Background	Localic real-valued functions	Insertion theorems	Extension theorems	References

# Localic real functions: a general setting

Making the ring of continuous localic real functions into a subring of all localic real functions

#### Javier Gutiérrez García

Department of Mathematics, University of the Basque Country, SPAIN

- joint work with Tomasz Kubiak (Poznan) and Jorge Picado (Coimbra)

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References

"The set C(X) of all continuous, real-valued functions on a topological space X will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection  $\mathbb{R}^X$  of all functions from X into the set  $\mathbb{R}$  of real numbers. [...]

In fact, it is clear that  $\mathbb{R}^{X}$  is a commutative ring with unity element (provided that X is non empty). [...]

Therefore C(X) is a commutative ring, a subring of  $\mathbb{R}^X$ ."



L. Gillman and M. Jerison. **Rings of Continuous Functions** 

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Background ●○○○○○	Localic real-valued functions	Insertion theorems	Extension theorems	Referen
Frames				

## The category of frames (locales)

#### pointfree topology

$$(X, \mathcal{O}X) \longrightarrow (\mathcal{O}X, \subseteq)$$

$$i \qquad i$$

 $f^{-1}$  preserves igvee and  $\wedge$ 

 $(Y, \mathcal{O}Y)$  $(\mathcal{O}Y, \subseteq)$ TopFrm $\mathsf{Top}(X, \Sigma L) \simeq \mathsf{Frm}(L, \mathcal{O}X)$ 

Gutiérrez García-Kubiak-Picado Localic real function

Localic real functions: a general setting

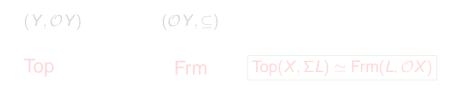
Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
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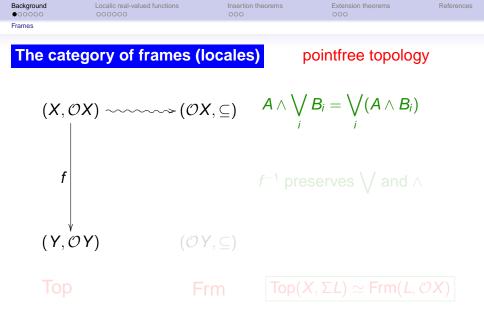
### The category of frames (locales)

pointfree topology

$$(X, \mathcal{O}X) \longrightarrow (\mathcal{O}X, \subseteq) \quad A \land \bigvee_i B_i = \bigvee_i (A \land B_i)$$

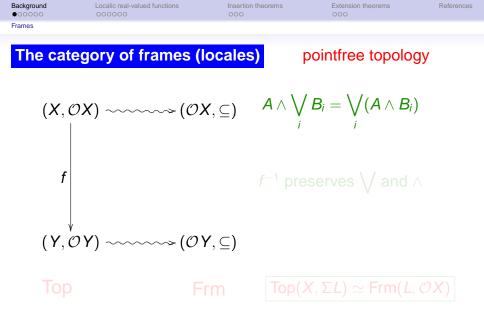
 $^{f^{-1}}$  preserves igvee and  $\wedge$ 





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Localic real functions: a general setting



Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

Background	Localic real-valued functions	Insertio	n theorems	Extension theorems	References
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The cate	gory of frames (I	ocales	s) poi	ntfree topol	ogy
(X, O)	X)> (O)	<b>⟨</b> ,⊆)	$A \wedge \bigvee_i B_i$	$=\bigvee_{i}(A\wedge I)$	B <sub>i</sub> )
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(Y, O)	Y)> (O)	 ∕,⊆)			
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Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

