perception induces a kind of vertical wedge type form as it can be seen in figure 5.

¹²They are, in fact, correlated similarities: an extension of the similarity relations defined by Rubinstein (1988). Rather than being constant, correlated similarities depend on the value of some relevant parameter. For more details, see Uriarte (1999) and (2007).

2. d_{ki} builds the λ_{ki} function, which, in turn, is used to define on F_{ki} correlated similarity relations of the ratio-type. The λ_{ki} function is defined as follows: given d_{ki} and $f_{ki} \in (0, 1)$, then for all expected payoffs $\pi_{ki} > d_{ki}(f_{ki})$,

$$\lambda_{ki}(\pi_{ki}, f_{ki}) = \frac{\pi_{ki}}{\pi_{ki} - d_{ki}(f_{ki})} > 1$$

Thus, for each $f_{ki} \in (0, 1)$, when $\pi_{ki} > d_{ki}(f_{ki})$, the λ_{ki} function defines a correlated similarity relation on F_{ki} whose similarity interval for that f_{ki} is:

$$[f_{ki}/\lambda_{ki}(\cdot), f_{ki} \cdot \lambda_{ki}(\cdot)]$$

If $\pi_{ki} \leq d_{ki}(f_{ki})$, then we would have instead a *degenerate* similarity relation; that is, a similarity relation with intervals in which the greatest number similar to f_{ki} is 1, and the smallest number similar to f_{ki} is 0. In other words, the similarity intervals for f_{ki} will always be $[0, 1]$. Notice that $\pi_{ki} \leq d_{ki}(f_{ki})$ would happen in the case where all agents play in the *doubt-full mode*, as it happens in Proposition 4; that is, when $d_{ki}(f_{ki})$ is almost 1 for all $f_{ki} \in (0, 1)$, $k \in K$ and $i \in S_k$. The condition $\pi_{ki} > d_{ki}(f_{ki})$, for all $f_{ki} \in (0, 1)$, would be satisfied when all agents are playing in the *doubt-less mode*, as in Proposition 5; that is, when $d_{ki}(f_{ki})$ is almost 0 for all $f_{ki} \in (0, 1)$, $k \in K$ and $i \in S_k$. Indeed, $\pi_{ki} > d_{ki}(f_{ki})$ would also happen, for some $f_{ki} \in (0, 1)$, when we consider doubt functions between the above two extreme modes.

Note the following nice properties of $\lambda_{ki}(\pi_{ki}, f_{ki})$:

(i) given f_{ki} and d_{ki} , if π_{ki} increases, $\lambda_{ki}(\pi_{ki}, f_{ki})$ decreases and thus, the similarity interval shrinks. This means that when the expected payoffs at stake increase, the discrimination efforts on the frequency space, F_{ki} , increases (generating a kind of horizontal wedge type form as it is shown in figure 5)

(ii) keeping the function d_{ki} , and π_{ki} constant, if the frequency f_{ki} increases (that is, if the accumulated experience with strategy i increases), $\lambda_{ki}(\pi_{ki}, f_{ki})$ decreases and so the similarity interval of the higher frequency shrinks.

Based on a model developed in Uriarte (1999), Uriarte (2007) shows how these two correlated similarity relations are used to build a (non-complete and non-transitive) preference-indifference relation defined on the space of expected payoffs and frequencies, $\Pi_{ki} \times F_{ki}$, attached to pure strategy $i \in S_k$. The distance to the upper contour set, denoted $U =$

