The metric and compact hedgehogs pointfreely

(and cardinal generalizations of normality)

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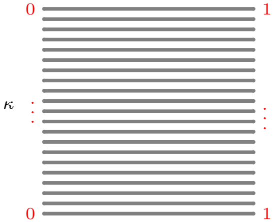
[1] Joint work with I. Arrieta, I. Mozo Carollo, J. Picado, and J. Walters-Wayland.

- J.G.G., I. Mozo Carollo, J. Picado, J. Walters-Wayland,
  Hedgehog frames and a cardinal extension of normality,
  J. Pure Appl. Algebra 23 (2019)

- I. Arrieta, J.G.G., J. Picado,
  Frame presentations of compact hedgehogs and their properties,
  Quaestiones Mathematicae (2022) in press.

Let I be a set of cardinality κ and consider the disjoint union $\bigcup_{i\in I}[0,1]\times\{i\}$ of κ copies of the real unit interval.



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Now we identify all the copies (the spines) of the real unit interval at the origin and obtain the hedgehog $J(\kappa)$.

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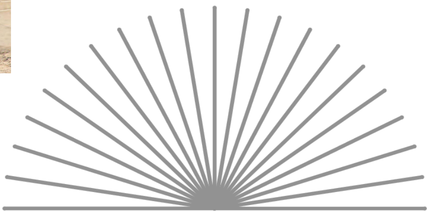

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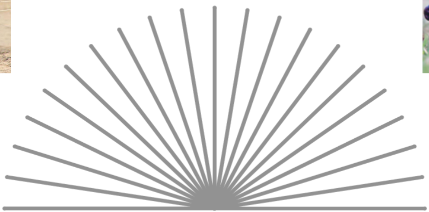

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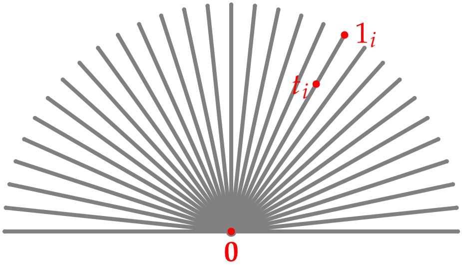
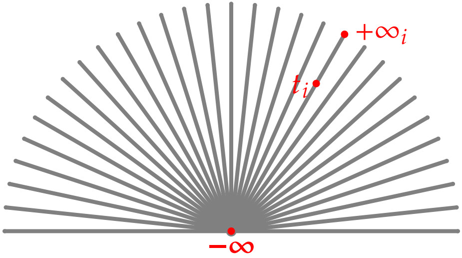


We denote by $\mathbf{0}$ the class of all elements of the form $(0,i)$ and by t_i the class of the element (t,i) (for $t\in(0,1]$).

Obviously, we can also perform precisely the same construction starting with the extended real line instead of the unit interval.

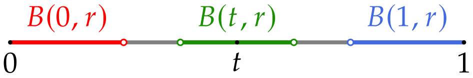


We denote by $-\infty$ the class of all elements of the form $(-\infty, i)$ and by t_i the class of the element (t, i) (for $t \in (0, +\infty]$).

The hedgehog

The usual topology on the unit interval $[0, 1]$ can naturally be introduced in two completely different ways:

(1) It is the metric topology induced by the euclidean metric on $[0, 1]$.



(Base)

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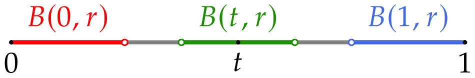
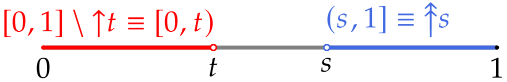


(Base)

(2) It is the Lawson topology induced by the linear order on $[0,1]$.



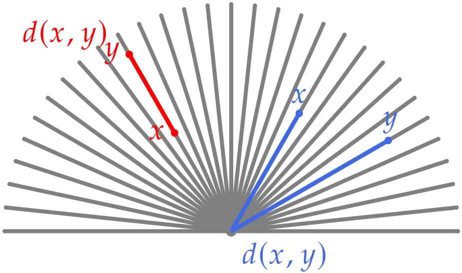
(Subbase)

These two approaches can be used to topologize the hedgehog, but in contrast with the case of unit interval $[0,1]$, they induce two different topological spaces.

