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On strict and double insertion theorems

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In mathematics there are many results concerning the possibility of inserting a function (or a pair of functions) between a given pair of comparable real-valued functions. In this note we are mainly concerned with strict and ultra-strict insertion (this is the terminology of Blatter and Seever [1]).

Let us start by first recalling the insertion scheme. Let $\mathbf{U}$, $\mathbf{L}$ and $\mathbf{C} = \mathbf{U} \cap \mathbf{L}$ be certain classes of real-valued functions on a set X.

The insertion scheme.

(I) Given $u \in \mathbf{U}$ and $l \in \mathbf{L}$ with $u \leq l$, there is $f \in \mathbf{C}$ such that $u \leq f \leq l$.

The prototype of this situation is the following theorem of Katětov [3] and Tong [7] in which X is a topological space, and $\mathbf{U}$, $\mathbf{L}$, and $\mathbf{C}$ consist of all upper semicontinuous, lower semicontinuous, and continuous real-valued functions on X.

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Theorem 1 (Katětov-Tong)**.** *A space X is normal if and only if, whenever $u, l\colon X \to \mathbb{R}$ are such that $u \leq l$, u is upper semicontinuous and l is lower semicontinuous, there exists a continuous $f\colon X \to \mathbb{R}$ such that $u \leq f \leq l$.*

We have also the dual result ([4]):

Theorem 2 (Lane)**.** *A space X is extremaly disconnected if and only if, whenever $l, u\colon X \to \mathbb{R}$ are such that $l \leq u$, l is lower semicontinuous and u is upper semicontinuous, there exists a continuous $f\colon X \to \mathbb{R}$ such that $l \leq f \leq u$.*

For insertion of strict type we have the following two schemes:

The strict insertion scheme.

(SI) Given $u \in \mathbf{U}$ and $l \in \mathbf{L}$ with $u < l$, there is $f \in \mathbf{C}$ such that $u < f < l$ [here and elsewhere $u < l$ means $u(x) < l(x)$ for all $x \in X$].

The ultra-strict insertion scheme.

(USI) Given $u \in \mathbf{U}$ and $l \in \mathbf{L}$ with $u \leq l$, there is $f \in \mathbf{C}$ such that $u \leq f \leq l$ and $u(x) < f(x) < l(x)$ whenever $u(x) < l(x)$.

Classical illustrations of schemes **(SI)** and **(USI)** are provided by the following two theorems of Dowker [2] and Michael [6].

Theorem 3 (Dowker)**.** *A space X is normal and countably paracompact if and only if, whenever $u, l\colon X \to \mathbb{R}$ are such that $u < l$, u is upper semicontinuous and l is lower semicontinuous, there exists a continuous $f\colon X \to \mathbb{R}$ such that $u < f < l$.*

Theorem 4 (Michael)**.** *A space X is perfectly normal if and only if, whenever $u, l\colon X \to \mathbb{R}$ are such that $u \leq l$, u is upper semicontinuous and l is lower semicontinuous, there exists a continuous $f\colon X \to \mathbb{R}$ such that $u \leq f \leq l$ and $u(x) < f(x) < l(x)$ whenever $u(x) < l(x)$.*

It is now evident that it is countable paracompactness (in Theorem 3) and perfectness (in Theorem 4) which are responsible for the strictness and ultra-strictness of the insertion.

It is therefore of interest to know how insertion theorems for countably paracompact spaces or perfect spaces (without normality) may look like.

The case of countable paracompactness was treated already in 1970 by Mack [5] who proved the following.

Theorem 5 (Mack)**.** *A space X is countably paracompact if and only if, given a lower semicontinuous function $l\colon X \to (0, +\infty)$ there exist $l_1, u\colon X \to (0, \infty)$ with l_1 lower semicontinuous and u upper semicontinuous such that $0 < l_1 \leq u < l$.*

In fact, Mack has $0 < l_1 \leq u \leq l$. It has been observed very recently by Xie and Yan [8, Theorem 2.3] that Mack's result continues to hold if one requires $0 < l_1 \leq u < l$.

In the class of normal spaces, paracompactness is equivalent to metacompactness, and in this respect there is another result in [8], viz.:

Theorem 6 (Xie-Yan)**.** *A space X is countably metacompact if and only if, given a lower semicontinuous function $l\colon X \to (0, +\infty)$ there exists an upper semicontinuous $u\colon X \to (0, \infty)$ such that $0 < u < l$.*

The case of perfectness has recently been treated by Yan and Yang [9] who proved the following (we recall that a space X is perfectly normal if it is normal and perfect, i.e. closed sets are G_δ-sets or open sets are F_σ-sets).

Theorem 7 (Yan-Yang)**.** *A space X is perfect if and only if, given a lower semicontinuous function $l\colon X \to [0, +\infty)$ there exist an upper semicontinuous $u\colon X \to [0, \infty)$ such that $0 \leq u \leq l$ and $0 < u(x) < l(x)$ whenever $l(x) > 0$.*

In Theorems 6 and 7, u is inserted between the constant function with value 0 and the function l. As it will be seen this restriction is unnecessary. Moreover, it is possible to strengthen the insertion conditions of Theorem 5 and 6 according to the following schemes:

The strict double insertion scheme.

