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Topological properties based on continuous extension of total preorders

Javier Gutiérrez García [a,1], *Esteban Induráin* [b,2] *and Tomasz Kubiak* [c,3]

[a] *Departamento de Matemáticas, Universidad del País Vasco-Euskal Herriko Unibertsitatea, Apdo. 644, 48080, Bilbao, Spain* (javier.gutierrezgarcia@ehu.es)

[b] *Departamento de Matemáticas, Universidad Pública de Navarra. Campus Arrosadía, 31006 Pamplona, Spain.* (steiner@unavarra.es)

[c] *Wydzial Matematyki i Informatyki, Uniwersytet im. Adama Mickiewicza, ul. Umultowska 87, 61-614 Poznan, Poland* (tkubiak@amu.edu.pl)

Abstract

We consider topological properties based on continuous extensions of total preorders. First we study extension properties based on closed subsets of topological spaces. Thus, we analyze the Yi's extension property for which any total preorder defined on a closed subset of a topological space, and continuous with respect to the relative topology, admits a continuous extension to the whole space. We show that the Yi's extension property implies the Tietze's extension property, so that it is a particular case of normality. In the second part of this paper, we study extension properties based on open subsets of topological spaces, showing that they are related to topological properties of extremal disconnectedness.

1. Introduction

A classical characterization of normality in topological spaces is the Tietze extension theorem, so that in a normal topological space a continuous real valued map defined on a closed subset has a continuous extension to the whole space.

At this stage, given a topological space (X, τ) we may observe that any continuous real-valued map $f : X \to \mathbb{R}$ immediately defines a continuous total preorder $\precsim$ on X by declaring that $x \precsim y \iff f(x) \leq f(y) \ \ (x, y \in X)$.

This obvious fact suggests to analyze a generalization of the Tietze's extension property, studying the topological spaces (X, τ) for which any total preorder defined on a closed subset A and continuous with respect to the relative topology on A, admits a continuous extension to the whole set X. This is the so-called *Yi's extension property* ([12, 3]).

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As a matter of fact Yi's extension property implies Tietze's extension property, but the converse does not hold, in general. Therefore, this new extension property of topological spaces, based on the consideration of total preorders defined on its closed subsets, is actually a more restrictive variant of the normality property.

As far as we know, Yi's property has not been identified yet as some of the classical notions related to normality.

As a partial result, it can be proved that on *metric* topological spaces that are *separably connected* it coincides with "normality plus separability", but we do not know which is the particular class of topological spaces where normality plus separability amounts to the Yi's extension property.

Another topological property related to extension of real-valued maps is the *extremal disconnectedness* (see e.g [8], p. 368). A topological space (X, τ) is extremally disconnected if and only if every bounded continuous real valued map defined on an *open* subset has a continuous extension to the whole space (see e.g. [9], pp. 22-23).

As in the case of the Yi's extension property considered above, this fact also suggests to introduce a *new topological property*, studying the topological spaces (X, τ) for which any total preorder defined on an *open* subset A and continuous with respect to the relative topology on A, admits a continuous extension to the whole set X.

The structure of the paper goes as follows: After the introductory section 1 and the necessary definitions and previous results (section 2), we analyze the *Yi's extension property* (section 3) and its relationship with normality. In section 4 we introduce a new extension property based on total preorders defined on *open* subsets of a topological space. We analyze the possible relationship between this new property and extremal disconnectedness. To conclude, other topological properties related to the continuous extension of total preorders are analyzed in the final section 5, whereas in the final section 6 some suggestions for further research are outlined.

2. Preliminaries

Let X be a nonempty set. For a preference $\precsim$ on the set X we will understand a total preorder (i.e., a reflexive, transitive and complete binary relation) defined on X. (If $\precsim$ is also antisymmetric, it is said to be a total order). We denote $x \prec y$ instead of $\neg(y \precsim x)$. Also $x \sim y$ will stand for $(x \precsim y) \wedge (y \precsim x)$ for every $x, y \in X$.