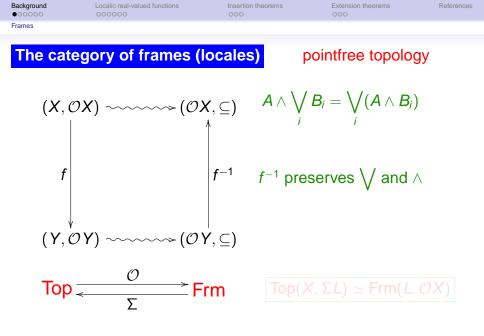
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f		f <sup>-1</sup>	<i>f</i> <sup>−1</sup> prese	rves	$\wedge$
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Background	Localic real-valued functions	Insertio	n theorems	Extension theorems	References
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(Y, O)	Y)> (O)	<b>/</b> ,⊆)			
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Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
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(X ())	( <i>OX</i> , <u>OX</u> ,	$ A \land \backslash B$	$_{i} = \bigvee (A \wedge B_{i})$	
	$(\mathcal{O}\mathcal{X}, \underline{Y})$	=) <b>v</b>	i	
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(Y O)	$\prime$ ) $\sim \mathcal{O} Y, O$	-)		
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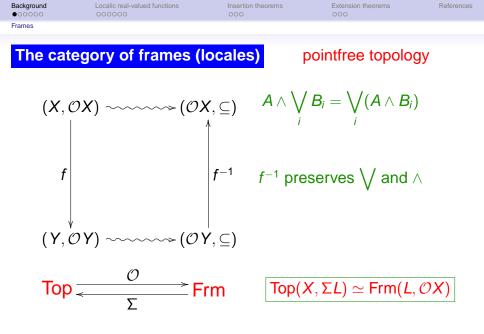
Gutiérrez García-Kubiak-Picado Localic real functi

Localic real functions: a general setting



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Localic real functions: a general setting



Gutiérrez García–Kubiak–Picado

Localic real functions: a general setting

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
The frame of reals				

The *frame of reals* is the frame  $\mathfrak{L}(\mathbb{R})$  generated by all ordered pairs (p, q), where  $p, q \in \mathbb{Q}$ , subject to the relations:

$$\begin{array}{ll} (\mathsf{R1}) & (p,q) \land (r,s) = (p \lor r,q \land s) \\ (\mathsf{R2}) & p \le r < q \le s \Rightarrow (p,q) \lor (r,s) = (p,s) \\ (\mathsf{R3}) & (p,q) = \bigvee \{(r,s) \mid p < r < s < q\} \\ (\mathsf{R4}) & \bigvee \{(p,q) \mid p,q \in \mathbb{Q}\} = 1. \end{array}$$

$$(-,q) := \bigvee_{p \in \mathbb{Q}} (p,q)$$

 $\mathfrak{L}_{l}(\mathbb{R}) = \langle (-,q) \mid q \in \mathbb{Q} \rangle$ 

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Background	Localic real-va		sertion theorems	Extension theorems	References
Real-valued fun	ictions				
		Тор	Frm		
	continuous	$f:X \to (\mathbb{R}, \mathcal{T}_e)$	$h: \mathfrak{L}(\mathbb{R}) \to L$		
		$f:X \to (\mathbb{R}, \mathcal{T}_l)$	$h: \mathfrak{L}_l(\mathbb{R}) \to L$		
		$f: X \to (\mathbb{R}, \mathcal{T}_u)$	$h: \mathfrak{L}_u(\mathbb{R}) \to L$		
		$Top(X,\mathcal{T}_e)$	$\simeq Frm(\mathfrak{L}(\mathbb{R}), \mathfrak{c})$	OX)	

B. Banaschewski, *The real numbers in pointfree topology* Textos de Matemática, Série B, 12, Univ. de Coimbra, 1997.

Background	Localic real-va		nsertion theorems	Extension theorems	References
Real-valued fur	nctions				
		Тор	Frm	]	
	continuous	$f: X \to (\mathbb{R}, \mathcal{T}_e)$	$h: \mathfrak{L}(\mathbb{R}) \to L$		
	usc	$f:X \to (\mathbb{R}, \mathcal{T}_l)$	$h: \mathfrak{L}_{l}(\mathbb{R}) \to L$	satisfying()	
	lsc		$h: \mathfrak{L}_u(\mathbb{R}) \to L$		
			???		

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Background	Localic real-va		nsertion theorems	Extension theorems	References
Real-valued fun	ctions				
		Тор	Frm	١	
	continuous	$f: X \to (\mathbb{R}, \mathcal{T}_e)$	$h: \mathfrak{L}(\mathbb{R}) \to L$		
	usc	$f: X \to (\mathbb{R}, \mathcal{T}_l)$	$h: \mathfrak{L}_{l}(\mathbb{R}) \to L$	satisfying()	
	lsc	$f: X \to (\mathbb{R}, \mathcal{T}_u)$	$h: \mathfrak{L}_u(\mathbb{R}) \to L$	satisfying()	
			???		
		$Top(X, \mathcal{T}_l)$	$ \simeq \operatorname{Frm}(\mathfrak{L}_l(\mathbb{R}),) $	<i>OX</i> ) !!!	

