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An invitation to localic real functions[1]

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Abstract

In the pointfree topological context of locales and frames, real functions on a locale L are represented as localic morphisms $\mathcal{S}(L) \to \mathfrak{L}(\mathbb{R})$ (i.e. frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$) where $\mathcal{S}(L)$ stands for the frame of all sublocales of L and $\mathfrak{L}(\mathbb{R})$ denotes the frame of reals. This is reminiscent of dealing with (not necessarily continuous) real functions $X \to \mathbb{R}$ as with continuous functions $\mathfrak{D}(X) \to \mathbb{R}$ where $\mathfrak{D}(X)$ is the discrete space on the underlying set of X. But it is deeper: the structure of $\mathcal{S}(L)$ is rich enough to provide a nice common framework for the three types of continuity (lower semicontinuity, upper semicontinuity and continuity) as well as general (not necessarily continuous) real functions. The aim of this expository note is to provide a short overview of the theory of pointfree real functions and the strength of its applications.

1. Introduction

Given a topological space $(X, \mathcal{O}X)$, the lattice $\mathcal{O}X$ of open sets is complete since any union of open sets is an open set; of course the *infinite distribution law*

$$A \wedge \bigvee_{i \in I} B_i = \bigvee_{i \in I} (A \wedge B_i)$$

holds in $\mathcal{O}X$ since the operations $\wedge$ (being a finite meet) and $\bigvee$ coincide with the usual set-theoretical operations of $\cap$ (intersection) and $\bigcup$ (union), respectively. Moreover, if $f : (X, \mathcal{O}X) \to (Y, \mathcal{O}Y)$ is a continuous map, f^{-1} defines a map of $\mathcal{O}Y$ into $\mathcal{O}X$ that clearly preserves the operations $\wedge$ and $\bigvee$. Therefore, defining a *frame* as a complete lattice L satisfying the infinite distribution law

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i),$$

and defining a *frame homomorphism* $h : L \to M$ as a map from L in M such that $h(\bigwedge_{i \in F} a_i) = \bigwedge_{i \in F} h(a_i)$ for every finite F (in particular, for $F = \varnothing$, $h(1) = 1$) and $h(\bigvee_{i \in I} a_i) = \bigvee_{i \in I} h(a_i)$ for every I (in particular, for $I = \varnothing$, $h(0) = 0$), we have the category Frm of frames and frame homomorphisms and a contravariant functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}$ defined by $(X, \mathcal{O}X) \mapsto \mathcal{O}X$ and $(f : (X, \mathcal{O}X) \to (Y, \mathcal{O}Y)) \mapsto (f^{-1} : \mathcal{O}Y \to \mathcal{O}X)$.

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Because of contravariance, to keep the original geometrical motivation it is necessary to introduce the dual category $\mathsf{Frm}^{\mathrm{op}}$, making functor $\mathcal{O}$ covariant. This is the genesis of the category Loc of locales and localic maps: it is precisely the category $\mathsf{Frm}^{\mathrm{op}}$. So, a locale is the same thing as a frame, but morphisms diverge: localic morphisms are defined abstractly, as morphisms of frames acting in the opposite direction.

The category of *locales* is a category set up to behave like the familiar one of topological spaces. One speaks about *sublocales* and, in particular, of *closed, open* and *dense sublocales*. One speaks about *continuous maps between locales* and, in particular, of *proper* and *open maps*. One speaks about *compact locales* and, analogously, many other separation axioms have their versions in locales: e.g, one speaks of *compact Hausdorff locales*, *regular locales*, *normal locales*, etc. But there is an important new aspect: the dual category of Loc (that is, the category Frm of frames) is an algebraic category, with all the nice properties and tools available in any category of algebras ([15]).

This analogy between the theory of locales and the theory of topological spaces is not quite exact; otherwise, the two theories would be indistinguishable and locale theory would be redundant. What exists is a translating device between the two theories: each topological space X defines naturally a locale $\mathcal{O}(X)$ (specifically, its topology). And given a locale L there exists a topological space $\Sigma(L)$ naturally associated to L. More precisely, there is a categorical adjunction between the category Top of topological spaces and continuous maps and the category Loc of locales, defined by the *open-sets functor* $\mathcal{O} : \mathsf{Top} \to \mathsf{Loc}$ and the *spectrum functor* $\Sigma : \mathsf{Loc} \to \mathsf{Top}$ (see [15] or [24] for details).

Each frame L has associated with it the well-known ring $C(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ of its continuous real functions ([1]). This is a commutative archimedean (strong) f-ring with unit. Since the spectrum $\Sigma(\mathfrak{L}(\mathbb{R}))$ of the frame of reals is homeomorphic to the usual space of reals, by the adjunction

$$\mathsf{Top} \underset{\Sigma}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{Loc}$$

mentioned above there is a bijection

$$\mathsf{Top}(X, \mathbb{R}) \simeq \mathsf{Loc}(\mathcal{O}X, \mathfrak{L}(\mathbb{R})) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X). \tag{1}$$

Thus the classical ring $C(X)$ ([7]) is naturally isomorphic to $C(\mathcal{O}X)$ and the correspondence $L \rightsquigarrow C(L)$ for frames extends that for spaces.