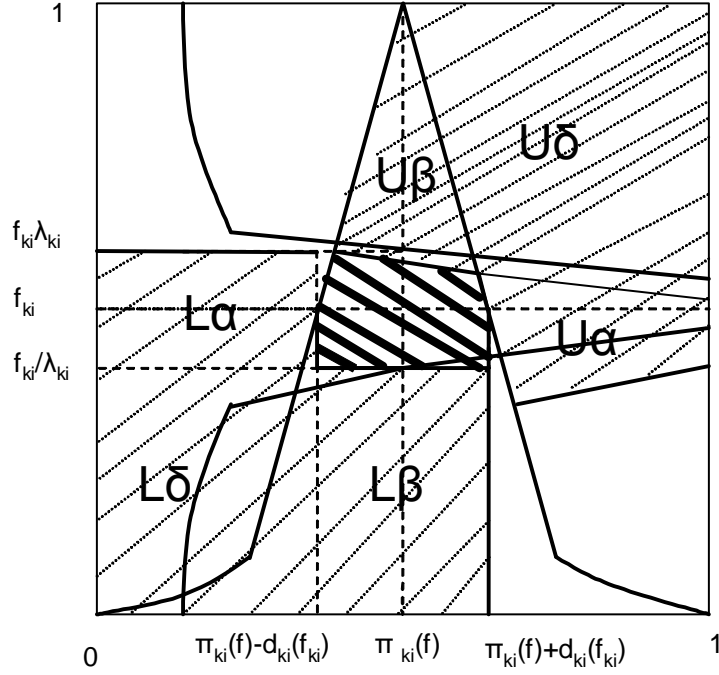


Figure 1: Figure 5. Preference-indifference relation compatible with correlated similarities. Relative to (π_{ki}, f_{ki}) , $U = U_{\alpha} \cup U_{\beta} \cup U_{\delta}$ denotes the upper-contour (or aspiration) set, $L = L_{\alpha} \cup L_{\beta} \cup L_{\delta}$ the lower contour set and the darker area is the indifference set.

$U_{\alpha} \cup U_{\beta} \cup U_{\delta}$, (which might be thought of as the agent's aspiration set) depends on how thick the indifference set of (π_{ki}, f_{ki}) is. We assume that agents are preference-satisficers; that is, they choose a strategy to reduce the distance from (π_{ki}, f_{ki}) to U . This can be achieved by reducing doubts and/or increasing expected payoffs. The smaller (greater) that distance the more satisfied (dissatisfied) the ki -agent will be with his current strategy.

It can be seen that the properties (i) and (ii) of λ_{ki} capture the changes in the thickness of the indifference sets. Hence, the λ_{ki} function can be thought of as an index of how satisfied the ki -agent is. See figure 5.

B Properties of the Index for Dissatisfied Agents

The properties we require for the α_{ki} function follow naturally from the properties of λ_{ki} when α_{ki} takes the following form:

$$\alpha_{ki} = \frac{\lambda_{ki} - 1}{\sum_{i=1}^{m_k} \lambda_{ki}} = \frac{\lambda_{ki} - 1}{\lambda_k}$$

That is,

1. The proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should decrease if expected payoffs to strategy $i \in S_k$, π_{ki} , increase.

$$\frac{\partial \alpha_{ki}}{\partial \pi_{ki}} = \frac{\frac{\partial \lambda_{ki}}{\partial \pi_{ki}} \lambda_k - \frac{\partial \lambda_{ki}}{\partial \pi_{ki}} (\lambda_{ki} - 1)}{\lambda_k^2} = \frac{\frac{-d_{ki}(f_{ki})}{(\pi_{ki} - d_{ki}(f_{ki}))^2} (\lambda_k - \lambda_{ki} + 1)}{\lambda_k^2} < 0$$

2. The proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should increase if expected payoffs to strategy $j \in S_k$, π_{kj} , increase.

$$\frac{\partial \alpha_{ki}}{\partial \pi_{kj}} = \frac{-\frac{\partial \lambda_{kj}}{\partial \pi_{kj}} (\lambda_{ki} - 1)}{\lambda_k^2} = \frac{\frac{-d_{kj}(f_{kj})}{(\pi_{kj} - d_{kj}(f_{kj}))^2} (1 - \lambda_{ki})}{\lambda_k^2} > 0$$

3. If agents ki 's doubts decrease, because the popularity of strategy i , f_{ki} , has increased, the proportion of dissatisfied should decrease too.