J. Gutiérrez García and J. Picado On the algebraic representation of semicontinuity Journal of Pure and Applied Algebra, 210 (2007) 299–306.

Background	Localic real-va	Localic real-valued functions Inser		Extension theorems	References
Real-valued fur	nctions				
		Тор	Frm		
	continuous	$f: X \to (\mathbb{R}, \mathcal{T}_e)$	$h: \mathfrak{L}(\mathbb{R}) \to L$		
	usc	$f:X\to (\mathbb{R}, \mathcal{T}_l)$	$h: \mathfrak{L}_{l}(\mathbb{R}) \to L$	satisfying()	
	lsc	$f: X \to (\mathbb{R}, \mathcal{T}_u)$	$h: \mathfrak{L}_u(\mathbb{R}) \to L$	satisfying()	
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Real-valued fund	ictions				
		Тор	Frm	]	
	continuous	$f: X \to (\mathbb{R}, \mathcal{T}_e)$	$h: \mathfrak{L}(\mathbb{R}) \to L$		
	usc	$f:X \to (\mathbb{R}, \mathcal{T}_l)$	$h: \mathfrak{L}_{l}(\mathbb{R}) \to L$	satisfying()	
	lsc	$f: X \to (\mathbb{R}, \mathcal{T}_u)$	$h:\mathfrak{L}_u(\mathbb{R})\to L$	satisfying()	
	C(X) = USC	$C(X) \cap LSC(X)$	???		

Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

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Background	Localic real-v		nsertion theorems	Extension theorems	References
Real-valued func	tions				
		Тор	Frm		
	continuous	$f: X \to (\mathbb{R}, \mathcal{T}_e)$	$h: \mathfrak{L}(\mathbb{R}) \to L$		
	usc	$f: X \to (\mathbb{R}, \mathcal{T}_{l})$	$h: \mathfrak{L}_l(\mathbb{R}) \to L$	satisfying()	
	lsc	$f: X \to (\mathbb{R}, \frac{T_u}{U})$	$h: \mathfrak{L}_u(\mathbb{R}) \to L$	satisfying()	

 $C(X) = USC(X) \cap LSC(X)$ 

???

Background	Localic real-va	Localic real-valued functions Inse 000000 00			extension theorems	I	References
Real-valued functions							
		Тор		Frm			
со	ntinuous	$f:X ightarrow (\mathbb{R},\mathcal{T}_{e})$	h: $\mathfrak{L}(\mathbb{R})$	$\rightarrow L$			
	1100	$f \cdot \mathbf{V} \to (\mathbb{D} \cdot \mathbf{T})$	$() h \cdot e(\mathbb{D})$	. 1	coticfuina(	)	

 $t: X \to (\mathbb{R}, \mathcal{T}_l) \mid h: \mathfrak{L}_l(\mathbb{R}) \to L \quad satisfying(\dots)$ usc  $f: X \to (\mathbb{R}, \mathcal{T}_u) \mid h: \mathfrak{L}_u(\mathbb{R}) \to L \quad satisfying(\dots)$ lsc ???

 $C(X) = USC(X) \cap LSC(X)$ 

(Q1) How to remedy this?

Background ○○○●○○	Localic real-valued functions	Insertion theorems	Extension theorems	References
Real-valued functions				
	Тор	[	Frm	
lsc a	Every $f: X \to \mathbb{R}$ ad and usc regularizat aking $f$ more "regul	ions ???		

(Q2)

How can we speak about general localic real functions?

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Real-valued functions				
	Тор		Frm	
E	very $f: X \to \mathbb{R}$ adn	nits		

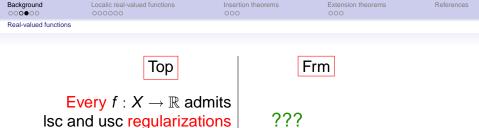
lsc and usc regularizations
(making f more "regular")

???

(Q2)

How can we speak about general localic real functions?

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(making f more "regular")

How can we speak (Q2) about general localic real functions?