The first (metric) approach is the best known:

(1) The natural extension of the euclidean metric to a metric on $J(\kappa)$ is given by

$$d(x,y) = \begin{cases} |t-s|, & \text{if } x = t_i \text{ and } y = s_i, \\ t+s, & \text{if } x = t_i \text{ and } y = s_j \text{ with } j \neq i. \end{cases}$$



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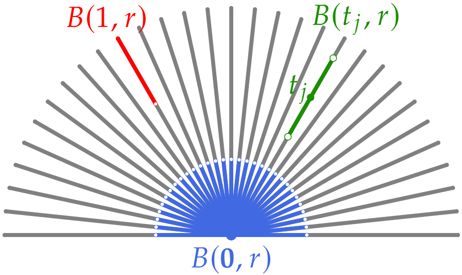

The first (metric) approach is the best known:

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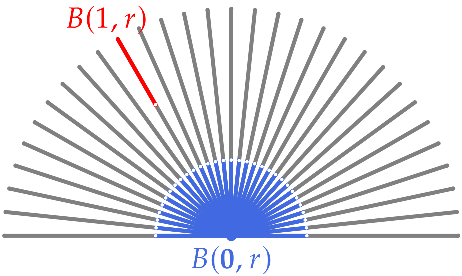
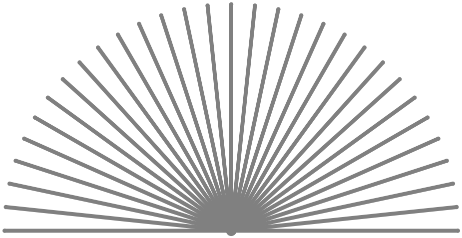
$$\{B(\mathbf{0}, r) \mid r \in \mathbb{Q} \cap (0,1)\} \cup \{B(1, r) \mid r \in \mathbb{Q} \cap (0,1) \text{ and } i \in I\}$$

form a subbase of τ_{metric}.



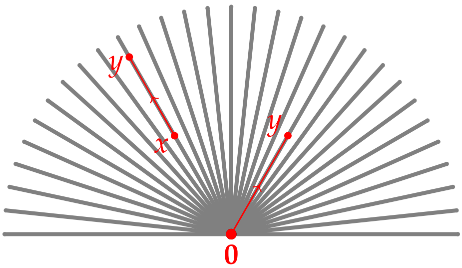
The compact hedgehog

The order approach is not so well known, but it is of particular interest when one is interested in order-theoretic notions like upper and lower semicontinuity.



(2) There exists a natural partial order on $J(\kappa)$ given by

$$x \leq y \iff x = \mathbf{0} \quad \text{or} \quad (x = t_i,\, y = s_i \text{ and } t \leq s)$$



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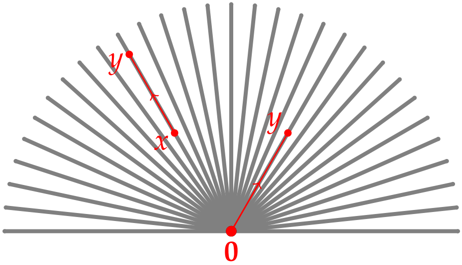
$(J(\kappa), \leq)$ is a bounded complete domain (= complete continuous semilattice).



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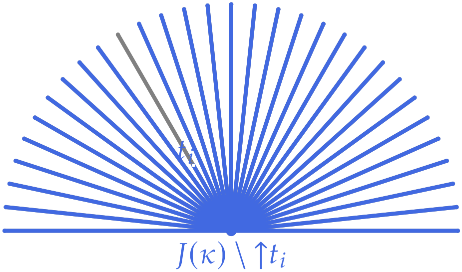
Hence the subsets of the form

$$J(\kappa) \setminus {\uparrow} t_i = \{s_j \in J(\kappa) \mid t_i \not\leq s_j\}$$

form a subbase for the lower topology.



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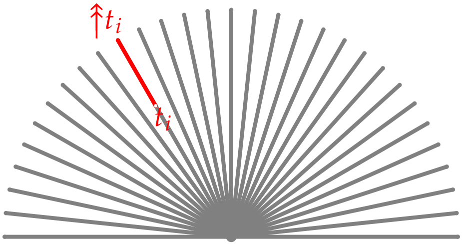
Hence the subsets of the form

$$\Uparrow t_i = \{s_j \mid t_i \ll s_j\}$$

form a subbase for the Scott topology.