$$\frac{\partial \alpha_{ki}}{\partial f_{ki}} = \frac{\frac{\partial \lambda_{ki}}{\partial f_{ki}} \lambda_k - \frac{\partial \lambda_{ki}}{\partial f_{ki}} (\lambda_{ki} - 1)}{\lambda_k^2} = \frac{\frac{\pi_{ki} \frac{\partial d_{ki}(f_{ki})}{\partial f_{ki}}}{(\pi_{ki} - d_{ki}(f_{ki}))^2} (\lambda_k - \lambda_{ki} + 1)}{\lambda_k^2} < 0$$

4. If the popularity of strategy $j \in S_k$, f_{kj} , increases, the proportion of dissatisfied agents with their current pure strategy $i \in S_k$ should increase.

$$\frac{\partial \alpha_{ki}}{\partial f_{kj}} = \frac{-\frac{\partial \lambda_{kj}}{\partial f_{kj}} (\lambda_{ki} - 1)}{\lambda_k^2} = \frac{\frac{\pi_{kj} \frac{\partial d_{kj}(f_{kj})}{\partial f_{kj}}}{(\pi_{kj} - d_{kj}(f_{kj}))^2} (1 - \lambda_{ki})}{\lambda_k^2} > 0$$

C Proofs of propositions

Let

	(y)	L	R
(x)U	a_{11}, b_{11}	a_{12}, b_{12}	
D	a_{21}, b_{21}	a_{22}, b_{22}	

denote the 2×2 constant-sum game G , and $I^* \equiv [(x^*, 1 - x^*), (y^*, 1 - y^*)]$, with $x^* > 0$ and $y^* > 0$, the Mixed strategy Nash Equilibrium of G . To get this equilibrium, we may assume, without loss of generality, that $a_{11} > a_{21}$, $b_{11} < b_{12}$, $a_{12} < a_{22}$, and $b_{22} < b_{21}$. Recall that payoffs are normalized so that they take values on $(0, 1]$. Recall too that to avoid the use of four different doubt parameters, we shall assume that the four doubt functions are the same: $d_D = d_U = d_R = d_L = d$. The doubt-based selection dynamics are represented by the following system:

$$\begin{aligned}
 \dot{x} &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)} (\pi_U d_D - \pi_D d_U) & (C.1) \\
 &= \frac{x(1-x)}{\pi_U(\pi_D - d_D) + \pi_D(\pi_U - d_U)} ((a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x)) \\
 &\equiv G_1(x, y)F_1(x, y)
 \end{aligned}$$

$$\begin{aligned}
 \dot{y} &= \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)} (\pi_L d_R - \pi_R d_L) & (C.2) \\
 &= \frac{y(1-y)}{\pi_L(\pi_R - d_R) + \pi_R(\pi_L - d_L)} ((b_{11}x + b_{21}(1-x))d_R(1-y) - (b_{12}x + b_{22}(1-x))d_L(y)) \\
 &\equiv G_2(x, y)F_2(x, y)
 \end{aligned}$$

Proof of Proposition 2:

We must first show that a Mixed Strategy Nash Equilibrium (MSNE) converges to Mixed Strategy Doubt-Full Equilibrium (MSDFE) as δ converges to 0 in $d_{ki} = d^{1-\delta} \in D^{1-\delta} \subset D$ (see Remark 1). Note that, by construction of the λ_{ki} function, the denominators of the system (C.1)-(C.2)) are positive.

1. An interior rest point of (C.1)-(C.2), (i.e. a MSDE), satisfies:

$$\begin{aligned} (a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x) &= 0 \\ (b_{11}x + b_{21}(1-x))d_R(1-y) - (b_{12}x + b_{22}(1-x))d_L(y) &= 0 \end{aligned}$$

Then, if $d_i \in D^{1-\delta}$ for $i \in \{U, D, L, R\}$,

$$\lim_{\delta \rightarrow 0} \frac{d_U(x)}{d_D(1-x)} = \lim_{\delta \rightarrow 0} \frac{d_L(y)}{d_R(1-y)} = 1, \text{ for all } (x, y) \in (0, 1) \times (0, 1)$$