The total preorder $\precsim$ is said to be *representable* if there exists a real-valued order-preserving isotony (also called *utility function*) $f : X \to \mathbb{R}$. Thus $x \precsim y \iff f(x) \leq f(y)$ $(x, y \in X)$. This fact is characterized (see e.g. [2], p.23) by equivalent conditions of "order-separability" that the preorder $\precsim$ must satisfy. Thus, the total preorder $\precsim$ is said to be *order-separable in the sense of Debreu* if there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exists an element $d \in D$ such that $x \precsim d \precsim y$. Such subset D is said to be *order-dense in* $(X, \precsim)$.

If X is endowed with a topology τ, the total preorder $\precsim$ is said to be *continuously representable* if there exists a utility function f that is continuous with respect to the topology τ on X and the usual topology on the real line $\mathbb{R}$. The total preorder $\precsim$ is said to be τ-*continuous* if the sets $U(x) = \{y \in X : x \prec y\}$ and $L(x) = \{y \in X : y \prec x\}$ are τ-open, for every $x \in X$. In this case, the topology τ is said to be *natural or compatible* with the preorder $\precsim$ (see [2], p.19). The coarsest natural topology is the *order topology* θ whose subbasis is the collection $\{L(x) : x \in X\} \cup \{U(x) : x \in X\}$.

A powerful tool to obtain continuous representations of an order-separable totally preordered set $(X, \precsim)$ endowed with a natural topology τ is the *Debreu's open gap lemma* (see[6], or Ch. 3 in [2]). To this extent, let T be a subset of the real line $\mathbb{R}$. A *lacuna* L corresponding to T is a nondegenerate interval of $\mathbb{R}$ that has both a lower bound and an upper bound in T and that has no points in common with T. A maximal lacuna is said to be a *Debreu gap*. In this direction, *Debreu's open gap lemma* states that if S is a subset of the extended real line $\bar{\mathbb{R}}$, then there exists a strictly increasing function $g : S \to \mathbb{R}$ such that all the Debreu gaps of $g(S)$ are open. Using Debreu's open gap lemma, the classical process to get a *continuous* real-valued isotony goes as follows: First, one can easily construct a (non necessarily continuous!) isotony f representing $(X, \precsim)$ when $\precsim$ is order-separable (see e.g. [1], Theorem 24 on p.200, or else [2], Theorem 1.4.8 on p.14). Once we have an isotony f, Debreu's open gap is applied to find a strictly increasing function $g : f(X) \to \mathbb{R}$ such that all the Debreu gaps of $g(f(X))$ are open. Consequently, the composition $F = g \circ f : X \to \mathbb{R}$ is also a utility function representing $(X, \precsim)$, but now F is *continuous with respect to any given natural topology* τ *on* X.

3. Yi's extension property and normal topological spaces

Definition 1. A topological space (X, τ) is said to have *Yi's extension property* ([12, 3]) if an arbitrary continuous total preorder defined on an arbitrary τ-closed subset of X has a continuous extension to the whole X.

Definition 2. Let (X, τ) be a topological space. (X, τ) is said to be *normal* if for each pair of disjoint τ-closed subsets $A, B \subseteq X$ there exists a pair of disjoint τ-open subsets $A^*, B^* \subseteq X$ such that $A \subseteq A^*, B \subseteq B^*$. (For basic topological definitions see e.g. [7, 8]).

It is well-known that this property of being normal is equivalent to an extension property for continuous real-valued functions. This is the "*Tietze's extension theorem*" (see e.g [11], 15.8).

Theorem 3 (Tietze's extension theorem)**.** *Let* (X, τ) *be a topological space. Then* (X, τ) *is normal if and only if for every* τ*-closed subset* $A \subseteq X$*, each continuous map* $f : A \to \mathbb{R}$ *admits a continuous extension* $F : X \to \mathbb{R}$*. Moreover, if* $f(X) \subseteq [-a, a]$ *for some* $a > 0$*, then* F *can be chosen so that* $F(X) \subseteq [-a, a]$*. (This topological property is known as the Tietze's extension property).*

Now suppose that (X, τ) is a normal topological space. An immediate corollary of Tietze's extension theorem states that continuous and representable preferences defined on closed subsets of X can be continuously extended to the entire set X.

