On lattice-valued frames

Javier Gutiérrez García

(*joint work with Ulrich Höhle and M. Angeles de Prada Vicente*)

Riga, December 17, 2010

eman ta zabal zazu

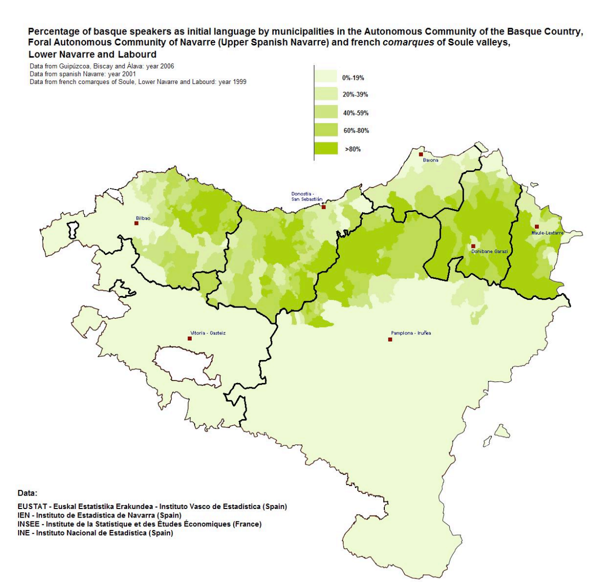
universidad del país vasco

euskal herriko unibertsitatea

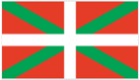
UNIVERSITAS LATVIENSIS

SCIENTIAE ET PATRIAE


Golfe de
Gascogne
France
Biarritz
Bayonne
San Sebastián
Bilbao
Pamplona
Vitoria
Logroño
Spain


Percentage of basque speakers as initial language by municipalities in the Autonomous Community of the Basque Country, Foral Autonomous Community of Navarre (Upper Spanish Navarre) and french *comarques* of Soule valleys, Lower Navarre and Labourd
Data from Guipúzcoa, Biscay and Álava: year 2006
Data from spanish Navarre: year 2001
Data from french comarques of Soule, Lower Navarre and Labourd: year 1999
0%-19%
20%-39%
40%-59%
60%-80%
>80%
Baiona
Donostia - San Sebastián
Bilbao
Maule-Lextarre
Donibane Garazi
Vitoria - Gasteiz
Pamplona - Iruñea
Data:
EUSTAT - Euskal Estatistika Erakundea - Instituto Vasco de Estadística (Spain)
IEN - Instituto de Estadística de Navarra (Spain)
INSEE - Institute de la Statistique et des Études Économiques (France)
INE - Instituto Nacional de Estadística (Spain)

Basque Country
Euskal Herria


Flag


Coat of arms

(and largest city)	Bilbao
Official language(s)	Basque, Spanish, French

Area	
- Total	20,947 km^2 8,088 sq mi
Population	
- estimate	about 3,000,000

Republic of Latvia
Latvijas Republika


Flag


Coat of arms

Capital (and largest city)	Riga 56°57′N 24°6′E
Official language(s)	Latvian

Area	
- Total	64,589 km^2 (124th) 24,938 sq mi
- Water (%)	1.57% (1,014 km^2)
Population	
- July 2010 estimate	▼ 2,217,969 [3] (143rd)
- 2000 ppl census	2,377,383

(Information taken from Wikipedia)

Origin of the work

- 29th Linz Seminar on Fuzzy Set Theory:

  "*Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology*"

  Linz, (Austria), February 4 to 9, (2008)

  Attendants:
  Rodabaugh, Höhle, de Prada Vicente, Shostak...

A. Pultr, S.E. Rodabaugh,
Lattice-valued frames, functor categories, and classes of sober spaces,
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- Visit of Ulrich Höhle to Bilbao, April 4 to 11, (2008)

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Introduction

- Pointfree (Pointless) topology, Frame (Locale) theory
- *L*-valued (Fuzzy) topology

Pointfree topology

Pointfree topology

Motivation

$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq)$$

Pointfree topology

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$$(X, \mathcal{O}X) \rightsquigarrow (\mathcal{O}X, \subseteq) \qquad A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$$

Pointfree topology

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f

$(Y, \mathcal{O}Y)$

Pointfree topology

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$$
\begin{array}{ccc}
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\Big\downarrow f & & \\
(Y, \mathcal{O}Y) & \rightsquigarrow & (\mathcal{O}Y, \subseteq)
\end{array}
$$

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$$\begin{array}{ccc}
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f^{-1} preserves $\bigcup$ and $\cap$