Now suppose that we are in the MSNE, $(x^*, y^*) \in (0, 1) \times (0, 1)$, of G and that $d_i \in D^{1-\delta}$. Then, the strategies available to each player get the same expected payoff; that is $a_{11}y^* + a_{12}(1-y^*) = a_{21}y^* + a_{22}(1-y^*)$ and $b_{11}x^* + b_{21}(1-x^*) = b_{12}x^* + b_{22}(1-x^*)$. Thus,

$$\lim_{\delta \rightarrow 0} \frac{(a_{11}y^* + a_{12}(1-y^*))d_D(1-x^*)}{(a_{21}y^* + a_{22}(1-y^*))d_U(x^*)} = \lim_{\delta \rightarrow 0} \frac{(b_{11}x^* + b_{21}(1-x^*))d_R(1-y^*)}{(b_{12}x^* + b_{22}(1-x^*))d_L(y^*)} = 1$$

This, plus continuity, establishes the result.

2. We show that for all $(x', y') \in (0, 1) \times (0, 1)$, there exists a sequence of functions $d^\delta \in D^\delta$ and a δ low enough that the rest point of (C.1)-(C.2) cannot be any $C \neq [(x', 1-x'), (y', 1-y')]$ for any $\delta \leq \delta'$ and then the result follows.

An interior rest point of (C.1)-(C.2) must satisfy:

$$\begin{aligned} (a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x) &= 0 \\ (b_{11}x + b_{21}(1-x))d_R(1-y) - (b_{12}x + b_{22}(1-x))d_L(y) &= 0 \end{aligned}$$

For interior rest points, this implies that

$$\begin{aligned} (a_{11}y + a_{12}(1-y)) \frac{d_D(1-x)}{d_U(x)} - (a_{21}y + a_{22}(1-y)) &= 0 \\ (b_{11}x + b_{21}(1-x)) \frac{d_R(1-y)}{d_L(y)} - (b_{12}x + b_{22}(1-x)) &= 0 \end{aligned}$$

Let $C = [(x^*, 1-x^*), (y^*, 1-y^*)]$, with $x^* > x'$ and $y^* > y'$. We construct the doubt function $d_{ki}^z(f_{ki})$ in D as follows:

$$d_{ki}^z(f_{ki}) = \begin{cases} 1 - \frac{1-\delta}{\delta} f_{ki} & \text{if } f_{ki} \leq \delta \\ \delta \frac{f_{ki}}{\delta} & \text{if } \delta < f_{ki} \leq z \\ \frac{\delta \frac{f_{ki}}{\delta}}{(1-z)^{1/\delta}} (1-f_{ki})^{1/\delta} & \text{if } f_{ki} > z \end{cases}$$

where $z \in \{x', y'\}$ and $\delta > 0$. Note that as δ approaches 0, the graph of d_{ki}^z function approaches both the horizontal and vertical axes and the agent is said to be in a *doubt-less* mode.

Now, for $x > x' (> 0.5)$ and $y > y' (> 0.5)$, there exists a δ' low enough such tat for all $\delta \leq \delta'$,

$$\begin{aligned}\frac{d_D(1-x)}{d_U(x)} &= \frac{\delta^{\frac{1-x}{\delta}}}{\frac{\delta^{\frac{x}{\delta}}}{(1-x')^{1/\delta}} (1-x)^{1/\delta}} = \delta^{\frac{1-2x}{\delta}} \left(\frac{1-x'}{1-x} \right)^{1/\delta} \\ \frac{d_R(1-y)}{d_L(y)} &= \frac{\delta^{\frac{1-y}{\delta}}}{\frac{\delta^{\frac{y}{\delta}}}{(1-y')^{1/\delta}} (1-y)^{1/\delta}} = \delta^{\frac{1-2y}{\delta}} \left(\frac{1-y'}{1-y} \right)^{1/\delta}\end{aligned}$$