Corollary 4. *Let* (X, τ) *be a normal topological space. Let* $S \subseteq X$ *be a* τ*-closed subset of* X*. Let* $\precsim_S$ *be a continuous total preorder defined on* S*. Then if* $\precsim_S$ *is representable through a continuous utility function* $u_S : S \to \mathbb{R}$*, it can also be extended to a continuous total preorder* $\precsim_X$ *defined on the whole* X*.*

Proof. Just observe that, by Tietze's theorem, the utility function u_S admits a continuous extension to a map $u_X : X \to \mathbb{R}$. Then define $\precsim_X$ on X as $x \precsim_X y \iff u_X(x) \leq u_X(y)$ $(x, y \in X)$. □

Yi's extension property was initially understood as an strengthening of Tietze's extension property, in a direction in which we are not interested in extending utility functions, but only preferences. Observe that Yi's and Tietze's extension properties are *not* equivalent in the general case. This is because preferences could fail to be representable.

Let us see that, indeed, Yi's extension property is stronger than Tietze's extension property, as claimed before. First we introduce a necessary definition.

Definition 5. A topological space (X, τ) is said to be *separably connected* if for every $a, b \in X$ there exists a connected and separable subset $C_{a,b} \subseteq X$ such that $a, b \in C_{a,b}$.

Theorem 6. *Let (X, τ) be a topological space that satisfies Yi's extension property. Then (X, τ) is normal. The converse is not true, in general.*

Proof. Let $A, B \subseteq X$ be two (nonempty) τ-closed subsets of X. Let $S = A \cup B$. S is obviously τ-closed. Consider the total preorder $\precsim_S$ defined on S as $a_1 \sim a_2$ for every $a_1, a_2 \in A$, $b_1 \sim b_2$ for every $b_1, b_2 \in B$ and $a \prec b$ for every $a \in A, b \in B$. It is plain that the total preorder $\precsim_S$ is continuous on S. Applying Yi's extension property, there exists a continuous total preorder $\precsim_X$ defined on the whole set X, and extending $\precsim_S$. We distinguish two possible situations:

1. In the first case, we assume that there exists some element $c \in X \setminus S$ such that $a \prec_X c \prec_X b$ for every $a \in A, b \in B$. We observe that $B \subseteq U(c) = \{x \in X : c \prec_X x\}$ and also $A \subseteq L(c) = \{x \in X : x \prec_X c\}$. Since $\precsim_X$ is τ-continuous, the sets $L(c), U(c)$ are τ-open. In addition, they are disjoint by its own definition.
2. Suppose that there is no element $c \in X \setminus S$ such that $a \prec_X c \prec_X b$ for every $a \in A, b \in B$. In this case, if we fix an element $\alpha \in A$ and also an element $\beta \in B$, by definition of $\precsim_S$ we immediately observe that $A \subseteq L(\beta) = \{x \in X : x \prec_X \beta\}$ and in the same way, $B \subseteq U(\alpha) = \{x \in X : \alpha \prec_X x\}$. Since $\precsim_X$ is τ-continuous, the sets $L(\beta), U(\alpha)$ are τ-open. In addition, they are disjoint because $\alpha \prec_S \beta$ by hypothesis.

Thus we see that X is a normal topological space. But normality is a property that is equivalent to Tietze's extension property.

To see that the converse is not true in general, we should have at hand a counterexample. In [5] (see also [3]) it was proved that on separably connected metric spaces Yi's extension property is equivalent to separability. On the other hand, metric spaces are always normal. Thus, an example of a separably connected metric space that is not separable would fit our purposes. This is easy: consider a non-separable Banach space (e.g., $\ell_2(\mathbb{R})$) endowed with its norm topology.

□

4. Extremal disconnectedness and continuous extension of total preorders defined on open subsets of a topological space

Roughly speaking, the topological property known as "extremal disconnectedness" is inspired by the definition of the normality property, but changing the roles of open and closed subsets.