(SDI) Given $u \in \mathbf{U}$ and $l \in \mathbf{L}$ with $u < l$, there are $l_1 \in \mathbf{L}$ and $u_1 \in \mathbf{U}$ such that $u < u_1 \leq l_1 < l$.

The ultra-strict double insertion scheme.

(USDI) Given $u \in \mathbf{U}$ and $l \in \mathbf{L}$ with $u \leq l$, there are $u_1 \in \mathbf{U}$ and $l_1 \in \mathbf{L}$ such that $u \leq u_1 \leq l_1 \leq l$ and $u(x) < u_1(x)$ and $l_1(x) < l(x)$ whenever $u(x) < l(x)$.

It is now obvious to check that we have the following implications:

$$\textbf{(SDI)} + \textbf{(I)} \implies \textbf{(SI)}$$
$$\textbf{(USDI)} + \textbf{(I)} \implies \textbf{(USI)}$$

In our talk we will explain how these schemes can be used to provide simpler proofs of some well-known results and also to obtain new results.

For example, in the particular case when X is a topological space, and **U**, **L**, and **C** consist of all upper semicontinuous, lower semicontinuous, and continuous real-valued functions on X, we have the following results:

Theorem 8. *A space X is countably metacompact if and only if, whenever $u, l\colon X \to \mathbb{R}$ are such that $u < l$, u is upper semicontinuous and l is lower semicontinuous, there exist $u_1, l_1\colon X \to \mathbb{R}$ with u_1 upper semicontinuous and l_1 lower semicontinuous such that $u < u_1 \le l_1 < l$.*

Theorem 9. *A space X is perfect if and only if, whenever $u, l\colon X \to \mathbb{R}$ are such that $u \le l$, u is upper semicontinuous and l is lower semicontinuous, there exist $u_1, l_1\colon X \to \mathbb{R}$ with u_1 upper semicontinuous and l_1 lower semicontinuous such that $u \le u_1 \le l_1 \le l$ and $u(x) < u_1(x)$ and $l_1(x) < l(x)$ whenever $u(x) < l(x)$.*

The results of Dowker (Theorem 3) and Michael (Theorem 4) follow immediately from Theorems 8 and 9 and Theorem 1.

On the other hand, we can also use these insertion results to obtain some extension results:

Corollary 10. *A space X is perfect if and only if for every closed $A \subset X$ and every continuous $f\colon A \to [0,1]$ there exist two extensions $u, l\colon X \to [0,1]$ of f such that $u \le l$, u is upper semicontinuous, l is lower semicontinuous and $0 < u(x)$ and $l(x) < 1$ whenever $x \in X \setminus A$.*

Once again, we can combine Corollary 10 with Theorem 1 in order to obtain the following Tietze-type theorem.

Corollary 11. *A space X is perfectly normal if and only if for every closed $A \subset X$ and every continuous $f\colon A \to [0,1]$ there exists a extension $F\colon X \to [0,1]$ of f such that $u \le l$, u is upper semicontinuous, l is lower semicontinuous and $0 < F(x) < 1$ whenever $x \in X \setminus A$.*

Among the new results we will present we have the dual versions of Theorems 8 and 9. We first note that the dual notions of perfectness and countably metacompactness are the following:

A topological space X is called *almost discrete* if every open set is clopen. X is called *cocountably metacompact* if for every non-decreasing sequence $\{G_n\}_{n\in\mathbb{N}}$ of closed subsets satisfying $\bigcup_{n\in\mathbb{N}} G_n = X$ there is a non-decreasing sequence $\{W_n\}_{n\in\mathbb{N}}$ of open sets such that $W_n \subset G_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} W_n = X$.

Theorem 12. *A space X is cocountably metacompact if and only if, whenever $l, u\colon X \to \mathbb{R}$ are such that $l < u$, l is lower semicontinuous and u is upper semicontinuous, there exist $l_1, u_1\colon X \to \mathbb{R}$ with l_1 lower semicontinuous and u_1 upper semicontinuous such that $l < l_1 \leq u_1 < u$.*

Theorem 13. *A space X is almost discrete if and only if, whenever $l, u\colon X \to \mathbb{R}$ are such that $l \leq u$, l is lower semicontinuous and u is upper semicontinuous, there exist $l_1, u_1\colon X \to \mathbb{R}$ with l_1 lower semicontinuous and u_1 upper semicontinuous such that $l \leq u_1 \leq u_1 \leq u$ and $l(x) < l_1(x)$ and $u_1(x) < u(x)$ whenever $l(x) < u(x)$.*

Corollary 14. *A space X is extremally disconnected and cocountably metacompact if and only, whenever $l, u\colon X \to \mathbb{R}$ are such that $l < u$, l is lower semicontinuous and u is upper semicontinuous, there exists a continuous $f\colon X \to \mathbb{R}$ such that $l < f < u$.*

Corollary 15. *A space X is almost discrete if and only if, whenever $l, u\colon X \to \mathbb{R}$ are such that $l \leq u$, l is lower semicontinuous and u is upper semicontinuous, there exists a continuous $f\colon X \to \mathbb{R}$ such that $l \leq f \leq u$ and $l(x) < f(x) < u(x)$ whenever $l(x) < u(x)$.*

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