Note that we can make $\delta^{\frac{1-2x}{\delta}} > 0$ sufficiently big by choosing a sufficiently small δ and that $1-x' > 1-x$. Then

$$\begin{aligned}\delta^{\frac{1-2x}{\delta}} \left(\frac{1-x'}{1-x} \right)^{1/\delta} &> \frac{(a_{21}y + a_{22}(1-y))}{(a_{11}y + a_{12}(1-y))} \\ \delta^{\frac{1-2y}{\delta}} \left(\frac{1-y'}{1-y} \right)^{1/\delta} &> \frac{(b_{12}x + b_{22}(1-x))}{(b_{11}x + b_{21}(1-x))}\end{aligned}$$

and thus

$$\begin{aligned}(a_{11}y + a_{12}(1-y)) \delta^{\frac{1-2x}{\delta}} \left(\frac{1-x'}{1-x} \right)^{1/\delta} - (a_{21}y + a_{22}(1-y)) &> 0 \\ (b_{11}x + b_{21}(1-x)) \delta^{\frac{1-2y}{\delta}} \left(\frac{1-y'}{1-y} \right)^{1/\delta} - (b_{12}x + b_{22}(1-x)) &> 0\end{aligned}$$

Now suppose that $0.5 > x > x'$ and $0.5 > y > y'$. Then,

$$\begin{aligned}\frac{d_D(1-x)}{d_U(x)} &= \frac{\frac{\delta^{\frac{1-x}{\delta}}}{(1-x')^{1/\delta}} x^{\frac{1}{\delta}}}{\frac{\delta^{\frac{x}{\delta}}}{(1-x')^{1/\delta}} (1-x)^{1/\delta}} = \delta^{\frac{1-2x}{\delta}} \left(\frac{x}{1-x} \right)^{1/\delta} \\ \frac{d_R(1-y)}{d_L(y)} &= \frac{\frac{\delta^{\frac{1-y}{\delta}}}{(1-y')^{1/\delta}} y^{\frac{1}{\delta}}}{\frac{\delta^{\frac{y}{\delta}}}{(1-y')^{1/\delta}} (1-y)^{1/\delta}} = \delta^{\frac{1-2y}{\delta}} \left(\frac{y}{1-y} \right)^{1/\delta}\end{aligned}$$

If we choose a sufficiently small δ , both of the above ratios can be made sufficiently small so that

$$\begin{aligned}\delta^{\frac{1-2x}{\delta}} \left(\frac{x}{1-x} \right)^{1/\delta} &< \frac{(a_{21}y + a_{22}(1-y))}{(a_{11}y + a_{12}(1-y))} \\ \delta^{\frac{1-2y}{\delta}} \left(\frac{y}{1-y} \right)^{1/\delta} &< \frac{(b_{12}x + b_{22}(1-x))}{(b_{11}x + b_{21}(1-x))}\end{aligned}$$

and thus

$$\begin{aligned}(a_{11}y + a_{12}(1-y)) \delta^{\frac{1-2x}{\delta}} \left(\frac{x}{1-x} \right)^{1/\delta} - (a_{21}y + a_{22}(1-y)) &< 0 \\ (b_{11}x + b_{21}(1-x)) \delta^{\frac{1-2y}{\delta}} \left(\frac{y}{1-y} \right)^{1/\delta} - (b_{12}x + b_{22}(1-x)) &< 0\end{aligned}$$

If $x < x' < 0.5$,

$$\frac{d_D(1-x)}{d_U(x)} = \delta^{\frac{1-2x}{\delta}} < 1$$

If $0.5 < x < x'$,

$$\frac{d_D(1-x)}{d_U(x)} = \delta^{\frac{1-2x}{\delta}} > 1$$

We proceed in the same manner for y' . In both cases, we can choose a δ sufficiently small so that

$$\begin{aligned}(a_{11}y + a_{12}(1-y)) \delta^{\frac{1-2x}{\delta}} - (a_{21}y + a_{22}(1-y)) &\neq 0 \\ (b_{11}x + b_{21}(1-x)) \delta^{\frac{1-2y}{\delta}} - (b_{12}x + b_{22}(1-x)) &\neq 0\end{aligned}$$