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Hence the subsets of the form

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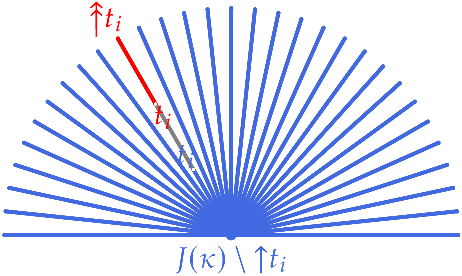
form a subbase for the Lawson topology τ_{Lawson}.



The metric and the compact hedgehog

– For κ finite both topologies coincide, but in general the metric topology is finer than the Lawson topology.

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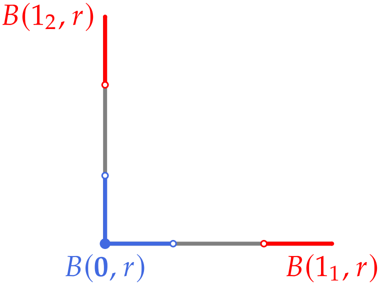

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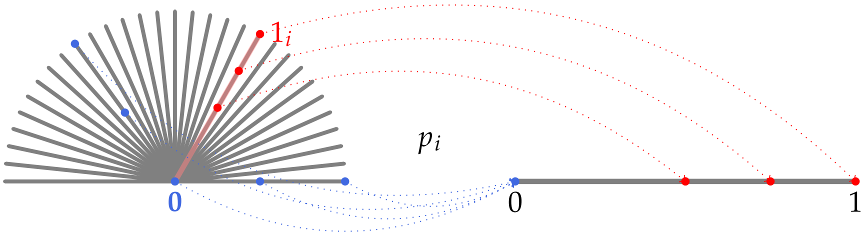
- $(J(\kappa), \tau_{\text{Lawson}})$ is metrizable if and only if $\kappa \leq \aleph_0$.

The projections

There are two different types of projections from the hedgehog onto the extended real line playing a key role in the study of the hedgehog:

(1) For each $i \in I$ the i-th projection $p_i \colon J(\kappa) \longrightarrow ([0,1], \tau_u)$



$$p_i(t_j) = \begin{cases} t, & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases}$$

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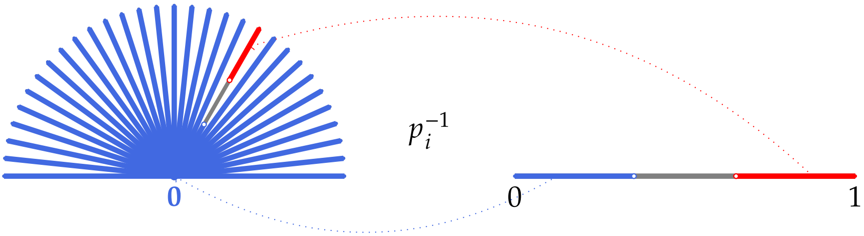

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Given a continuous map $f \colon (X, \tau_X) \longrightarrow (J(\kappa), \tau_{\text{Lawson}})$, the composition $\pi_i \circ f \colon (X, \tau_X) \longrightarrow ([0,1], \tau_u)$ is continuous for each $i \in I$.

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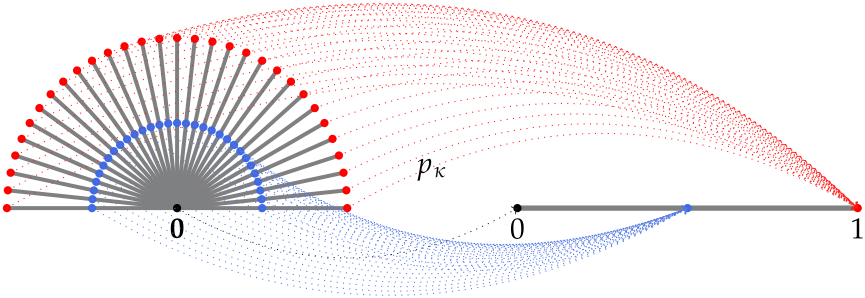
Consequently, the family

$$\left\{(\pi_i \circ f)^{-1}((0,1])\right\}_{i \in I}$$

is a pairwise disjoint family of cozero sets in X.