Definition 7. A topological space (X, τ) is said to be *extremally disconnected* (see e.g. [10], p. 33) if for each pair of disjoint τ-open subsets $C, D \subseteq X$ there exists a pair of disjoint τ-closed subsets $C^*, D^* \subseteq X$ such that $C \subseteq C^*, D \subseteq D^*$. As a matter of fact, this is equivalent to say that for each pair of disjoint τ-open subsets $C, D \subseteq X$, the closures $\bar{C}$ and $\bar{D}$ are also disjoint. Moreover, this is also equivalent to say that the interior of every τ-closed set is also τ-closed.

In the Introduction, we gave another different definition of a extremally disconnected topological space. This is due to the next Theorem 8 that states a key equivalence, dual to the equivalence between the usual definition of normality and the Tietze's extension property, but now changing the roles of open and closed subsets.

Theorem 8. *Let (X, τ) be a topological space. Then (X, τ) is extremally disconnected if and only if for every τ-open subset $A \subseteq X$, every continuous and bounded real-valued function $f : A \to \mathbb{R}$ has a continuous extension $F : X \to \mathbb{R}$ to the whole set X. (In particular, $F(a) = f(a)$ for every $a \in A$).*

Proof. See e.g. [9], pp. 22-23 or [11], 15G. □

This inspires the following definition, paralell to that of Yi's extension property, but again changing the role of open and closed subsets.

Definition 9. Let (X, τ) a topological space. We say that τ has the *extension property for total preorders on open subsets* if for every τ-open subset $S \subseteq x$ it holds that any continuous total preorder $\precsim_S$ defined on S has a continuous extension $\precsim_X$ to the entire set X.

As expected, we obtain the following result, parallel to Corollary 4. (To do so, we must observe that a continuous utility function can be assumed to be *bounded*, without loss of generality).

Corollary 10. *Let (X, τ) be a extremally disconnected topological space. Let $S \subseteq X$ be a τ-open subset of X. Let $\precsim_S$ be a continuous total preorder defined on S. Then if $\precsim_S$ is representable through a continuous utility function $u_S : S \to \mathbb{R}$, it can also be extended to a continuous total preorder $\precsim_X$ defined on the whole X.*

In addition, we also obtain the following result, in the spirit of Theorem 6.

Theorem 11. *Let (X, τ) be a topological space such that τ has the extension property for total preorders on open subsets. Then (X, τ) is extremally disconnected.*

Proof. Let $C, D \subseteq X$ be two (nonempty) τ-open subsets of X. Let $T = C \cup D$. T is obviously τ-closed. Consider the total preorder $\precsim_T$ defined on T as $c_1 \sim c_2$ for every $c_1, c_2 \in C$, $d_1 \sim d_2$ for every $d_1, d_2 \in D$ and $c \prec d$ for every $c \in C, d \in D$. It is plain that the total preorder $\precsim_T$ is continuous on T. Applying the hypothesis about the extension property on open subsets, there exists a continuous total preorder $\precsim_X$ defined on the whole set X, and extending $\precsim_T$. Now we observe that $C \subseteq C^* = \{x \in X : x \precsim_X c\}$, for any fixed $c \in C$. Similarly $D \subseteq D^* = \{x \in X : x \precsim_X d\}$, for any fixed $d \in D$. By continuity of $\precsim_X$, the sets C^* and D^* are τ-closed. By definition, they are disjoint. This proves that τ is extremally disconnected.

□

5. Other topological properties related to the continuous extension of total preorders

As mentioned in the proof of Theorem 6, on separably connected metric spaces the following important result is in order (see [5, 3]):

Theorem 12. *Let (X, d) be a separably connected metric space where d stands for the metric defined on X. Let τ_d denote the corresponding metric topology on X. Then (X, τ_d) has Yi's extension property if and only if it is separable.*

Now we may ask ourselves about the possibility of characterizing other classes of topological spaces, different from the separably connected metric ones, where Yi's extension property coincides with separability.

Moreover, since on metric spaces Yi's separability and second countability are equivalent properties, we may also ask ourselves about a characterization of the category of topological spaces in which second countability amounts to Yi's extension property.

These questions are still *open*.

As analyzed in [3], Yi's extension property could be also related to properties of continuous representability of total preorders defined on a topological space, in the direction of the next Definition 13. Moreover, as stated in next Theorem 14, continuous representability properties lean on covering properties, namely the fulfillment of the second countability axiom.

