Inserting continuous functions with values in bounded complete domains and hedgehog-like structures

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1. Bounded complete domains

Basic reference:

COMPENDIUM (2003) = Continuous Lattices and Domains.
G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott.

Given a poset $L = (L, \leq)$ and $a, b \in L$, one writes $a \ll b$ iff, given a directed subset $D \subset L$ for which the sup $\bigvee D$ exists and $b \leq \bigvee D$, there is $d \in D$ with $a \leq d$.
Notation: $\downarrow a = \{b \in L : b \leq a\}$, $\twoheaddownarrow a = \{b \in L : b \ll a\}$, and dually for $\uparrow a$ and $\twoheaduparrow a$.

Definition

A poset L is a **bounded complete domain** if:

(1) each directed subset of L has a sup (i.e., L a **dcpo**);

(2) each subset of L that is bounded above has a sup (i.e., L is **bounded complete**);

(3) L satisfies the **axiom of approximation** with respect to $\ll$, i.e.,

$$a = \bigvee \{b \in L : b \ll a\} \text{ for all } a \in L.$$

Remarks [COMPENDIUM (2003)]

(1) L is a **bounded complete domain** iff L is a **complete continuous $\wedge$-semilattice**.

(2) Every bounded complete domain L has the **insertion property**:

$$a \ll b \text{ implies } a \ll c \ll b \text{ for some } c \in L.$$

2. Bases of bounded complete domains

Definition

Let L be a bounded complete domain. A subset $D \subset L$ is called a $\ll$-**basis** of L iff:

(1) $a = \bigvee(D \cap \downarrow a)$ for all $a \in L$,

(2) if $a \in L$ and $d_1 \ll a$ with $d_1 \in D$, then there exists a $d_2 \in D$ s. t. $d_1 \ll d_2 \ll a$.

Remark

A $D \subset L$ is a basis in the sense of [Compendium (2003)], if (2) is replaced by

(2') that the set $D \cap \downarrow a$ be directed.

Then D is a basis in the sense of [Compendium (2003)] iff, whenever $a \ll b$ in L, there exists $d \in D$ with $a \ll d \ll b$.

Example

A basis in the sense of [Compendium (2003)] is a $\ll$-basis in our sense, but not conversely: the axes of $L = [0, 1]^2$ form a basis in our sense, which fails to be a basis in the sense of [Compendium (2003)].

General assumption

In what follows – if not otherwise stated – L is a bounded complete domain and X is a nonempty set.

3. Scales of subsets

The set L^X of all maps from X into L is ordered pointwise:

$$f \leq g \text{ in } L^X \text{ if and only if } f(x) \leq g(x) \text{ in } L \text{ for each } x \in X.$$

Notation

For an $a \in L$, we let:

$$[f \geq a] = f^{-1}(\uparrow a) \text{ and } [f \gg a] = f^{-1}(\uparrow\uparrow a).$$

Definition

Let $\varnothing \neq D \subset L$. A family $\mathcal{F} = \{F_d \subset X : d \in D\}$ is called a **prescale** if

(1) $\{d \in D : x \in F_d\}$ is bounded in L for all $x \in X$.

If one additionally assumes:

(2) for any $C \subset D$ for which $\bigvee C$ does exist, one has $\bigcap_{c \in C} F_c \subset F_d$ whenever $d \ll \bigvee C$,

then $\mathcal{F}$ is called a **scale**.

Remark

Each scale is **$\ll$-antitone**, i.e., $F_{d_1} \supset F_{d_2}$ whenever $d_1 \ll d_2$.

4. Generating bounded domain-valued functions

Lemma

Let $\varnothing \neq D \subset L$ and let $\mathcal{F} = \{F_d \subset X : d \in D\}$. Then:

(1) *If $\mathcal{F}$ is a prescale, then $f : X \to L$ defined by*

$$f(x) = \bigvee \{d \in D : x \in F_d\}$$

is a well-defined function.

(2) $\mathcal{F}$ is a scale iff there exists a function $f : X \to L$ such that for every $d \in D$:

$$[f \gg d] \subset F_d \subset [f \geq d].$$

(3) *If D is a $\ll$-basis consisting of coprimes and $\mathcal{F}$ is a prescale, then $\mathcal{F}$ is a scale iff it is $\ll$-antitone.*

Definition

Let $D \subset L$ and $\mathcal{F} = \{F_d \subset X : d \in D\}$ be a (pre)scale. The function $f \in L^X$ defined by

$$f(x) = \bigvee \{d \in D : x \in F_d\}$$

is said to be **generated** by the (pre)scale $\mathcal{F}$.

4. Generating bounded domain-valued functions

Lemma

Let $D \subset L$ be a $\ll$-basis in L. The following hold:

(1) *Let $f, g \in L^X$ be generated by the scales $\{F_d\}_{d \in D}$ and $\{G_d\}_{d \in D}$. Then $f \leq g$ iff $F_{d_1} \subset G_{d_2}$ whenever $d_2 \ll d_1$.*

(2) *If $\{H_d\}_{d \in D}$ is a scale, then there exists a unique $h \in L^X$ such that $[h \gg d] \subset H_d \subset [h \geq d]$. Moreover,* if *$a \in L$ and $d_1 \in D$, then:*

$$[h \geq a] = \bigcap_{d \ll a} H_d, \quad \text{and} \quad [h \gg d_1] = \bigcup_{d_1 \ll d} H_d.$$

5. Lower semicontinuity and lower limit functions

We need three topologies on a bounded complete domain L:

- the **Scott topology** $\sigma(L)$ with $\{\Uparrow a : a \in L\}$ as a base,
- the **lower topology** $\omega(L)$ with $\{L \setminus \downarrow a : a \in L\}$ as a subbase,
- the **Lawson topology** $\lambda(L) = \sigma(L) \vee \omega(L)$.

Notation

$\Sigma L = (L, \sigma(L))$, $\Omega L = (L, \omega(L))$, and $\Lambda L = (L, \lambda(L))$.

Definition

For X a topological space, define $(\cdot)_* : L^X \to L^X$ by

$$f_*(x) = \bigvee_{U \in \mathcal{N}_x} \bigwedge f(U), \quad x \in X,$$

where $\mathcal{N}_x$ = all open neighbourhoods of x. We call f_* the **lower limit function** of f.

Proposition

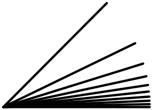
Let X be a topological space. Then:

(1) $f_{**} = f_*$.

(2) $C(X, \Sigma L) = \{f \in L^X : f = f_*\}$.

9. Hedgehog and its mutants as $\ll$-separable bounded complete domains

(1) HEDGEHOG WITH COUNTABLY MANY SPINES. This is the union of countably many disjoint copies of the unit interval after the identification of zero points of each interval:



(2) A SEQUENCE OF HEDGEHOGS. One may join a sequence of hedgehogs (with countably many spines) as follows:

(3) THE BOOK WITH INFINITELY MANY PAGES. This is the product of a hedgehog with countably many spines with the unit interval.