Proceedings of the Workshop in Applied
Topology WiAT'09, pp. 111 – 117

Separation Theorems for Lattice-valued Functions on Preordered Topological Spaces

Javier Gutiérrez García, Iraide Mardones-Pérez and María Ángeles de Prada Vicente [1]

Departamento de Matemáticas, Universidad de País Vasco (UPV-EHU), 48080 Bilbao, Spain
(javier.gutierrezgarcia@ehu.es, iraide.mardones@ehu.es, mariangeles.deprada@ehu.es)

Abstract

In this work, we study the possibility of inserting an increasing continuous lattice-valued function between two comparable semicontinuous functions on a preordered topological space. Depending on the monotonicity conditions imposed on the semicontinuous functions, new characterizations of different classes of preordered topological spaces are obtained. Among them are characterizations of normally preordered and extremally preorder-disconnected spaces. Conditions for the continuous and increasing extension of lattice-valued maps of the same type are also investigated.

1. Introduction

The insertion of continuous real-valued functions between pairs of comparable functions has been a useful tool to characterize separation and countability properties of topological spaces, where different assumptions on the functions lead to characterizations of different topological properties. Classical results are, for example, Katětov-Tong's characterization of normal spaces [7, 15], Dowker's characterization [1] of normal and countably paracompact spaces and Michael's [9] characterization of perfectly normal spaces.

Soon after the work of Nachbin [10] on (pre)ordered topological spaces, Priestley [12] studied Katětov-Tong type insertion theorems in this new context, and characterized normally ordered spaces in these terms. Recently, Edwards [4] investigated insertion properties of real-valued functions that surround extremally preorder-disconnected spaces. All these results reduce to classical insertion theorems from general topology when the (pre)order is trivial.

Clearly, every insertion theorem necessitates the range space to be endowed with a partial order. Influenced in part by other areas of mathematics and also by computer science, functions with values in a more general poset rather than in the real line have deserved attention, such as, for example, functions with values in a partially ordered vector lattice, a $\triangleleft$-separable completely distributive lattice, a complete domain... Insertion theorems for functions with values in a $\triangleleft$-separable completely distributive lattice were

[1]This research was supported by the Ministry of Education and Science of Spain and FEDER under grant MTM2006-14925-C02-02 and by the UPV-EHU under grant GIU07/27.

studied, among others, by Gutiérrez García, Kubiak and de Prada Vicente in [6]. This work deserves special mention because the techniques developed in that paper will be crucial in our study.

The present work deals with lattice-valued insertion on preordered topological spaces. We shall study the possibility of inserting a lattice-valued continuous increasing function between comparable semicontinuous functions defined on preordered topological spaces. The techniques established in [6] will allow us to give lattice-valued counterparts of some results for real-valued functions on preordered topological spaces given in [12] and [4]. First, we shall investigate conditions under which the desired insertion takes place between a pair of comparable upper and lower semicontinuous functions. Depending on the monotonicity conditions imposed on the initial semicontinuous functions, new different characterizations (in terms of insertion) of normal-type preordered topological spaces will be given, such as, for example, normally preordered spaces. Then, we shall undertake the dual study, i.e., we shall investigate conditions which allow to insert a continuous and increasing lattice-valued function between a pair of comparable lower and upper semicontinuous functions. A new characterization of extremally preorder-disconnected spaces will be given on the way. Finally, extension properties of lattice-valued functions on preordered spaces will also be studied.

This presentation is a summary of the work, and therefore, only main results will be stated, and their proofs omitted. The entire work will be published elsewhere.

2. Preliminaries

2.1. **Lattices.** In the sequel L denotes a completely distributive lattice (with bounds 0 and 1).

We shall not use here the equational characterization of complete distributivity. Alternatively, we shall use the description of complete distributivity in terms of an extra order with the approximation property [11]:

Given a complete lattice L and $a, b \in L$, we write $a \lhd b$ if and only if, whenever $A \subset L$ and $b \leq \bigvee A$, there is $c \in A$ with $a \leq c$. The lattice L is then *completely distributive* if and only if $\lhd$ has the approximation property, i.e., $a = \bigvee\{b \in L : b \lhd a\}$ for each $a \in L$. We shall use the following properties of the extra order: (1) $a \lhd b$ implies $a \leq b$; (2) $c \leq a \lhd b \leq d$ implies $c \lhd d$; (3) $a \lhd b$ implies $a \lhd c \lhd b$ for some $c \in L$ (Interpolation Property).