The projections

(2) The global projection $p_\kappa \colon J(\kappa) \longrightarrow ([0,1], \tau_u)$



$$p_\kappa(t_j) = t = \bigvee_{i \in I} p_i(t_j)$$

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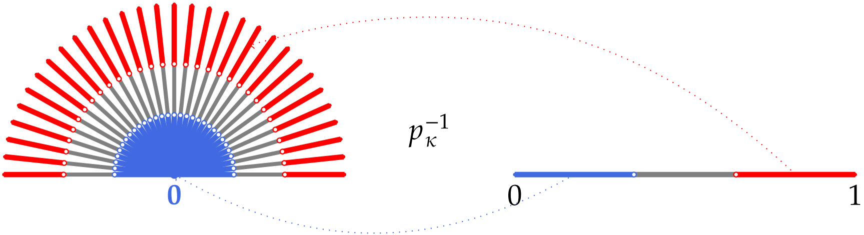

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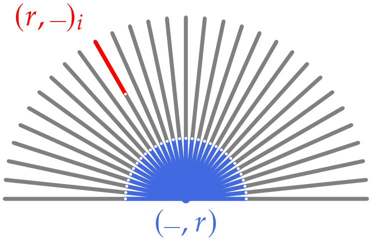
We will say that $\left\{(\pi_i \circ f)^{-1}((0,1])\right\}_{i\in I}$ is a κ-family of cozero sets in X

The frames of the metric and the compact hedgehogs

One of the differences between point-set topology and pointfree topology is that one may present frames by generators and relations.

In particular, we may now present frames of the metric and compact hedgehogs by using generators and relations; without any notion of real number involved.

The frame of the metric hedgehog with κ spines is the frame $\mathfrak{L}(J(\kappa))$ presented by generators $(r,\text{—})_i$ and $(\text{—},r)$ for $r \in \mathbb{Q}$ and $i \in I$, subject to the defining relations:



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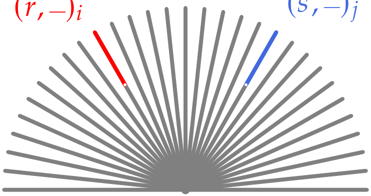

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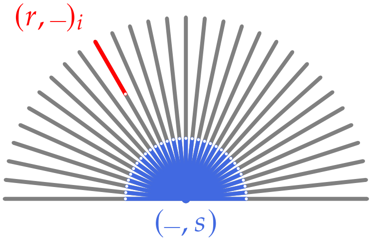
(h1) $(r,\text{–})_i \wedge (\text{–},s) = 0$ whenever $r \geq s$ and $i \in I$,



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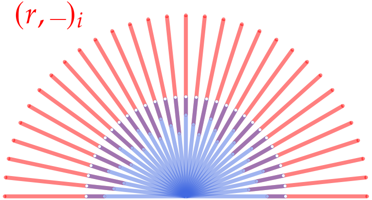

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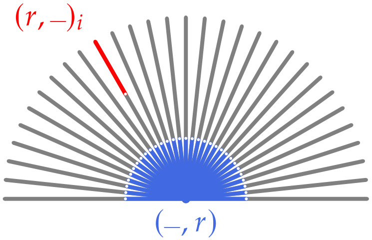
(h3) $(r,\text{--})_i = \bigvee_{s>r} (s,\text{--})_i$, for every $r \in \mathbb{Q}$ and $i \in I$,



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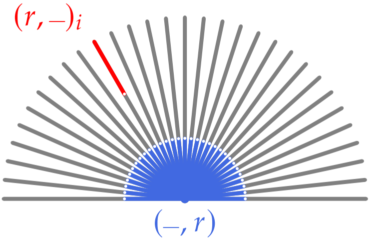
(h4) $(\text{–},r) = \bigvee_{s<r}(\text{–},s)$, for every $r \in \mathbb{Q}$.



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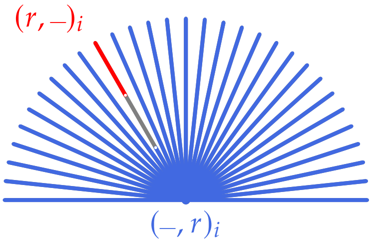
Analogously, the frame of the compact hedgehog with κ spines is the frame $\mathfrak{L}(cJ(\kappa))$ presented by generators $(r,-)_i$ and $(-,r)_i$ for $r \in \mathbb{Q}$ and $i \in I$, subject to the defining relations:



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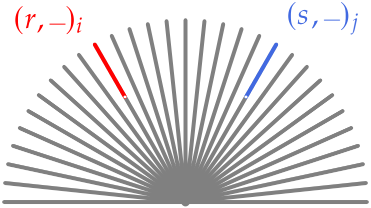

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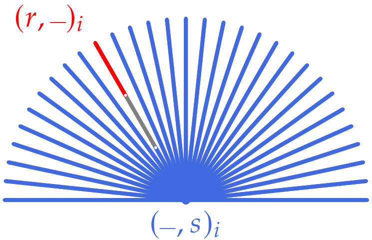

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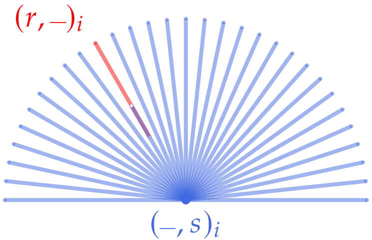

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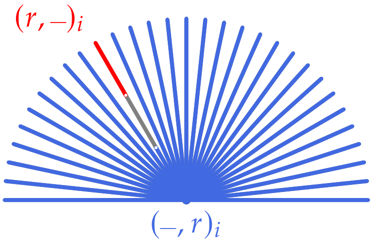

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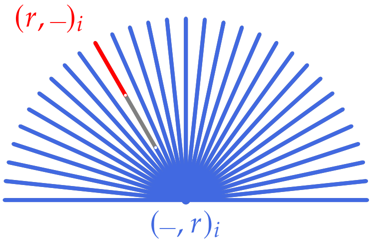

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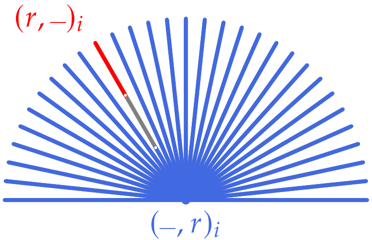

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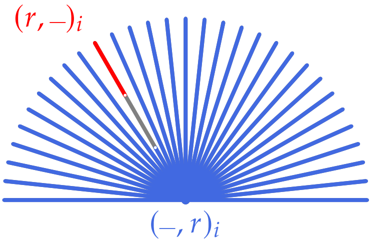

- $\mathfrak{L}(J(1)) = \mathfrak{L}(cJ(1)) = \mathfrak{L}(\overline{\mathbb{R}})$.

(1) $(r,-) \wedge (-,s) = 0$ whenever $r \geq s$,
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(3) $(r,-) = \bigvee_{s>r}(s,-)$, for every $r \in \mathbb{Q}$,
(4) $(-,r) = \bigvee_{s<r}(-,s)$, for every $r \in \mathbb{Q}$.

Consequently both frames $\mathfrak{L}(J(\kappa))$ and $\mathfrak{L}(cJ(\kappa))$ are cardinal extensions of the frame of the extended real line $\mathfrak{L}(\overline{\mathbb{R}})$.

▸ B. Banaschewski, J.G.G. and J. Picado, Extended real functions in pointfree topology, *J. Pure Appl. Algebra* 216 (2012) 905–922.

The frames of the metric and the compact hedgehogs

- $\mathfrak{L}(J(1)) = \mathfrak{L}(cJ(1)) = \mathfrak{L}(\overline{\mathbb{R}})$.
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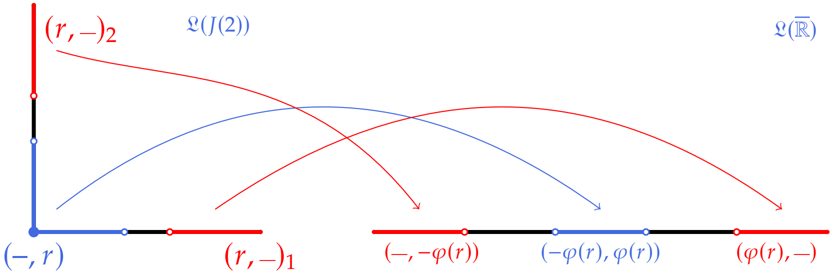
  The isomorphism is induced by the following correspondence (where φ denotes any increasing bijection between $\mathbb{Q}$ and $\mathbb{Q}^+$):



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- For $\kappa_1, \kappa_2 > 2$, if $\kappa_1 \neq \kappa_2$ then
  $$\mathfrak{L}(J(\kappa_1)) \not\simeq \mathfrak{L}(J(\kappa_2)) \quad \text{and} \quad \mathfrak{L}(cJ(\kappa_1)) \not\simeq \mathfrak{L}(cJ(\kappa_2))$$

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▸ T. Dube, A short note on separable frames, *Comment. Math. Univ. Carolin.* 37 (2012) 375–377.