1. The argument is equivalent for y' . ■

Proof of Proposition 4

Let $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)] \in (0, 1) \times (0, 1)$ be an interior Mixed Strategy Nash Equilibrium (MSNE) of G . In this equilibrium, expected payoffs are equalized across strategies; that is, $\pi_U = \pi_D$ and $\pi_L = \pi_R$. From Proposition 2, we also know that an MSNE is a Mixed Strategy Doubt-Full equilibrium (MSDFE); that is, $\pi_U d_D(1-x^*) = \pi_D d_U(x^*)$ and $\pi_L d_R(1-y^*) = \pi_R d_L(y^*)$. Hence, an interior MSNE is a stationary state of the system (C.1)-(C.2) if all agents behave in a *doubt-full mode*.

Thus, $F_1(x^*, y^*) = 0$ and $F_2(x^*, y^*) = 0$, where

$$\begin{aligned}
F_1(x, y) &= \pi_U d_D(1-x) - \pi_D d_U(x) \\
&= (a_{11}y + a_{12}(1-y))d_D(1-x) - (a_{21}y + a_{22}(1-y))d_U(x) \\
F_2(x, y) &= \pi_L d_R(1-y) - \pi_R d_L(y) \\
&= (b_{11}x + b_{21}(1-x))d_R(1-y) - (b_{12}x + b_{22}(1-x))d_L(y)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F_1(x, y)}{\partial x} &= \pi_U \frac{\partial d_D(1-x)}{\partial x} - \pi_D \frac{\partial d_U(x)}{\partial x} \\
\frac{\partial F_1(x, y)}{\partial y} &= (a_{11} - a_{12})d_D(1-x) + (a_{22} - a_{21})d_U(x) \\
\frac{\partial F_2(x, y)}{\partial x} &= (b_{11} - b_{21})d_R(1-y) + (b_{22} - b_{12})d_L(y) \\
\frac{\partial F_2(x, y)}{\partial y} &= \pi_L \frac{\partial d_R(1-y)}{\partial y} - \pi_R \frac{\partial d_L(y)}{\partial y}
\end{aligned}$$

On the other hand, the Jacobian $J(x, y)$ of the dynamic system (C.1)-(C.2) evaluated at the steady state (x^*, y^*) is:

$$J(x^*, y^*) = \begin{bmatrix} G_1(x^*, y^*) \left. \frac{\partial F_1(x, y)}{\partial x} \right|_{I^*} & G_1(x^*, y^*) \left. \frac{\partial F_1(x, y)}{\partial y} \right|_{I^*} \\ G_2(x^*, y^*) \left. \frac{\partial F_2(x, y)}{\partial x} \right|_{I^*} & G_2(x^*, y^*) \left. \frac{\partial F_2(x, y)}{\partial y} \right|_{I^*} \end{bmatrix}$$

In an MSNE, $\pi_U = \pi_D$, $\pi_L = \pi_R$. If, on the other hand, agents are playing in a *doubt-full mode*, (that is, $d_i \in D^{1-\delta}$ for $i \in \{U, D, L, R\}$ with $\lim_{\delta \rightarrow 0} d_U(x) = \lim_{\delta \rightarrow 0} d_D(1-x) = \lim_{\delta \rightarrow 0} d_L(y) = \lim_{\delta \rightarrow 0} d_R(1-y)$ and being nearly 1, for all $(x, y) \in (0, 1) \times (0, 1)$). Then, writing $d_i(\cdot) = 1$, we would also have $\pi_U d_D = \pi_D d_U$ and $\pi_L d_R = \pi_R d_L$.