We shall need yet another extra order in L defined as follows: if $a, b \in L$, then $a \blacktriangleleft b$ if and only if, whenever $\bigwedge A \leq a$ for some $A \subset L$, there exists a $c \in A$ with $c \leq b$. Clearly enough, the extra order $\blacktriangleleft$ has the dual properties (1), (2) and (3), and a lattice L is completely distributive if and only if $a = \bigwedge\{b \in L : a \blacktriangleleft b\}$ for every $a \in L$.

A subset $D \subset L$ is called *join-dense* (or a *base*) if $a = \bigvee\{d \in D : d \leq a\}$ for each $a \in L$. An element $a \in L$ is called *supercompact* if $a \lhd a$ holds. As in [6], any completely distributive lattice which has a countable join-dense subset free of supercompact elements will be called $\lhd$-*separable.*

2.2. **Semicontinuous lattice-valued functions.** Given a set X, L^X denotes the complete lattice of all maps from X into L ordered pointwisely, i.e., $f \leq g$ in L^X if and only if $f(x) \leq g(x)$ in L for each $x \in X$. Given $f \in L^X$ and $a \in L$, we write $[f \geq a] = \{x \in X : a \leq f(x)\}$ and similarly for $[f \leq a]$, $[f \rhd a]$ and $[f \blacktriangleleft a]$.

For any topological space X and any $f \in L^X$ let

$$f_*(x) = \bigvee_{U \in \mathcal{N}_x} \bigwedge_{y \in U} f(y) \quad \text{and} \quad f^*(x) = \bigwedge_{U \in \mathcal{N}_x} \bigvee_{y \in U} f(y)$$

where $\mathcal{N}_x$ is the family of all open neighborhoods of x. It is said that f is *lower* [*upper*] *semicontinuous* if and only if $f = f_*$ [$f = f^*$] (cf. [6, 8, 14]).

The collections of all lower and upper semicontinuous functions of L^X will be denoted by $LSC(X, L)$ and $USC(X, L)$, respectively. Elements of $C(X, L) = LSC(X, L) \cap USC(X, L)$ are called *continuous*.

We shall need the following characterizations of semicontinuity (cf. [6]):

(P3) $f \in USC(X, L)$ iff $[f \geq a]$ is closed in X for each $a \in L$,
iff $[f \blacktriangleleft a]$ is open for each $a \in L$.

(P4) $f \in LSC(X, L)$ iff $[f \leq a]$ is closed for each $a \in L$,
iff $[f \rhd a]$ is open for each $a \in L$.

2.3. **Preorders.** Let X be a set. By a preorder on X we understand a reflexive and transitive relation on X. If X is a preordered topological space, the preorder is called *closed* if its graph is closed in $X \times X$.

If $(X, \preceq)$ is preordered, $A \subseteq X$ is said to be *increasing* if $x \preceq y$ together with $x \in A$ imply $y \in A$. Similarly, A is said to be *decreasing* if $y \preceq x$ together with $x \in A$ implies $y \in A$. Note that A is increasing if and only if $X - A$ is decreasing.

For any $x \in X$ and $A \subseteq X$ we standardly write $\uparrow x = \{y \in X : x \preceq y\}$, $\downarrow x = \{y \in X : y \preceq x\}$, $\uparrow A = \bigcup_{x \in A} \uparrow x$ and $\downarrow A = \bigcup_{x \in A} \downarrow x$. Note that $A \subseteq X$ is increasing (resp. decreasing) if and only if $\uparrow A = A$ (resp. $\downarrow A = A$) .

A map $f \in L^X$ is called *increasing* if $f(x) \leq f(y)$ whenever $x \preceq y$.