Now, replace the space X in (1) by a discrete space $\mathfrak{D}(X)$. We get

$$\mathbb{R}^X \simeq \mathsf{Top}(\mathfrak{D}(X), \mathbb{R})) \simeq \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathfrak{D}(X)).$$

For any L in the category Frm, the role of the lattice $\mathfrak{D}(X)$ of all subspaces of X is taken by the lattice $\mathcal{S}(L)$ of all sublocales of L, which justifies to think of the members of $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L)) = C(\mathcal{S}(L))$ as *arbitrary not necessarily continuous* real functions on the frame L ([11]). The real functions on a frame L are thus the continuous real functions on the sublocale lattice of L and therefore, from the results of [1], constitute a commutative archimedean (strong) f-ring with unit that we denote by $F(L)$. It is partially ordered by

$$\begin{aligned} f \le g \quad &\equiv \quad f(r,—) \le g(r,—) \quad \text{for all } r \in \mathbb{Q} \\ &\Leftrightarrow \quad g(—,r) \le f(—,r) \quad \text{for all } r \in \mathbb{Q}. \end{aligned}$$

Since any L is isomorphic to the subframe $\mathfrak{c}L$ of $\mathcal{S}(L)$ of all closed sublocales, the ring $C(L)$ may be equivalently viewed as the subring of $F(L)$ of all real functions $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ for which $f(p,q)$ is a closed sublocale for every p, q (i.e. $f(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$). So our ring $F(L)$ embodies the ring $C(L)$ in a nice way. We shall refer to these $f \in C(L)$ (indistinctly regarded as elements of $F(L)$ or as elements of $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$) as the *continuous* real functions on L and will use the same notation $C(L)$ for denoting both classes.

Besides continuity, $F(L)$ allows also to distinguish the two types of semicontinuity: $f \in F(L)$ is *lower semicontinuous* if $f(p, —)$ is always closed, and f is *upper semicontinuous* if $f(—, q)$ is always closed. We shall denote by $LSC(L)$ and $USC(L)$ the classes of lower and upper semicontinuous functions respectively. Hence, the ring $F(L)$ provides an appropriate level of generality for $C(L)$, $LSC(L)$ and $USC(L)$ and

$$\textit{lower semicontinuous} \;+\; \textit{upper semicontinuous} \;=\; \textit{continuous}.$$

The first right approach to semicontinuity in pointfree topology was presented in [12]. The approach here considered, succinctly described above, has wider scope and was introduced recently in [11] (see also [13]).

2. Preliminaries: the lattice of sublocales and the frame of reals

For general information on frames and locales the reader is referred to [15], [24] or [25].

One of the fundamental differences between Top and Loc relies on their lattices of subobjects. In fact, sublocale lattices are much more complicated than their topological counterparts (complete atomic Boolean algebras): they are in general *co-frames* (i.e., complete lattices satisfying the distribution law $S \vee \bigwedge_{i\in I} T_i = \bigwedge_{i\in I}(S \vee T_i)$, dual to the distribution law that characterizes frames). Even the sublocale lattice of a topology $\mathcal{O}X$ can be much larger than the Boolean algebra of the subspaces of X; e.g., $\mathbb{Q}$ considered as a subspace of $\mathbb{R}$ (with the usual euclidean topology) has $2^{\mathfrak{c}}$ many non-isomorphic sublocales.

Let L be a locale. The *sublocales* $j : M \rightarrowtail L$ of L, that is, the regular monomorphisms in Loc with codomain L (or still, the quotients or surjective frame homomorphisms $L \twoheadrightarrow M$ with domain L) can be described in several equivalent ways (cf. [25] or [24]). Here we shall use the approach of [24]: a subset S of L is a *sublocale* of L if:

1. For each $A \subseteq S$, $\bigwedge A \in S$.
2. For any $a \in L$ and $s \in S$, $a \to s \in S$.

Since any intersection of sublocales is a sublocale, the set of all sublocales of L is a complete lattice. This is a co-frame, in which $\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i$, $\bigvee_{i\in I} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in I} S_i\}$, $0 = \{1\}$ and $1 = L$. It will be convenient to work with the corresponding dual lattice, hence a frame, that we shall denote by $\mathcal{S}(L)$.

Each sublocale S is itself a frame with $\bigwedge$ and $\to$ as in L (the top of S coincides with the one of L but the bottom 0_S may differ from the one of L).

In spite of $\mathcal{S}(L)$ not being in general a Boolean algebra, it contains many complemented elements. For example, for each $a \in L$, the sets

$$\mathfrak{c}(a) := \uparrow a = \{b \in L \mid a \le b\} \quad \text{and} \quad \mathfrak{o}(a) := \{a \to b \mid b \in L\}$$

define sublocales of L, complemented to each other, i.e. $\mathfrak{c}(a) \vee \mathfrak{o}(a) = 1$ and $\mathfrak{c}(a) \wedge \mathfrak{o}(a) = 0$. The former are the so-called *closed sublocales*, while the latter are the *open sublocales*.