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
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The congruence frame				

- sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
- sublocale sets.

A *congruence* on a frame *L*, is an equivalence relation  $\theta$  on *L* which is a subframe of  $L \times L$  in the obvious sense.

The lattice of frame congruences on *L* under set inclusion is a frame, denoted by  $\mathcal{C}L$ .

Open and closed congruences:

$$\Delta_a = \{(a,b) \in L \times L \mid a \land x = b \land x\}$$
  
$$\nabla_a = \{(a,b) \in L \times L \mid a \lor x = b \lor x\}$$

Complemented:

$$eg \Delta_a = 
abla_a$$

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The lattice of frame congruences on *L* under set inclusion is a frame, denoted by  $\mathcal{C}L$ .

Open and closed congruences:

$$\Delta_a = \{(a,b) \in L \times L \mid a \land x = b \land x\}$$
  
$$\nabla_a = \{(a,b) \in L \times L \mid a \lor x = b \lor x\}$$

Complemented:

$$eg \Delta_a = 
abla_a$$

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
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The congruence frame				

Quotients in Frm (equivalently, subobjects in  $Loc = Frm^{op}$ ):

- sublocale maps (i.e. onto frame homomorphisms),
- congruences,
- nuclei
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The congruence frame				

The correspondence  $x \mapsto \nabla_x$  defines an isomorphism  $L \to \nabla L$ .

 $abla : L \xrightarrow{\simeq} \nabla L \subset \mathfrak{C}L$ 

Closure and interior of a congruence:

 $\overline{\theta} = \bigvee \{ \nabla_{a} : \nabla_{a} \le \theta \} \qquad \overset{\circ}{\theta} = \bigwedge \{ \Delta_{a} : \theta \le \Delta_{a} \}.$ 

$$\overline{\Delta_a} = 
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Background 000000	Localic real-valued functions	Insertion theorems	Extension theorems	References
Motivation				

 $\mathsf{Top}\left(X,\left(\mathbb{R},\mathcal{T}_{\mathsf{e}}\right)\right)\simeq\mathsf{Frm}\left(\mathfrak{L}\left(\mathbb{R}\right),\mathcal{O}X\right)$ 

 $\mathrm{F}\left(X,\mathbb{R}
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Slogan of pointfree topology:

Congruences in  $L \equiv$  generalized subspaces

 $\mathrm{F}(X,\mathbb{R})\simeq\mathrm{Frm}(\mathfrak{L}(\mathbb{R}),\mathcal{P}(X))\longrightarrow\mathrm{Frm}(\mathfrak{L}(\mathbb{R}),\mathfrak{C}L)$ 

A *localic real function* on *L* is a frame homomorphism  $\mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ .

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#### Definition

A *localic real function* on *L* is a frame homomorphism  $\mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ .

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Continuous function	S			
• /	${\sf F}:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C} L$	general	F( <i>L</i> )	
	$F:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L$ t. $F(\mathfrak{L}_l(\mathbb{R}))\subseteq  abla L$		USC(L)	
	$F:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L\ \mathfrak{t}.\ F(\mathfrak{L}_d(\mathbb{R}))\subseteq  abla L$		LSC(L)	
	$F:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L$ t. $F(\mathfrak{L}(\mathbb{R}))\subseteq  abla L$		C( <i>L</i> )	
		C(I) =	$\operatorname{USC}(I) \cap \operatorname{USC}(I)$	

$$\mathcal{C}(L) = \mathrm{USC}(L) \cap \mathrm{LSC}(L)$$

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Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Continuous function	15			
• /	${\sf F}:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C} L$	general	F( <i>L</i> )	
	$F:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L$ t. $F(\mathfrak{L}_l(\mathbb{R}))\subseteq  abla L$		USC( <i>L</i> )	
	$F:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L$ t. $F(\mathfrak{L}_d(\mathbb{R}))\subseteq  abla L$		LSC(L)	
	$F:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L$ t. $F(\mathfrak{L}(\mathbb{R}))\subseteq  abla L$		C( <i>L</i> )	
		C(I) –	$USC(I) \cap ISC(I)$	$\left( \right)$