Hence, in an MSNE as an MSDFE :

$$\begin{aligned}
G_1(x^*, y^*) &= \frac{x^*(1-x^*)}{2\pi_U\pi_D - \pi_U d_D(1-x^*) - \pi_D d_U(x^*)} \\
&= \frac{x^*(1-x^*)}{\pi_U(2\pi_U - d_D(1-x^*) - d_U(x^*))} \\
&= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} \\
G_2(x^*, y^*) &= \frac{y^*(1-y^*)}{2\pi_L(\pi_L - 1)}
\end{aligned}$$

Thus, the elements of the Jacobian matrix are the following:

$$\begin{aligned}
j_{11} &= G_1(x^*, y^*) \left. \frac{\partial F_1(x, y)}{\partial x} \right|_{I^*} \\
&= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} \left(\pi_U \frac{\partial d_D(1-x)}{\partial x} - \pi_U \frac{\partial d_U(x)}{\partial x} \right) \Big|_{I^*} \\
&= \frac{x^*(1-x^*)}{2(\pi_U - 1)} \left(\frac{\partial d_D(1-x)}{\partial x} - \frac{\partial d_U(x)}{\partial x} \right) \Big|_{I^*}
\end{aligned}$$

$$\begin{aligned}
j_{12} &= G_1(x^*, y^*) \left. \frac{\partial F_1(x, y)}{\partial y} \right|_{I^*} \\
&= \frac{x^*(1-x^*)}{2\pi_U(\pi_U - 1)} ((a_{11} - a_{12}) d_D(1-x^*) + (a_{22} - a_{21}) d_U(x^*))
\end{aligned}$$

$$\begin{aligned}
j_{21} &= G_2(x^*, y^*) \left. \frac{\partial F_2(x, y)}{\partial y} \right|_{I^*} \\
&= \frac{y^*(1-y^*)}{2\pi_L(\pi_L - 1)} ((b_{11} - b_{21}) d_R(1-y^*) + (b_{22} - b_{12}) d_L(y^*))
\end{aligned}$$

$$\begin{aligned}
j_{22} &= G_2(x^*, y^*) \left. \frac{\partial F_2(x, y)}{\partial y} \right|_{I^*} \\
&= \frac{y^*(1-y^*)}{2(\pi_L - 1)} \left(\frac{\partial d_R(1-y)}{\partial y} - \frac{\partial d_L(y)}{\partial y} \right) \Big|_{I^*}
\end{aligned}$$

Recall that the real part of the eigenvalues of $J(x^*, y^*)$ only depends on the sum of the diagonal terms (the trace of the matrix):

$$\begin{aligned} \text{Trace of } J(x^*, y^*) &= G_1(x^*, y^*) \left. \frac{\partial F_1(x, y)}{\partial x} \right|_{I^*} + G_2(x^*, y^*) \left. \frac{\partial F_2(x, y)}{\partial y} \right|_{I^*} \\ &= \frac{x^*(1-x^*)}{2(\pi_U - 1)} \left(\left. \frac{\partial d_D(1-x)}{\partial x} - \frac{\partial d_U(x)}{\partial x} \right)_{I^*} \right) \\ &\quad + \frac{y^*(1-y^*)}{2(\pi_L - 1)} \left(\left. \frac{\partial d_R(1-y)}{\partial y} - \frac{\partial d_L(y)}{\partial y} \right)_{I^*} \right) \end{aligned}$$

Note that the expected values $\pi_U = a_{11}y^* + a_{12}(1-y^*)$ and $\pi_L = b_{11}x^* + b_{21}(1-x^*)$ are smaller than 1 because, by assumption, payoffs take values in $(0, 1]$ and we are considering interior mixed equilibria, $(x^*, y^*) \in (0, 1) \times (0, 1)$. Thus, both $\frac{x^*(1-x^*)}{2(\pi_U-1)}$ and $\frac{y^*(1-y^*)}{2(\pi_L-1)}$ are negative. The sign of $\left(\left. \frac{\partial d_D(1-x)}{\partial x} - \frac{\partial d_U(x)}{\partial x} \right)_{I^*} \right)$ and $\left(\left. \frac{\partial d_R(1-y)}{\partial y} - \frac{\partial d_L(y)}{\partial y} \right)_{I^*} \right)$ is clearly positive (that is, the signs of the derivatives of $d_D(1-x)$ and $d_R(1-y)$ with respect to x and y , respectively, are positive and those of $d_U(x)$ and $d_L(y)$ are negative). Thus, $j_{11} < 0$ and $j_{22} < 0$ and so the sign of the trace is negative

$$\text{sign} \left[G_1(x^*, y^*) \left. \frac{\partial F_1(x, y)}{\partial x} \right|_{I^*} + G_2(x^*, y^*) \left. \frac{\partial F_2(x, y)}{\partial y} \right|_{I^*} \right] < 0$$