Here is a list of some of the most significative properties of $\mathcal{S}(L)$ ([24, 25]):

- (S_1) $\mathfrak{c}L := \{\mathfrak{c}(a) \mid a \in L\}$ is a subframe of $\mathcal{S}(L)$ isomorphic to L; the isomorphism $\mathfrak{c} : L \to \mathfrak{c}L$ is given by $a \mapsto \mathfrak{c}(a)$. In particular, $\mathfrak{c}(\bigvee_{i\in I} a_i) = \bigvee_{i\in I} \mathfrak{c}(a_i)$ and $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \wedge \mathfrak{c}(b)$.
- (S_2) Let $\mathfrak{o}L$ denote the subframe of $\mathcal{S}(L)$ generated by $\{\mathfrak{o}(a) \mid a \in L\}$. The map $L \to \mathfrak{o}L$ defined by $a \mapsto \mathfrak{o}(a)$ is a dual lattice embedding. In particular, we have $\mathfrak{o}(\bigvee_{i\in I} a_i) = \bigwedge_{i\in I} \mathfrak{o}(a_i)$ and $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \vee \mathfrak{o}(b)$.
- (S_3) $\mathfrak{c}(a) \le \mathfrak{o}(b)$ iff $a \wedge b = 0$, and $\mathfrak{o}(a) \le \mathfrak{c}(b)$ iff $a \vee b = 1$.
- (S_4) The *closure* $\overline{S} := \bigvee\{\mathfrak{c}(a) \mid \mathfrak{c}(a) \le S\}$ and the *interior* $S^\circ := \bigwedge\{\mathfrak{o}(a) \mid S \le \mathfrak{o}(a)\}$ of a sublocale S satisfy the following properties, where $(\cdot)^*$ stands for the pseudocomplementation operator: $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$, $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$, $(\overline{S^*})^* = S^\circ$.

Since Frm is an algebraic category, we have at our disposal the familiar procedure from traditional algebra of *presentation* of objects by *generators and relations* ([15, 1]). The *frame of reals* ([1]) is the frame $\mathfrak{L}(\mathbb{R})$ generated by ordered pairs (p,q), with $p,q\in\mathbb{Q}$, and relations

(R_1) $(p,q)\wedge(r,s)=(p\vee r,q\wedge s)$,
(R_2) $(p,q)\vee(r,s)=(p,s)$ whenever $p\le r<q\le s$,
(R_3) $(p,q)=\bigvee_{p<r<s<q}(r,s)$,
(R_4) $\bigvee_{p,q\in\mathbb{Q}}(p,q)=1$.

Notice that a map from the generating set of $\mathfrak{L}(\mathbb{R})$ into L defines a frame homomorphism $\mathfrak{L}(\mathbb{R})\to L$ if and only if it transforms relations (R_1)–(R_4) of $\mathfrak{L}(\mathbb{R})$ into identities in L.

Let $(p,\text{—})=\bigvee_{q\in\mathbb{Q}}(p,q)$ and $(\text{—},q)=\bigvee_{p\in\mathbb{Q}}(p,q)$. With $(p,\text{—})$ and $(\text{—},q)$ taken as the primitive notions, $\mathfrak{L}(\mathbb{R})$ may be equivalently defined ([21]) as the frame generated by elements $(p,\text{—})$ and $(\text{—},q)$, with $p,q\in\mathbb{Q}$, and relations

(R'_1) $(p,\text{—})\wedge(\text{—},q)=0$ whenever $p\ge q$,
(R'_2) $(p,\text{—})\vee(\text{—},q)=1$ whenever $p<q$,
(R'_3) $(p,\text{—})=\bigvee_{r>p}(r,\text{—})$,
(R'_4) $(\text{—},q)=\bigvee_{s<q}(\text{—},s)$,
(R'_5) $\bigvee_{p\in\mathbb{Q}}(p,\text{—})=1$,
(R'_6) $\bigvee_{q\in\mathbb{Q}}(\text{—},q)=1$.

3. Constructing real functions: scales

In order to define a real function $f\in F(L)$ it suffices to consider two maps from $\mathbb{Q}$ to $\mathcal{S}(L)$ that turn the defining relations (R'_1)–(R'_6) above into identities in $\mathcal{S}(L)$.

This can be easily done with scales ([8]; trails in [1]): here by a *scale* in $\mathcal{S}(L)$ is meant a family $(S_p)_{p\in\mathbb{Q}}$ of sublocales of L satisfying (1) $S_p\vee {S_q}^*=1$ whenever $p<q$, and (2) $\bigvee_{p\in\mathbb{Q}}S_p=1=\bigvee_{p\in\mathbb{Q}}{S_p}^*$.

The following lemma, essentially proved in [10], plays a key role.

Lemma 1. *For each scale $(S_r)_{r\in\mathbb{Q}}$ the formulas*

$$f(p,\text{—})=\bigvee_{r>p}S_r\quad\text{and}\quad f(\text{—},q)=\bigvee_{r<q}{S_r}^*,\quad p,q\in\mathbb{Q}$$

determine an $f\in F(L)$. Moreover, if every S_r is closed (resp. open, resp. clopen) then $f\in LSC(L)$ (resp. $f\in USC(L)$, resp. $f\in C(L)$). □

Let us mention two basic examples of real functions.