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Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Continuous function	S			
• F	$\bar{T}:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C}L$	general	F( <i>L</i> )	
	$F: \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ t. $F(\mathfrak{L}_l(\mathbb{R})) \subseteq \nabla L$	USC	USC( <i>L</i> )	
	$m{F}: \mathfrak{L}(\mathbb{R})  ightarrow \mathfrak{C}L$ t. $F(\mathfrak{L}_{\iota}(\mathbb{R})) \subseteq  abla L$		LSC( <i>L</i> )	
	$ar{F}:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L$ t. $F(\mathfrak{L}(\mathbb{R}))\subseteq  abla L$		C( <i>L</i> )	
		C(1)		(1)

 $C(L) = USC(L) \cap LSC(L)$ 

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Continuous function	s			
• F	$\exists:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C}L$	general	F( <i>L</i> )	
	$F: \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ t. $F(\mathfrak{L}_{l}(\mathbb{R})) \subseteq \nabla L$	USC	USC( <i>L</i> )	
	$F:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L$ t. $F(\mathfrak{L}_d(\mathbb{R}))\subseteq  abla L$		LSC( <i>L</i> )	
	$egin{aligned} & oldsymbol{\in} \mathfrak{L}(\mathbb{R})  o \mathfrak{C}L \ &  ext{t. } F(\mathfrak{L}(\mathbb{R})) \subseteq  abla L \end{aligned}$		C(L)	
		$\sim$ (1)		

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Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Continuous function	s			
• /	$\exists:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C}L$	general	F( <i>L</i> )	
	$ f: \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L $ t. $F(\mathfrak{L}_{l}(\mathbb{R})) \subseteq \nabla L $	USC	USC( <i>L</i> )	
	$oldsymbol{\in}:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C}L$ t. $F(\mathfrak{L}_d(\mathbb{R}))\subseteq  abla L$	Isc	LSC( <i>L</i> )	
	$ar{T}:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C} L$ t. $F(\mathfrak{L}(\mathbb{R}))\subseteq  abla L$		C( <i>L</i> )	
		C(I) =	$USC(I) \cap LSC(I)$	

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Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Continuous function	S			
• /	$\exists:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C}L$	general	F( <i>L</i> )	
	$ f: \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L $ t. $F(\mathfrak{L}_l(\mathbb{R})) \subseteq \nabla L $	USC	USC( <i>L</i> )	
	$F: \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ t. $F(\mathfrak{L}_u(\mathbb{R})) \subseteq \nabla L$	Isc	LSC( <i>L</i> )	
	$ar{F}:\mathfrak{L}(\mathbb{R}) o\mathfrak{C}L,$ t. $F(\mathfrak{L}(\mathbb{R}))\subseteq  abla L$		C( <i>L</i> )	
		C(L) =	$\mathrm{USC}(L)\cap\mathrm{LSC}(L)$	_)

Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

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Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Continuous function	S			
• /	$\exists:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C}L$	general	F( <i>L</i> )	
• /	$\exists:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C}L$	USC	USC(L)	
S.	t. $F(\mathfrak{L}_{l}(\mathbb{R})) \subseteq \nabla L$			
• /	$\exists:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C}L$	lsc	LSC(L)	
S.	t. $F(\mathfrak{L}_u(\mathbb{R})) \subseteq \nabla L$			
• /	$\exists:\mathfrak{L}(\mathbb{R}) ightarrow\mathfrak{C} L$	continuous	C(L)	
S.	t. $F(\mathfrak{L}(\mathbb{R})) \subseteq \nabla L$			
		C(L) =	$USC(L) \cap LSC($	L)

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Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Continuous function	15			
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• /	$\exists : \mathfrak{L}(\mathbb{R})  ightarrow \mathfrak{C}L$	USC	USC(L)	
	( )			
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		100		
S.	t. $F(\mathfrak{L}_u(\mathbb{R})) \subseteq \nabla L$			
		aantinuuuu	C(1)	
	$\exists : \mathfrak{L}(\mathbb{R})  ightarrow \mathfrak{C}L$	continuous	C(L)	
S.	t. $F(\mathfrak{L}(\mathbb{R})) \subseteq \nabla L$			
		C(L) = C(L)	$\mathrm{USC}(L)\cap\mathrm{LSC}(L)$	_)

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Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
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Characteristic functions				