3. Separating upper and lower semicontinuous functions

The objective of this section is to characterize different kinds of normal-type ordered topological spaces in terms of insertion properties of lattice-valued functions. Therefore, the results in this section will extend the ones obtained by Priestley in [12] to lattice-valued context.

We first recall some normality type axioms on preordered topological spaces which come from [10] and [12].

A preordered topological space X is said to be *normally preordered* if for any disjoint closed sets F_1 and F_2, decreasing and increasing, respectively, there exist two disjoint open sets U_1 and U_2, decreasing and increasing, respectively, such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

A preordered topological space X is said to be an *N_i-space* if for any disjoint closed sets F_1 and F_2, such that F_2 is increasing, there exist two disjoint open sets U_1 and U_2, decreasing and increasing, respectively, such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

A preordered topological space X is said to be an *N_d-space* if for any disjoint closed sets F_1 and F_2, such that F_1 is decreasing, there exist two disjoint open sets U_1 and U_2, decreasing and increasing, respectively, such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$.

Clearly, N_i-spaces and N_d-spaces are normally preordered.

A preordered topological space X is said to be an *N-space* if given any closed sets F_1 and F_2 such that $x_2 \not\preceq x_1$ for any $x_1 \in F_1$ and $x_2 \in F_2$, there exist disjoint open sets U_1

and U_2, decreasing and increasing, respectively, such that $F_1 \subseteq U_1$ and $F_2 \subseteq U_2$. It was shown by Priestley [12] that a preordered topological space X is an N-space iff it is both an N_i-space and an N_d-space and the preorder is closed.

As we said before, the objective is to characterize, in terms of insertion, the preordered topological spaces satisfying the normality type axioms stated in the beginning of the section. This insertion is going to be made in between a comparable upper and lower semicontinuous function. Depending on wether we assume that both/a fixed one/one/none of the semicontinuous functions we wish to separate are/is increasing, the right kind of spaces in which the insertion of an increasing continuous function takes place changes. This becomes apparent in the following results.

Theorem 1. *Let $(X, \preceq)$ be a preordered topological space and L be a $\lhd$-separable completely distributive lattice. The following statements are equivalent:*

(1) X is normally preordered;
(2) For any $f \in USC(X, L)$ and $g \in LSC(X, L)$, both increasing, such that $f \leq g$, there exists an increasing $h \in C(X, L)$ such that $f \leq h \leq g$.
(3) For any $f \in USC(X, L)$ and $g \in LSC(X, L)$, both increasing, such that $f \leq g$, there exist $l \in LSC(X, L)$ and $u \in USC(X, L)$, both increasing, such that $f \leq l \leq u \leq g$.

Theorem 2. *Let $(X, \preceq)$ be a preordered topological space and L be a $\lhd$-separable completely distributive lattice. The following statements are equivalent:*

(1) X is an N_i-space;
(2) For any $f \in USC(X, L)$ and $g \in LSC(X, L)$ such that $f \leq g$ and f is increasing, there exists an increasing $h \in C(X, L)$ such that $f \leq h \leq g$.
(3) For any $f \in USC(X, L)$ and $g \in LSC(X, L)$ such that $f \leq g$ and f is increasing, there exist $h \in LSC(X, L)$ increasing such that $f \leq h \leq h^* \leq g$.

Theorem 3. *Let $(X, \preceq)$ be a preordered topological space and L be a $\lhd$-separable completely distributive lattice. The following statements are equivalent:*

(1) X is an N_d-space;
(2) For any $f \in USC(X, L)$ and $g \in LSC(X, L)$ such that $f \leq g$ and g is increasing, there exists an increasing $h \in C(X, L)$ such that $f \leq h \leq g$.
(3) For any $f \in USC(X, L)$ and $g \in LSC(X, L)$ such that $f \leq g$ and g is increasing, there exist an increasing $h \in USC(X, L)$ such that $f \leq h_* \leq h \leq g$.