Example 2 (Constant functions). For each $r\in\mathbb{Q}$, the family $(S^r_t)_{t\in\mathbb{Q}}$ defined by $S^r_t=1$ if $t<r$ and $S^r_t=0$ if $t\ge r$ is a scale. The corresponding function in $C(L)$ provided by Lemma 1 is given for each $p,q\in\mathbb{Q}$ by

$$\mathbf{r}(p,\text{—})=\begin{cases}1 & \text{if } p<r\\ 0 & \text{if } p\ge r\end{cases}\quad\text{and}\quad \mathbf{r}(\text{—},q)=\begin{cases}0 & \text{if } q\le r\\ 1 & \text{if } q>r.\end{cases}$$

Example 3 (Characteristic functions). Let S be a complemented sublocale of L with complement $\neg S$. Then $(S_r)_{r\in\mathbb{Q}}$ defined by $S_r=1$ if $r<0$, $S_r=\neg S$ if $0\le r<1$ and $S_r=0$ if $r\ge 1$, is a scale. We denote the corresponding real function in $F(L)$ by χ_S and refer to it as the *characteristic function* of S. It is defined for each $p,q\in\mathbb{Q}$ by

$$\chi_S(p,\text{—})=\begin{cases}1 & \text{if } p<0\\ \neg S & \text{if } 0\le p<1\\ 0 & \text{if } p\ge 1\end{cases}\quad\text{and}\quad \chi_S(\text{—},q)=\begin{cases}0 & \text{if } q\le 0\\ S & \text{if } 0<q\le 1\\ 1 & \text{if } q>1.\end{cases}$$

4. The algebraic structure of $F(L)$

For any frame L, the algebra $C(L)$ of continuous real functions on L has as its elements the frame homomorphisms $f:\mathfrak{L}(\mathbb{R})\to L$ (or, equivalently, as already pointed out, the frame homomorphisms $f:\mathfrak{L}(\mathbb{R})\to \mathcal{S}(L)$ for which $f(p,q)$ is closed for any $p,q\in\mathbb{Q}$). The operations are determined by the operations of $\mathbb{Q}$ as lattice-ordered ring as follows (see [1] for more details):

(1) For $\diamond = +,\cdot,\wedge,\vee$, $(f\diamond g)(p,q)$ is given by $\bigvee\{f(r,s)\wedge g(t,u)\mid \langle r,s\rangle\diamond\langle t,u\rangle\subseteq\langle p,q\rangle\}$, where $\langle\cdot,\cdot\rangle$ stands for open interval in $\mathbb{Q}$ and the inclusion on the right means that $x\diamond y\in\langle p,q\rangle$ whenever $x\in\langle r,s\rangle$ and $y\in\langle t,u\rangle$.

(2) $(-f)(p,q)=f(-q,-p)$.

(3) For all $\lambda>0$ in $\mathbb{Q}$, $(\lambda\cdot f)(p,q)=f(\frac{p}{\lambda},\frac{q}{\lambda})$.

Indeed, these stipulations define maps from $\mathbb{Q}\times\mathbb{Q}$ to L and turn the defining relations (R_1)–(R_4) of $\mathfrak{L}(\mathbb{R})$ into identities in L and consequently determine frame homomorphisms $\mathfrak{L}(\mathbb{R})\to L$ (the result that $C(L)$ is an f-ring follows then from the fact that any identity in these operations which is satisfied by $\mathbb{Q}$ also holds in $C(L)$). In particular, $F(L)=C(\mathcal{S}(L))$ is an f-ring. In the sequel we present alternative formulas for its operations, picked from [13], which were obtained with the help of scales and Lemma 1.

Given $f,g\in F(L)$, $(f(p,\text{—})\vee g(p,\text{—}))_{p\in\mathbb{Q}}$ and $(f(p,\text{—})\wedge g(p,\text{—}))_{p\in\mathbb{Q}}$ are scales that generate respectively the *supremum* $f\vee g\in F(L)$ and the *infimum* $f\wedge g\in F(L)$. Therefore $(f\vee g)(p,\text{—})=\bigvee_{r>p}\big(f(r,\text{—})\vee g(r,\text{—})\big)=f(p,\text{—})\vee g(p,\text{—})$, $(f\vee g)(\text{—},q)=\bigvee_{r<q}\big(f(r,\text{—})\vee g(r,\text{—})\big)^*=f(\text{—},q)\wedge g(\text{—},q)$, $(f\wedge g)(p,\text{—})=f(p,\text{—})\wedge g(p,\text{—})$ and, finally, $(f\wedge g)(\text{—},q)=f(\text{—},q)\vee g(\text{—},q)$. In summary, we have:

Proposition 4. *The poset $F(L)$ has binary joins and meets; $LSC(L)$, $USC(L)$ and $C(L)$ are closed under these joins and meets.* □

Now for each $p\in\mathbb{Q}$ define $S_p^{f+g}=\bigvee_{r\in\mathbb{Q}}\big(f(r,\text{—})\wedge g(p-r,\text{—})\big)$. The family $(S_p^{f+g})_{p\in\mathbb{Q}}$ is a scale that determines the *sum* $f+g\in F(L)$ of f and g. It is not hard to see that $(f+g)(p,\text{—})=\bigvee_{r\in\mathbb{Q}}\big(f(r,\text{—})\wedge g(p-r,\text{—})\big)$ for every $p\in\mathbb{Q}$ and $(f+g)(\text{—},q)=\bigvee_{s\in\mathbb{Q}}\big(f(\text{—},s)\wedge g(\text{—},q-s)\big)$ for every $q\in\mathbb{Q}$. Hence we have:

Proposition 5. *Let $f,g\in F(L)$. If $f,g\in LSC(L)$ (resp. $USC(L)$, resp. $C(L)$) then $f+g\in LSC(L)$ (resp. $USC(L)$, resp. $C(L)$).* □

Given $f,g\in F(L)$, with $f-g=f+(-g)$ we also have:

Proposition 6. *Let $f,g\in F(L)$. Then:*

(i) *$(f-g)(p,\text{—})=\bigvee_{r\in\mathbb{Q}}f(r,\text{—})\wedge g(\text{—},r-p)$ for every $p\in\mathbb{Q}$.*

(ii) *$(f-g)(\text{—},q)=\bigvee_{s\in\mathbb{Q}}f(\text{—},s)\wedge g(s-q,\text{—})$ for every $q\in\mathbb{Q}$.*

(iii) *If $f\in LSC(L)$ (resp. $USC(L)$, resp. $C(L)$) and $g\in USC(L)$ (resp. $LSC(L)$, resp. $C(L)$) then $f-g\in LSC(L)$ (resp. $USC(L)$, resp. $C(L)$).* □

Finally, with respect to the product $f\cdot g$, for the case $f,g\geq\mathbf{0}$ we have that, defining, for each $p\in\mathbb{Q}$, $S_p^{f\cdot g}=\bigvee_{r>0}\big(f(r,\text{—})\wedge g(\frac{p}{r},\text{—})\big)$ if $p\geq 0$ and $S_p^{f\cdot g}=1$ otherwise, then $(S_p^{f\cdot g})_{p\in\mathbb{Q}}$ is a scale generating $f\cdot g$. Therefore

$$(f\cdot g)(p,\text{—})=\begin{cases}\bigvee\limits_{r>0}\big(f(r,\text{—})\wedge g(\frac{p}{r},\text{—})\big) & \text{if } p\geq 0\\ 1 & \text{if } p<0\end{cases}$$

and

$$(f\cdot g)(\text{—},q)=\begin{cases}\bigvee\limits_{s>0}\big(f(\text{—},s)\wedge g(\text{—},\frac{q}{s})\big) & \text{if } q>0\\ 0 & \text{if } q\leq 0.\end{cases}$$

Proposition 7. *Both $LSC(L)$ and $USC(L)$ are closed under products of non-negative elements (and so does $C(L)$).*

In order to extend this result to the product of two arbitrary f and g in $F(L)$ let $f^+ = f \vee \mathbf{0}$ and $f^- = (-f) \vee \mathbf{0}$. Notice that $f = f^+ - f^-$. Since $C(\mathcal{S}(L))$ is an ℓ-ring, from general properties of ℓ-rings we have that $f \cdot g = \big(f^+ \cdot g^+\big) - (f^+ \cdot g^-) - (f^- \cdot g^+) + (f^- \cdot g^-)$. In particular, if $f, g \leq \mathbf{0}$, then $f \cdot g = f^- \cdot g^- = (-f) \cdot (-g)$. Hence:

Proposition 8. *$C(L)$ is closed under products. If $f, g \leq \mathbf{0}$ and $f, g \in LSC(L)$ (resp. $USC(L)$) then $f \cdot g \in USC(L)$ (resp. $LSC(L)$).* □

5. Upper and lower regularizations

A fact from the theory of real functions asserts that every real function $f : X \to \mathbb{R}$ on a topological space X admits the so-called lower semicontinuous regularization $f_* : X \to \overline{\mathbb{R}}$, given by the lower limit of f:

$$f_*(x) := \liminf_{y \to x} f(y) = \bigvee \{ \bigwedge f(U) \mid x \in U \in \mathcal{O}X \}.$$

This is the largest lower semicontinuous minorant of f: $f_* = \bigvee \{ g \in \mathsf{LSC}(X, \overline{\mathbb{R}}) \mid g \leq f \}$. For each $p \in \mathbb{Q}$ we have

$$f_*^{-1}(]p, +\infty[) = \bigcup_{r>p} (f^{-1}([r, +\infty[))^\circ = X \setminus \bigcap_{r>p} \overline{f^{-1}(] - \infty, r[)},$$

which means that the lower regularization f_* takes values in $\mathbb{R}$ if and only if it has a lower semicontinuous minorant; equivalently, if and only if $\bigcup_{r \in \mathbb{Q}} f_*^{-1}(]r, +\infty[) = X$, that is, $\bigcap_{r \in \mathbb{Q}} \overline{f^{-1}(] - \infty, r[)} = \varnothing$.

In the pointfree context, since we know already how to deal with generic real functions, the construction of the corresponding lower and upper regularizations can be performed in a surprisingly easy way ([9]) which we describe in the sequel.