The characteristic function  $\chi_{\theta} \in F(L)$ :

$$\boldsymbol{\chi}_{\boldsymbol{\theta}}(-,q) = \begin{cases} 0 & \text{if } q \leq 0 \\ \theta & \text{if } 0 < q \leq 1, \\ 1 & \text{if } q > 1 \end{cases} \quad \boldsymbol{\chi}_{\boldsymbol{\theta}}(\boldsymbol{p},-) = \begin{cases} 1 & \text{if } \boldsymbol{p} < 0 \\ \neg \theta & \text{if } 0 \leq \boldsymbol{p} < 1 \\ 0 & \text{if } \boldsymbol{p} \geq 1. \end{cases}$$

•  $\chi_{\theta} \in \text{USC}(L)$  if and only if  $\theta$  is closed

- $\chi_{\theta} \in \mathrm{LSC}(L)$  if and only if  $\theta$  is open.
- *χ*<sub>θ</sub> ∈ C(L) if and only if θ is clopen.

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Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
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Regularization of a g	eneral real function			

$$F^{\circ}(-,q) = \bigvee_{s < q} \neg \overline{F(s,-)}$$
 and  $F^{\circ}(p,-) = \bigvee_{r > p} \overline{F(r,-)}.$ 

$$F^{\circ} \leq F$$

$$F^{\circ \circ} = F^{\circ}$$

$$F^{\circ} \in LSC(L)$$

$$G \in LSC(L) \text{ and } G \leq F \quad \Rightarrow \quad G \leq F^{\circ}$$

$$(\chi_{\theta})^{\circ} = \chi_{\theta}^{\circ}$$

Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

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Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

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Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
Regularization of a ge	eneral real function			

$$F^{\circ}(-,q) = \bigvee_{s < q} \neg \overline{F(s,-)}$$
 and  $F^{\circ}(p,-) = \bigvee_{r > p} \overline{F(r,-)}.$ 

$$\begin{array}{l} F^{\circ} \leq F \\ F^{\circ \circ} = F^{\circ} \\ F^{\circ} \in \mathrm{LSC}(\mathcal{L}) \\ G \in \mathrm{LSC}(\mathcal{L}) \text{ and } G \leq F \quad \Rightarrow \quad G \leq F^{\circ} \\ \left(\chi_{\theta}\right)^{\circ} = \chi_{\frac{\theta}{\theta}} \end{array}$$

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$$F^{\circ\circ} = F^{\circ}$$

$$F^{\circ} \in LSC(L)$$

$$G \in LSC(L) \text{ and } G \leq F \quad \Rightarrow \quad G \leq F^{\circ}$$

$$(\chi_{\theta})^{\circ} = \chi_{\overset{\circ}{\theta}}$$

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
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Regularization of a	a general real function			

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Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

Background	Localic real-valued functions	Insertion theorems	Extension theorems	References
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Regularization of a genera	al real function			

For  $F \in \overline{F}(L)$  we define the *upper regularization*  $F^-$ :

$$F^{-}(-,q) = \bigvee_{s < q} \overline{F(-,s)}$$
 and  $F^{-}(p,-) = \bigvee_{r > p} \neg \overline{F(-,r)}.$ 

$$F \leq F^{-}$$

$$F^{--} = F^{-}$$

$$F^{-} \in USC(L)$$

$$G \in USC(L) \text{ and } F \leq G \quad \Rightarrow \quad F^{-} \leq G$$

$$(\chi_{\theta})^{-} = \chi_{\overline{\theta}}$$

Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

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Gutiérrez García-Kubiak-Picado Localic real functions: a general setting

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Background 000000	Localic real-valued functions	Insertion theorems	Extension theorems	References
Achievements				

# Localic real-valued functions

### Achievements

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- One can see semicontinuous functions as a particular kind of real-valued functions on the frame of congruences, with the same domain, namely L(R).
- Being all upper and lower semicontinuous functions particular kinds of real-valued functions on the frame of congruences, we can compare them.
- By considering the algebraic operations of the ring Frm(𝔅(ℝ), 𝔅L), we obtain, in particular, a way of defining the sum of upper and lower semicontinuous functions.
- The class of continuous functions is precisely the intersection of the classes of lower and upper ones.
- The situation with respect to regularization is precisely the same as in the topological setting.

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# Theorem (Katětov-Tong)

The following conditions on a frame L are equivalent:

(1) *L* is normal.