Corollary 4. *Let $(X, \preceq)$ be a preordered topological space and L be a $\lhd$-separable completely distributive lattice. If X is an N-space, then for any $f \in USC(X, L)$ and $g \in LSC(X, L)$ such that $f \leq g$ and f or g is increasing, there exists an increasing $h \in C(X, L)$ such that $f \leq h \leq g$.*

Moreover, if $\preceq$ is a closed preorder, the converse holds.

We now consider the situation in which none of the semicontinuous functions f, g is assumed to be increasing. It is clear that if one wants to insert a continuous increasing h in between, the minimun requirement that the maps f and g have to satisfy is that $f(x) \leq g(y)$ whenever $x \preceq y$. We shall see that, as for the case of real-valued functions [12], this minimum requirement is also sufficient when we have N-spaces.

Theorem 5. *Let $(X, \preceq)$ be an N-space and L be a $\lhd$-separable completely distributive lattice. Let $f \in USC(X, L)$ and $g \in LSC(X, L)$ be such that $f(x) \leq g(y)$ whenever $x \preceq y$. Then, there exists an increasing $h \in C(X, L)$ such that $f \leq h \leq g$.*

Remark 6. It is worth noting that the previous theorem also holds for normally preordered spaces which satisfy the additional condition:

$$F \text{ closed } \Rightarrow \uparrow F \text{ and } \downarrow F \text{ are also closed.} \qquad (\star)$$

Note that any N-space is, in particular, a normally preordered space satisfying the previous additional condition $(\star)$ (see [12]).

4. Separating lower and upper semicontinuous functions

We now turn to the dual situation and study in which kind of spaces it is possible to insert a continuous and increasing function between a pair of comparable lower and an upper semicontinuous functions.

We shall begin by imposing monotonicity conditions on both semicontinuous functions. We shall see that, in this case, the insertion property characterizes the so called extremally preorder-disconnected spaces.

A preordered topological space X is said to be *extremally preorder-disconnected* [4] if it is preordered and for any increasing open set U and any decreasing open set V of X, the sets $\overline{U}^{\uparrow}$ and $\overline{V}^{\downarrow}$ are also open (with $\overline{U}^{\uparrow}$, resp. $\overline{V}^{\downarrow}$, the smallest closed and increasing, resp. decreasing, set containing U, resp. V).

Remark 7. It is easy to show that extremally preorder-disconnected spaces are precisely those spaces characterized by the following property: For any U open increasing and F closed increasing with $U \subseteq F$ there exist G closed increasing and V open increasing such that $U \subseteq G \subseteq V \subseteq F$. This fact highlights, once again, the duality existing between normal and extremally disconnected-type topological spaces.

Theorem 8. *Let $(X, \preceq)$ be a preordered topological space and L be a $\lhd$-separable completely distributive lattice. The following statements are equivalent:*

(1) *X is extremally preorder-disconnected;*
(2) *For any $f \in LSC(X, L)$ and $g \in USC(X, L)$, both increasing, such that $f \leq g$, there exists an increasing $h \in C(X, L)$ such that $f \leq h \leq g$.*
(3) *For any $f \in LSC(X, L)$ and $g \in USC(X, L)$, both increasing, such that $f \leq g$, there exist $u \in USC(X, L)$ and $l \in LSC(X, L)$, both increasing, such that $f \leq u \leq l \leq g$.*

Remark 9. As it happened for normal-type preordered topological spaces, if we only assume that one of the semicontinuous functions is increasing, then characterizations of a stronger type of extremally preorder-disconnected spaces are obtained. We shall not include them here to avoid repetitions.

Let us finally consider the case in which none of the semicontinuous functions is assumed to be increasing. For the case of real-valued functions, this study was done by Edwards in [4].

The following theorem shows the type of spaces characterized by the insertion of a continuous and increasing function in this last situation.

Theorem 10. *Let $(X, \preceq)$ be a preordered topological space and L be a completely distributive lattice. The following statements are equivalent:*

(1) *For each open subset U of X, $\overline{\uparrow U}$ is open and increasing;*
(2) *For each $g \in LSC(X, L)$ and $f \in USC(X, L)$ such that $g(x) \leq f(y)$ whenever $x \preceq y$, there exists an increasing $h \in C(X, L)$ such that $g \leq h \leq f$.*

Remark 11. The previous theorem shows that, in this case, extremally preorder-disconnected spaces are not good enough to be characterized by the insertion property (2), since condition (1) of Theorem 10 is stronger than the extremal preorder-disconnectedness. However, as it happened in the previous section, if we add a certain topological conditions on the preorder of an extremally preorder-disconnected space, the insertion property will also be verified. This topological conditions on the preorder will once again be dual to those imposed in the case of normality.