Let $f \in F(L)$. The family $\big(\overline{f(r, —)}\big)_{r \in \mathbb{Q}}$ is a scale so, by Lemma 1, formulas

$$f^\circ(p, —) = \bigvee_{r>p} \overline{f(r, —)} \quad \text{and} \quad f^\circ(—, q) = \bigvee_{s<q} \neg\, \overline{f(s, —)}$$

determine an $f^\circ \in LSC(L)$, called the *lower regularization* of f. It satisfies, among others, the following properties ([9, 11]):

Proposition 9. *Let $f, g \in F(L)$. Then $f^\circ \leq f$, $(f \wedge g)^\circ = f^\circ \wedge g^\circ$ and $f^{\circ\circ} = f^\circ$. Moreover, $f^\circ = \bigvee \{ g \in LSC(L) \mid g \leq f \}$ and $LSC(L) = \{ f \in F(L) \mid f = f^\circ \}$.* □

Analogously, we can define the *upper regularization* of $f \in F(L)$ by

$$f^-(—, q) = \bigvee_{s<q} \overline{f(—, s)} \quad \text{and} \quad f^-(p, —) = \bigvee_{r>p} \neg\, \overline{f(—, r)}.$$

Thus $f^- = (-f)^\circ$ which with Proposition 9 yields the following:

Proposition 10. *Let $f, g \in F(L)$. Then $f^- \in USC(L)$, $f \leq f^-$, $(f \vee g)^- = f^- \vee g^-$, $f^{--} = f^-$, $f^- = \bigwedge \{ g \in USC(L) \mid f \leq g \}$ and $USC(L) = \{ f \in F(L) \mid f = f^- \}$.* □

6. Insertion-type results

Our aim now is to give evidence of the scope and usefulness of the ring $F(L)$ with a short review of its main applications to insertion-type results in normal or extremally disconnected frames. Their classical (particular) versions about the existence of continuous real functions in normal spaces or extremally disconnected spaces rank among the fundamental results of point-set topology and can be classified in three types: separation theorems (like

Urysohn's Lemma), extension theorems (like Tietze's Theorem), and insertion theorems (like Katětov-Tong Theorem). The latter are the most important since they imply the former two as corollaries.

We begin by the pointfree extension of Katětov-Tong insertion theorem which holds for *normal frames*, that is, frames in which, whenever $a \vee b = 1$, there exists $u \in L$ such that $a \vee u = 1 = b \vee u^*$. It is not difficult to show that a frame L is normal if and only if

> *for any countable $A, B \subseteq L$ satisfying $a \vee \bigwedge B = 1 = b \vee \bigwedge A$ for all $a \in A$ and $b \in B$, there exists $u \in L$ such that $a \vee u = 1 = b \vee u^*$ for all $a \in A$ and $b \in B$* ([23]).

Based on this characterization it is then possible to show the Katětov-Tong Theorem:

Theorem 11 (**Insertion: Katětov-Tong;** [12]). *For a frame L, the following are equivalent:*

(i) *L is normal.*
(ii) *For every $f \in USC(L)$ and $g \in LSC(L)$ satisfying $f \leq g$, there exists $h \in C(L)$ such that $f \leq h \leq g$.* □

Other insertion theorems were meanwhile obtained for other classes of frames ([5, 8, 9, 10, 13]). The following one is, in some sense, a dual version of the previous theorem; equivalence (i)⇔(v) generalizes Corollary 4 of [20] and all the others extend results of Kubiak-de Prada Vicente ([19]). Recall that a frame L is *extremally disconnected* if $a^* \vee a^{**} = 1$ for every $a \in L$. These frames are precisely those in which the second De Morgan law $(\bigwedge_{i\in I} a_i)^* = \bigvee_{i\in I} a_i^*$ holds (this is the reason why they are also referred to as *De Morgan frames*).

Theorem 12 (**Insertion: Lane, Kubiak-de Prada Vicente;** [9]). *For a frame L, the following are equivalent:*

(i) *L is extremally disconnected.*
(ii) *$C(L) = \{f^\circ \mid f \in USC(L)$ and $f^\circ \in LSC(L)\}$.*
(iii) *$C(L) = \{g^- \mid g \in LSC(L)$ and $g^- \in USC(L)\}$.*
(iv) *For every $f \in USC(L)$ and $g \in LSC(L)$, if $g \leq f$ then $g^- \leq f^\circ$.*
(v) *For every $f \in USC(L)$ and $g \in LSC(L)$ satisfying $g \leq f$, there exists $h \in C(L)$ such that $g \leq h \leq f$.* □

Our next result is the monotone version of Katětov-Tong Theorem and generalizes the (monotone insertion) theorem of Kubiak in [18]. First note that the definition of normality may be rephrased in the following way: let $\mathcal{D}_L = \{(a,b) \in L \times L \mid a \vee b = 1\}$; a frame L is normal if and only if there exists a map $\Delta : \mathcal{D}_L \to L$ satisfying $a \vee \Delta(a,b) = 1 = b \vee \Delta(a,b)^*$. Equipping $\mathcal{D}_L$ with the partial order $(\leq^{\rm op}, \leq)$ inherited from $L^{\rm op} \times L$, L is called *monotonically normal* in case it is normal and Δ is monotone ([8]). Let $\mathsf{UL}(L) = \{(f,g) \in USC(L) \times LSC(L) \mid f \leq g\}$ be partially ordered by the order inherited from $F(L)^{\rm op} \times F(L)$, i.e., $(f_1, g_1) \leq (f_2, g_2) \equiv f_2 \leq f_1$ and $g_1 \leq g_2$. Then:

Theorem 13 (**Monotone insertion: Kubiak;** [8]). *For a frame L, the following are equivalent:*

(i) *L is monotonically normal.*
(ii) *There is a monotone map $\Delta : \mathsf{UL}(L) \to C(L)$ such that $f \leq \Delta(f,g) \leq g$ for every $(f,g) \in \mathsf{UL}(L)$.* □

Now let $f, g \in F(L)$ and define $\iota(f,g) = \bigvee_{p\in\mathbb{Q}} \big(f(—,p) \wedge g(p,—)\big) \in \mathcal{S}(L)$. One writes $f < g$ whenever $\iota(f,g) = 1$. Note that the relation $<$ is indeed stronger than $\leq$.

The next insertion theorem in our list is the pointfree version of the (insertion) theorem of Dowker ([3]) for countably paracompact spaces. More generally, a frame L is said to be *countably paracompact* ([4]) if every countable non-decreasing cover $(a_j)_{j\in J}$ is shrinkable (i.e., there is a cover $(b_j)_{j\in J}$ such that $b_j^* \vee a_j = 1$ for all $j \in J$).

Theorem 14 (Strict insertion: Dowker; [10, 13]). *For a frame L, the following are equivalent:*

(i) *L is normal and countably paracompact.*
(ii) *For every $f \in USC(L)$ and $g \in LSC(L)$ satisfying $f < g$, there exists $h \in C(L)$ such that $f < h < g$.* □

The last two insertion results that we list here are the pointfree extensions of respectively the insertion theorem of Michael for perfectly normal spaces ([22]) and the insertion theorem of Kubiak for completely normal spaces ([17]). We recall from [10] that a frame L is *perfectly normal* if, for each $a \in L$, there exists a countable subset $B \subseteq L$ such that $a = \bigvee B$ and $b^* \vee a = 1$ for every $b \in B$; a frame L is *completely normal* if for each $a, b \in L$ there exist $u, v \in L$ such that $u \wedge v = 0$, $b \le a \vee u$ and $a \le b \vee v$ ([5]).

Theorem 15 (Bounded insertion: Michael; [10, 13]). *For a frame L, the following are equivalent:*

(i) *L is perfectly normal.*
(ii) *For every $f \in USC(L)$ and $g \in LSC(L)$ satisfying $f \le g$, there exists $h \in C(L)$ such that $f \le h \le g$ and $\iota(f,h) = \iota(h,g) = \iota(f,g)$.* □

Theorem 16 (General insertion: Kubiak; [5]). *For a frame L, the following are equivalent:*

(i) *L is completely normal.*
(ii) *For every $f, g \in F(L)$ satisfying $f^- \le g$ and $f \le g^\circ$, there exists $h \in LSC(L)$ such that $f \le h \le h^- \le g$.* □

It is worth mentioning that all the preceding theorems, when applied to $L = \mathcal{O}X$ (for the specific type of space X in question), yield the corresponding classical result. We illustrate this here with Katětov-Tong insertion: applying Theorem 11 to the topology $\mathcal{O}X$ of a normal space X, the implication "(i) ⇒ (ii)" provides the non-trivial implication of the classical Katětov-Tong Theorem ([16, 27]) as we describe next.

Let $f : X \to \mathbb{R}$ be an upper semicontinuous function and $g : X \to \mathbb{R}$ a lower semicontinuous one such that $f \le g$. The families $(\mathfrak{c}(f^{-1}(]-\infty, q[))_{q\in\mathbb{Q}}$ and $(\mathfrak{c}(p^{-1}(]p, +\infty[))_{p\in\mathbb{Q}}$ are scales in $\mathcal{S}(\mathcal{O}X)$. Then, by Lemma 1, the formulas

$$\begin{array}{rclcrcl} \widetilde{f}(\text{—}, q) &=& \mathfrak{c}(f^{-1}(]-\infty, q[)), && \widetilde{f}(p, \text{—}) &=& \bigvee_{r>p} \mathfrak{o}(f^{-1}(]-\infty, r[)), \\ \widetilde{g}(p, \text{—}) &=& \mathfrak{c}(g^{-1}(]p, +\infty[)), && \widetilde{g}(\text{—}, q) &=& \bigvee_{s<q} \mathfrak{o}(g^{-1}(]s, +\infty[)), \end{array}$$

establish functions $\widetilde{f}, \widetilde{g} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(\mathcal{O}X)$ with $\widetilde{f} \in USC(\mathcal{O}X)$ and $\widetilde{g} \in LSC(\mathcal{O}X)$. The condition $f \le g$ implies $f^{-1}(]-\infty, q[) \supseteq g^{-1}(]-\infty, q[)$ for every $q \in \mathbb{Q}$, thus $\widetilde{f} \le \widetilde{g}$. Consider $\widetilde{h} \in C(\mathcal{O}X)$ provided by Theorem 11, and the corresponding continuous map $h : X \to \mathbb{R}$ defined by $h(x) \in]p, q[$ iff $x \in \mathfrak{c}^{-1}\left(\widetilde{h}(p,q)\right)$. It is then clear that $f \le h \le g$.