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Cardinal generalizations of normality

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  A frame is κ-collectionwise normal if for any discrete family $\{F_i\}_{i\in I}$, $|I| \leq \kappa$, of closed sublocales, there is a pairwise disjoint family $\{U_i\}_{i\in I}$ of of open sublocales such that $F_i \subseteq C_i$ for all i.

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- Each metric frame is collectionwise normal.

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Normality can be characterized in terms of coz-onto frame maps:,
or, equivalently, in terms of z-embedded sublocales:

> *A frame is normal iff each of its closed quotients is coz-onto iff each of its closed sublocales is z-embedded.*

[1] T. Dube and J. Walters-Wayland, Coz-onto frame maps and some applications, *Appl. Categ. Structures* 15 (2007)

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Theorem

A locale L is κ-collectionwise normal if and only if every closed sublocale of L is z_κ-embedded.

In view of this, it is natural to introduce a further cardinal generalization of the notion of z-embedding and normality:

Cardinal extensions of normality: total collectionwise normality

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- A sublocale S of L is z^c_κ-embedded if for every pairwise disjoint κ-family of cozero elements $\{a_i\}_{i\in I}$ of S, there is a pairwise disjoint κ-family of cozero elements $\{b_i\}_{i\in I}$ of L such that $\nu_S(b_i) = a_i$ for every $i \in I$.

  The "c" in the notation stands for "compact"; the reason will be clear later.

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- A frame is totally κ-collectionwise normal if every closed sublocale is z^c_κ-embedded.
  A frame is totally collectionwise normal if it is totally κ-collectionwise normal for all κ.

Cardinal extensions of normality: total collectionwise normality

It is a reasonable cardinal extension of normality:

- Metrizable

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  Hereditary κ-collectionwise normality

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Yes. The idea is to use in each case the right cardinal generalization of the real line:

- The metric hedgehog in the case of collectionwise normality.
- The compact hedgehog in the case of total collectionwise normality.

Theorem (Urysohn's Lemma)

Let X be a topological space. TFAE:

(1) X is normal.

(2) For every disjoint closed sets F_1 and F_2, there exists a continuous $f\colon X \to \overline{\mathbb{R}}$ such that $F_1 \subseteq f^{-1}((-\infty, 0])$ and $F_2 \subseteq f^{-1}([1, +\infty))$.

Collectionwise normality and the metric hedgehog

Theorem (Localic Urysohn's Lemma)

Let L be a frame. TFAE:

(1) L is normal.

(2) For each pair $a_1, a_2 \in L$ such that $a_1 \vee a_2 = 1$, there exists a a frame homomorphism $h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $h((-,0)^*) \leq a_1$ and $h((1,-)^*) \leq a_2$.

- C.H. Dowker, D. Papert. On Urysohn's lemma. *Proc. Second Prague Topological Sympos.* 1966
- B. Banaschewski, *The real numbers in Pointfree Topology*, Textos de Matemática, Vol. 12, University of Coimbra, 1997.
- R. N. Ball, J. Walters-Wayland, C-and C^*-quotients in pointfree topology, *Diss. Math.* 412 (2002)

Theorem (Localic Urysohn's Lemma)

Let L be a frame. TFAE:

(1) L is normal.

(2) For each pair $a_1, a_2 \in L$ such that $a_1 \vee a_2 = 1$, there exists a a frame homomorphism $h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that $h((\text{–}, 0)^*) \leq a_1$ and $h((1, \text{–})^*) \leq a_2$.

Theorem (Cardinal extension)

Let L be a frame. TFAE:

(1) L is κ-collectionwise normal.

(2) For each co-discrete system $\{a_i\}_{i\in I}$, $|I| \leq \kappa$, there exists a a frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $h((0, \text{–})_i^*) \leq a_i$ for each $i \in I$.

Theorem (Tietze)

Let X be a topological space. TFAE:

(1) X is normal.

(2) For each closed subset F of X, each continuous $f\colon F \longrightarrow \overline{\mathbb{R}}$ has an extension to X.