The preorder of a topological space X is said to be *compliant* ([2, 3, 13]) if for all open $U \subseteq X$ the sets $\uparrow U$ and $\downarrow U$ are open.

Theorem 12. *Let $(X, \preceq)$ be a preordered topological space and L be a completely distributive lattice. If X is extremally preorder-disconnected and its preorder is compliant, then for each $g \in LSC(X, L)$ and $f \in USC(X, L)$ such that $g(x) \leq f(y)$ whenever $x \preceq y$, there exists an increasing $h \in C(X, L)$ such that $g \leq h \leq f$.* □

5. Extending increasing continuous real-valued functions

From the separation theorems obtained in the previous sections, it is possible to deduce easily the following Tietze-type extension theorems.

Theorem 13. *Let $(X, \preceq)$ be an N-space (or a normally preordered space satisfying condition $(\star)$) and L be a $\lhd$-separable completely distributive lattice. If F is closed in X and $f \in C(F, L)$ is increasing, then there exists an increasing $h \in C(X, L)$ such that $h|_F = f$*

Theorem 14. *Let $(X, \preceq)$ be a preordered topological space, L be a completely distributive lattice and assume that any of the equivalent conditions in Theorem 10 is satisfied. Then, if U is open in X and $f \in C(U, L)$ is increasing, there exists an increasing $h \in C(X, L)$ such that $h|_U = f$.*

Remark 15. In particular, the previous extension property holds in extremally preorder-disconnected spaces with a compliant preorder.

References

[1] C. H. Dowker, *On countably paracompact spaces*, Can. J. Math. **3** (1951), 219–232.
[2] D. A. Edwards, *On lattices of continuous functions on a Stonian space*, Math. Proc. Cambridge. Philos. Soc. **132** (2002), 169–180.
[3] D. A. Edwards, *Some characterizations of commutative subspace lattices*, Bull. London Math. Soc. **36** (2004), 252–262.
[4] D. A. Edwards, *A separation theorems for semicontinuous functions on preordered topological spaces*, Arch. Math. **84** (2005), 559–567.
[5] F. van Gool, *Lower semicontinuous functions with values in a continuous lattice*, Comment. Math. Univ. Carolinae **33** (1992), 505–523.
[6] J. Gutiérrez García, T. Kubiak, M.A. de Prada Vicente, *Insertion of lattice-valued and hedgehog-valued functions*, Topol. Appl. **153** (2006), no. 9, 1458–1475.
[7] M. Katětov, *On real-valued functions in topological spaces*, Fund. Math. **38** (1951), 85–91; Correction: Fund. Math. **40** (1953), 203–205.
[8] Y. M. Liu, M. K. Luo, *Lattice-valued Hahn-Dieudonné-Tong insertion theorem and stratification structure*, Topol. Appl. **45** (1992), 173–188.
[9] E. Michael, *Continuous selection I*, Ann. of Math. **63** (1956), 361–382.
[10] L. Nachbin, Topology and order, Van Nostrand (Princeton N. J., 1965).
[11] G. N. Raney, *A subdirect-union representation for completely distributive complete lattices*, P. Am. Math. Soc. **4** (1953), 518–522.
[12] H. A. Priestley, *Separation theorems for semi-continuous functions on normally ordered topological spaces*, J. London Math. Soc. **3** (1971), no. 2, 371–377.

[13] H. A. Priestley, *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. **24** (1972), no. 3, 507–530.

[14] D. S. Scott, Continuous lattices, in: *Lecture Notes in Math.*, vol. 274, Springer (Berlin, 1972), pp. 97–136.

[15] H. Tong, *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. **19** (1952), 289–292.