Without loss of generality, we may assume, for an interior equilibrium, that $a_{11} > a_{21}$, $b_{11} < b_{12}$, $a_{12} < a_{22}$, and $b_{22} < b_{21}$. Then it can be seen that the sign of $j_{21} \times j_{12}$ is negative, when the agents are playing in the *absent* or *doubt-full mode*:

$$j_{21} \times j_{12} = \left(\frac{y^*(1-y^*)}{2\pi_L(\pi_L-1)} ((b_{11} - b_{12}) + (b_{22} - b_{21})) \right) \times \left(\frac{x^*(1-x^*)}{2\pi_U(\pi_U-1)} ((a_{11} - a_{21}) + (a_{22} - a_{12})) \right) < 0$$

Thus, the determinant associated to $J(x^*, y^*)$, $\text{Det } J(x^*, y^*) = j_{11} \times j_{22} - j_{21} \times j_{12}$, has a positive sign. Therefore, when every agent is in the *absent* or *doubt-full mode of play*, the MSNE, $I^* \equiv [(x^*, 1-x^*), (y^*, 1-y^*)]$, is a *sink* and therefore is an asymptotically stable equilibrium. ■

Proof of Proposition 5

Recall that from Proposition 2, an interior MSNE is a stationary state for the system (C.1)-(C.2) if all the agents behave in a doubt-less mode. In that mode of play, $\pi_i > d_i$ for $i \in \{U, D, L, R\}$ and $d_i \in D^\delta$ taking values arbitrarily small in $(0, 1)$.

As in the proof of Proposition 4, we look at the Jacobian $J(x, y)$ of the dynamic system (C.1)-(C.2) evaluated at the MSNE (x^*, y^*) as an MSDLE. Since $\pi_i > d_i$ for $i \in \{U, D, L, R\}$, it can be seen that $j_{11} > 0$ and $j_{22} > 0$ and $j_{21} \times j_{12}$ is negative. Thus, the signs of the Trace of $J(x^*, y^*)$ and the Det $J(x^*, y^*) = j_{11} \times j_{22} - j_{21} \times j_{12}$, have positive signs. The MSDLE is therefore a *source*. ■

Proof Proposition 6:

(a) Let $S_k = \{1, 2\}$ be player population k 's strategy set. Without loss of generality, let us refer to the dynamics of strategy 1. Then, by equation (1), we have

$$\begin{aligned} \dot{f}_{k1} &= f_{k1} \bar{\phi}_k - \phi_{k1} \phi_k & (C.3) \\ &= \frac{\epsilon_k}{\pi_{k1}(\pi_{k2} - \epsilon_k) + \pi_{k2}(\pi_{k1} - \epsilon_k)} f_{k1} (\pi_{k1} - \bar{\pi}_k) \\ &= \frac{\epsilon_k}{D(f)} f_{k1} [\pi_{k1} - \bar{\pi}_k] \end{aligned}$$

where $D(f) \equiv \pi_{k1}(\pi_{k2} - \epsilon_k) + \pi_{k2}(\pi_{k1} - \epsilon_k) > 0$.

By equation (C.3), the growth rates $\frac{\dot{f}_{k1}}{f_{k1}}$ equal payoff differences $[\pi_{k1} - \bar{\pi}_k]$ multiplied by a (Lipschitz) continuous, positive function $\frac{\epsilon_k}{D(f)}$. This concludes the proof. (Note that, given ϵ_k , a payoff difference $[\pi_{k1} - \bar{\pi}_k]$ will have stronger dynamic effect if $D(f)$ is low than if it is high; if ϵ_k decreases, the dynamic effect of $[\pi_{k1} - \bar{\pi}_k]$ decreases).

(b) Easy. ■