Theorem (Localic Tietze)

Let L be a frame. TFAE:

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Collectionwise normality and the compact hedgehog

Theorem (Localic Tietze)

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Theorem (Cardinal extension)

Let L be a frame. TFAE:

(1) L is totally κ-collectionwise normal.

(2) For each closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(cJ(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.

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Yes (for total collectionwise normal locales).

In this case we have to use suitable notions of upper/lower semicontinuity, and the right cardinal generalization of the real line is the compact hedgehog.

A real-valued

- function on L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \longrightarrow \mathcal{S}(L)^{op}$;

▸ J. G.G., T. Kubiak, and J. Picado, Localic real functions: a general setting, *J. Pure Appl. Algebra.* 213 (2009)

Semicontinuous functions

A real-valued

- function on L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$;
- lower semicontinuous function on L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$ such that $f(r,\_)$ is a closed sublocale for every $r \in \mathbb{Q}$;

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- upper semicontinuous function on L is a frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$ such that $f(\text{–},r)$ is closed for every $r \in \mathbb{Q}$;

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Semicontinuous functions

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- upper semicontinuous function on L is a frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)^{op}$ such that $f(\text{–}, r)$ is closed for every $r \in \mathbb{Q}$;
- continuous function is a function which is both upper and lower semicontinuous.

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- continuous function is a function which is both upper and lower semicontinuous.

The corresponding classes of compact hedgehog-valued functions will be denoted by, respectively, LSC(L), USC(L) and C(L).

Semicontinuous functions

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- upper semicontinuous function on L is a frame homomorphism $f\colon \mathfrak{L}(cJ(\kappa)) \longrightarrow \mathcal{S}(L)^{op}$ such that $f((\text{—},r)_i)$ is closed for every $r \in \mathbb{Q}$;

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- upper semicontinuous function on L is a frame homomorphism $f\colon \mathfrak{L}(cJ(\kappa)) \longrightarrow \mathcal{S}(L)^{op}$ such that $f((\text{–},r)_i)$ is closed for every $r \in \mathbb{Q}$;
- continuous function is a function which is both upper and lower semicontinuous.

The corresponding classes of extended real-valued functions will be denoted by, respectively, $\mathsf{LSC}_\kappa(L)$, $\mathsf{USC}_\kappa(L)$ and $\mathsf{C}_\kappa(L)$.

Theorem (Katetov-Tong's Theorem)

Let X be a topological space. TFAE:

(1) X is normal.

(2) For every $f \in \mathsf{USC}(X, \mathbb{R})$ and $g \in \mathsf{LSC}(X, \mathbb{R})$ such that $f \leq g$, there exists $h \in \mathsf{C}(X, \mathbb{R})$ such that $f \leq h \leq g$.

Theorem (Localic Katetov-Tong's Theorem)

Let L be a frame. TFAE:

(1) L is normal.

(2) For every $f \in \mathrm{USC}(L)$ and $g \in \mathrm{LSC}(L)$ such that $f \leq g$, there exists $h \in \mathrm{C}(L)$ such that $f \leq h \leq g$.

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Theorem (Localic Katetov-Tong's Theorem)

Let L be a frame. TFAE:

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Theorem

Let L be a frame. TFAE:

(1) L is normal.

(2) For every κ, and every $f \in \mathsf{USC}_\kappa(L)$ and $g \in \mathsf{LSC}_\kappa(L)$ such that $f \leq g$, there exists $h \in \mathsf{C}_\kappa(L)$ such that $f \leq h \leq g$.

Theorem (Localic Katetov-Tong's Theorem)

Let L be a frame. TFAE:

(1) L is normal.

(2) For every $f \in \mathsf{USC}(L)$ and $g \in \mathsf{LSC}(L)$ such that $f \leq g$, there exists $h \in \mathsf{C}(L)$ such that $f \leq h \leq g$.

Theorem (Cardinal extension)

Let L be a frame. TFAE:

(1) L is totally κ-collectionwise normal.

(2) For each $a \in L$ and every $f \in \mathsf{USC}_\kappa(\mathfrak{c}(a))$ and $g \in \mathsf{LSC}_\kappa(\mathfrak{c}(a))$ such that $f \leq g$, there exists an $\overline{h} \in \mathsf{C}_\kappa(L)$ such that $f \leq \nu_{\mathfrak{c}(a)} \circ \overline{h} \leq g$.

Thank you for your attention!