For a more detailed discussion of the preceding results and more examples of results in this vein consult [5, 8, 11, 13].

Remark 17. For a unified presentation of insertion-type results regarding normal and extremally disconnected objects in the categories of topological spaces, bitopological spaces, ordered topological spaces and locales see [6].

7. Separation-type corollaries

Let L be a normal frame and consider $a, b \in L$ satisfying $a \vee b = 1$. By property (S$_3$) of Section 2, $\mathfrak{o}(a) \leq \mathfrak{c}(b)$. Therefore, $\chi_{\mathfrak{c}(b)} \leq \chi_{\mathfrak{o}(a)}$. Consequently, applying Theorem 11 we obtain a continuous $\widetilde{h} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\chi_{\mathfrak{c}(b)} \leq \widetilde{h} \leq \chi_{\mathfrak{o}(a)}$. Consider then the $h : \mathfrak{L}(\mathbb{R}) \to L$ given by $h = \mathfrak{c}^{-1} \circ \widetilde{h}$. Observing that $\chi_{\mathfrak{c}(b)} \leq \widetilde{h}$ iff $h(—, 0) = 0$ and $h(—, 1) \leq b$ and that, on the other hand, $\widetilde{h} \leq \chi_{\mathfrak{o}(a)}$ iff $h(0, —) \leq a$ and $h(1, —) = 0$, we get immediately the non-trivial implication of the following:

Corollary 18 (Separation: Urysohn; [11]). *A frame L is normal if and only if, for every $a, b \in L$ satisfying $a \vee b = 1$, there exists $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $h((—, 0) \vee (1, —)) = 0$, $h(0, —) \leq a$ and $h(—, 1) \leq b$.* □

The statement of Corollary 18 is precisely the statement of the (separation) lemma of Urysohn for frames (cf. [1], Prop. 5), that extends the famous Urysohn's Lemma of point-set topology. From Theorem 12 we can arrive, in a similar way, at the frame extension of the (separation) lemma for extremally disconnected spaces in Gillman and Jerison ([7, 1.H]):

Corollary 19 (Separation: Gillman and Jerison; [9]). *A frame L is extremally disconnected if and only if, for every $a, b \in L$ satisfying $a \wedge b = 0$, there exists $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $h((—, 0) \vee (1, —)) = 0$, $h(0, —) \leq a^*$ and $h(—, 1) \leq b^*$.* □

If we do a similar thing with Theorem 15 we arrive to the pointfree extension of a separation result due to Vedenissoff ([28]):

Corollary 20 (Bounded separation: Vedenissoff; [10]). *A frame L is perfectly normal if and only if, for every $a, b \in L$ satisfying $a \vee b = 1$, there exists $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $h((—, 0) \vee (1, —)) = 0$, $h(0, —) = a$ and $h(—, 1) = b$.* □

8. Extension-type corollaries

We conclude our journey through pointfree real functions with the question: when is it possible to extend a continuous functions from a sublocale of L to all of L?

For any sublocale S of L, let $c_S : L \twoheadrightarrow S$ denote the corresponding frame quotient, given by $c_S(x) = \bigwedge\{s \in S \mid x \leq s\}$. A continuous $\widetilde{h} \in C(L)$ is said to be a *continuous extension* of $h \in C(S)$ whenever $c_S \circ \mathfrak{c} \circ \widetilde{h} = \mathfrak{c} \circ h$ ([11]).

As outlined in [23], from Theorem 11 it also follows the well-known (extension) Theorem of Tietze for frames:

Corollary 21 (Extension: Tietze; [21, 23]). *For a frame L, the following are equivalent:*

(i) *L is normal.*
(ii) *For any closed sublocale S of L and any $h \in C(S)$, there exists a continuous extension $\widetilde{h} \in C(L)$ of h.* □

Dually, from Theorem 12 it readily follows:

Corollary 22 (Extension: Gillman and Jerison; [9]). *For a frame L, the following are equivalent:*

(i) *L is extremally disconnected.*
(ii) *For any open sublocale S of L and any $h \in C(S)$, there exists a continuous extension $\widetilde{h} \in C(L)$ of h.* □

A similar characterization holds for perfectly normal frames, in terms of the rings $C^*(L)$ and $C^*(S)$ of *bounded* functions (of course, an $f \in F(L)$ is bounded in case $\mathbf{0} \leq f \leq \mathbf{1}$):

Corollary 23 (**Bounded extension;** [10]). *For a frame L, the following are equivalent:*

(i) *L is perfectly normal.*

(ii) *For every closed sublocale S of L and any $h \in C^*(S)$, there exists a continuous extension $\widetilde{h} \in C^*(L)$ of h such that $\widetilde{h}(0,1) \geq S$.* □

Remark 24. Replacing the frame $\mathfrak{L}(\mathbb{R})$ of reals by the frame $\mathfrak{L}\left(\overline{\mathbb{R}}\right)$ of *extended reals* (defined by dropping conditions (R'_5) and (R'_6) in Section 2) we are able to deal with *rings of extended real functions*. This is the object of study of the ongoing research project [2].

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