Definition 13. Given a topological space (X, τ) the topology τ on X is said to have the *continuous representability property* (CRP) if every continuous total preorder $\precsim$ defined on X admits a representation by means of a continuous order-preserving isotony, and it is said to be *preorderable* if it is the order topology $\tau_{\precsim}$ of some total preorder $\precsim$ defined on X.

Theorem 14. *Let (X, τ) be a topological space. Then the topology τ satisfies CRP if and only if all its preorderable subtopologies are second countable.*

Proof. See [4], Th. 5.1. □

Theorem 15. *Let (X, d) be a separably connected metric space. Then the metric topology τ_d satisfies CRP if and only if it has Yi's extension property.*

Proof. See [3], Th. 3.3. □

Theorem 16. *Let (X, τ) be a Hausdorrf topological space that satisfies the continuous representability property (CRP) and has Yi's extension representability property. Then for every τ-closed subset $F \subseteq X$, the relative topology τ_F that τ induces on F satisfies CRP. (In other words, CRP is hereditary for closed subsets).*

Proof. Let $\precsim_F$ be a τ_F-continuous total preorder defined on F. Let $\precsim_X$ be a continuous extension of $\precsim_F$ to the whole X. This extension exists because (X, τ) has the Yi's extension property. Using CRP, let $u_X : X \to \mathbb{R}$ be a continuous utility function that represents $\precsim_X$. The restriction U_F of the map u_X to the closed set F is indeed a continuous representation of $\precsim_F$. Therefore (F, τ_F) also satisfies CRP. □

Theorem 17. *Let (X, τ) be a normal topological space that satisfies CRP hereditarily on closed subsets. Then (X, τ) has the Yi's extension property.*

Proof. See [3], Theorem 4.12. □

Combining the last two results and Theorem 6 we get the following Corollary 18.

Corollary 18. *Let (X, τ) be a topological space. Then (X, τ) satisfies CRP and has the Yi's extension property if and only if it is normal and satisfies CRP hereditarily for closed subsets.*

6. Some remarks, and suggestions for further research

First of all, we must point out that this note corresponds to an *unfinished* work, that the three co-authors are preparing jointly. But, to conclude the job, many things must be analyzed. In this section, we enumerate the main topics to which we are paying our interest. The results that we will obtain (if any) shall be included in a future paper, based on the items of the present note.

1. We have proved that topological spaces that have the Yi's extension property are actually normal. Since the converse is not true, we should *identify the special kind of normality that agrees with Yi's extension property.*
2. A topological space that has the extension property for total preorders defined on open subsets is extremally disconnected. Quite probably, the converse is *not* true, but in this note we have not provided yet a counterexample.
3. Topological spaces satisfying both the Yi's extension property and the extension property for total preorders defined on open subsets must have a particular feature. In particular, they must be normal as well as extremally disconnected. It could be interesting to have a *characterization* of these spaces. At first glance, one could think that these spaces are the *discrete* ones, so that a non-discrete example (if any) would be important at this stage.
4. We have seen that in the category of separably connected metric spaces Yi's extension property amounts to separability. It could be interesting to look for *other different categories of topological spaces having the same property.*
5. Also, in the category of separably connected metric spaces Yi's extension property amounts to CRP. Again, it could be interesting to search for *other different categories of topological spaces accomplishing this fact.*
6. We have obtained some partial result concerning the relationship between Yi's extension property and CRP. However, these properties are in general *independent* as analyzed in [3]. More implications between Yi's extension properties and CRP should be analyzed in this direction.
7. Yi's extension property has been put in relation with CRP. Moreover, in some category of topological spaces (e.g, separably connected metric spaces) they coincide. *Is there a similar result for topological spaces that have the extension property for total preorders defined on open subsets?*
8. In General Topology, normality, as well as the property of being extremally disconnected are *separation* properties, whereas second countability is a *covering* property. We have said that CRP is related to second countability. In this direction, *we should study the relationship (if any) between Yi's extension property, the extension property for total preorders on open subsets, and the usual covering properties on topological spaces.*

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