(2) For every F ∈ USC(L) and every G ∈ LSC(L) with F ≤ G, there exists H ∈ C(L) such that F ≤ H ≤ G.

# Theorem (Stone)

The following conditions on a frame L are equivalent:

(1) L is extremally disconnected.

(2)  $C(L) = \{F^- : F \in LSC(L)\}.$ 

 $(3) C(L) = \{ \mathbf{G}^\circ : \mathbf{G} \in \mathrm{USC}(L) \}.$ 

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Background 000000	Localic real-valued functions	Insertion theorems ●○○	Extension theorems	References

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Background	Localic real-valued functions	Insertion theorems ○●○	Extension theorems	References

Let  $UL(L) = \{(F, G) \in USC(L) \times LSC(L) : F \leq G\}$  with the order  $(F_1, G_1) \leq (F_2, G_2) \iff F_2 \leq F_1$  and  $G_1 \leq G_2$ .

## Theorem (Monotone Katětov-Tong)

For a frame L, the following are equivalent:

- (1) L is monotonically normal.
- (2) There exists a monotone function Λ : UL(L) → C(L) such that F ≤ Λ(F, G) ≤ G for all (F, G) ∈ UL(L).

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Background 000000	Localic real-valued functions	Insertion theorems ○●○	Extension theorems	References

Let  $UL(L) = \{(F, G) \in USC(L) \times LSC(L) : F \leq G\}$  with the order  $(F_1, G_1) \leq (F_2, G_2) \iff F_2 \leq F_1$  and  $G_1 \leq G_2$ .

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For a frame L, the following are equivalent:

- (1) L is monotonically normal.
- (2) There exists a monotone function  $\Lambda : UL(L) \to C(L)$  such that  $F \leq \Lambda(F, G) \leq G$  for all  $(F, G) \in UL(L)$ .

Background 000000	Localic real-valued functions	Insertion theorems ○○●	Extension theorems	References

#### Theorem

The following conditions on a frame L are equivalent:

- (1) L is completely normal.
- (2) L is hereditarily normal.
- (3) Each open sublocale of L is normal.
- (4) For every  $F, G \in F(L)$ , if  $F^- \leq G$  and  $F \leq G^\circ$ , then there exists an  $H \in LSC(L)$  such that  $F \leq H \leq H^- \leq G$ .

For each frame L the following are equivalent:

Strict insertion Michael insertion theorem for perfectly normal frames... Dowker insertion theorem for normal and countably paracompact frames...

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Each  $\theta \in \mathfrak{C}L$  determines a unique sublocale  $S_{\theta} \subseteq L$  and a unique frame quotient  $c_{\theta} \in \operatorname{Frm}(L, S_{\theta})$ .

 $H \in C(L)$  is said to be a *continuous extension* of  $H \in C(S_{\theta})$  if and only if the following diagram commutes



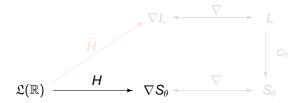
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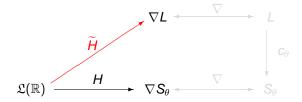


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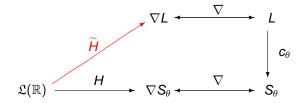
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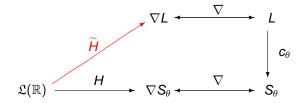


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## Theorem (Tietze)

The following conditions on a frame L are equivalent:

- (1) L is normal.
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Also versions for monotone normality, perfect normality, ...

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For a frame L, the following are equivalent:

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Background 000000	Localic real-valued functions	Insertion theorems	Extension theorems	References

#### J. Gutiérrez García and J. Picado

On the algebraic representation of semicontinuity Journal of Pure and Applied Algebra, 210 (2007) 299–306.

#### J. Gutiérrez García, T. Kubiak and J. Picado

Monotone insertion and monotone extension of frame homomorphisms

Journal of Pure and Applied Algebra, 212 (2008) 955–968.

Lower and upper regularizations of frame semicontinuous real functions To appear in: Algebra Universalis (2008)

To appear in: *Algebra Universalis*, (2008).

- Pointfree forms of Dowker and Michael insertion theorems To appear in: Journal of Pure and Applied Algebra, (2008).
  - Localic real-valued functions: a general setting Submitted.

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