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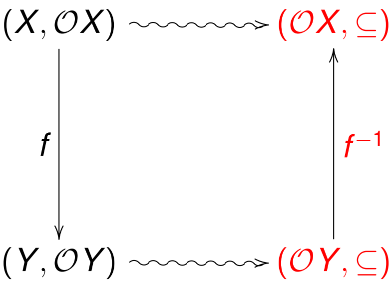

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TOPOLOGY $\rightsquigarrow$ Abstraction

Pointfree topology

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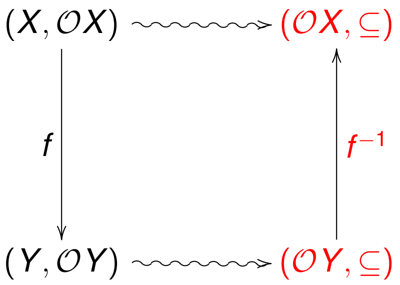

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TOPOLOGY ⇝ Abstraction ⇝ POINTFREE TOPOLOGY

Pointfree topology — the category of frames **Frm**

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- Morphisms, called frame homomorphisms, are those maps between frames that preserve arbitrary joins and finite meets.

Pointfree topology

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- $\mathcal{O}$: **Top** $\rightarrow$ **Frm** is a contravariant functor with $X \longmapsto \mathcal{O}X$ and $X \xrightarrow{f} Y \longmapsto \mathcal{O}Y \xrightarrow{f^{-1}} \mathcal{O}X$.

Pointfree topology — the dual category **Loc**=**Frm**^{op}

- The objects in **Loc** are frames, from now on, also called locales.

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Pointfree topology the dual category $\mathbf{Loc}=\mathbf{Frm}^{op}$

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Advantage: **Loc** can be thought of as a natural extension of (sober) spaces.

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Advantage: **Loc** can be thought of as a natural extension of (sober) spaces.

Disadvantage: Morphisms thought in this way may obscure the intuition.

Pointfree Topology, Pointless Topology, Frame Theory, Locale Theory…

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"*Locales not only "capture" or "model" the lattice theoretical behaviour of topological spaces, more importantly when we work in a universe where choice principles are not allowed, it is locales, not spaces, which provide the right context in which to do topology.*"

P.T. Johnstone, *The point of pointless topology*, Bull. Amer. Math. Soc. 8 (1983) 41-53.

Pointfree topology spatial frames and sober spaces

Apart from the functor $\mathcal{O}$: **Top** $\to$ **Frm**, there is a functor in the opposite direction, the spectrum functor

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Pointfree topology — spatial frames and sober spaces

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An element $p \in L \setminus \{1\}$ is called prime if for each $\alpha, \beta \in L$ with

$$\alpha \wedge \beta \leq p \quad \Longrightarrow \quad \alpha \leq p \text{ or } \beta \leq p.$$

We denote by $\mathrm{Spec}\, L$ the spectrum of L, i.e. the set of all prime elements of L.

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The functor Spec assigns to each frame L its spectrum $\mathrm{Spec}\, L$, endowed with the hull-kernel topology whose open sets are

$$\Delta_L(\alpha) = \{p \in \mathrm{Spec}\, L : \alpha \not\leq p\} = \mathrm{Spec}\, L \setminus {\uparrow}\alpha \qquad \text{for } \alpha \in L.$$

Pointfree topology spatial frames and sober spaces

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Recall that a topological space X is sober if the only prime opens are those of the form $X \setminus \overline{\{x\}}$ for some $x \in X$ and a frame L is spatial if L is generated by its prime elements, i.e. if

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The categories **Sob** of sober topological spaces and **SpatFrm** of spatial frames are dual under the restrictions of the functors $\mathcal{O}$ and Spec.

$$\mathbf{Sob} \sim \mathbf{SpatFrm}$$

L-valued topology

L-valued topology — the category L-**Top**

With L a complete lattice and X a set, L^X is the complete lattice of all maps from X to L, called L-sets, in which

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If $\alpha \in L$, the associated constant map is denoted $\underline{\alpha}$.

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If $f : X \to Y$ and $b \in L^Y$ we let $f^{-1}(b) = b \circ f \in L^X$.

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An L-valued topological space (shortly, an L-topological space) is a pair (X, τ) consisting of a set X and a subset τ of L^X (the L-valued topology or L-topology on the set X) closed under finite meets and arbitrary joins.

L-valued topology

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Given two L-topological spaces $(X, \tau), (Y, \sigma)$ a map $f : X \to Y$ is an L-continuous map if the correspondence $f^{-1}(b)$ maps σ into τ. The resulting category will be denoted by L-**Top**.

"*It is a natural and interesting question whether or not it is possible to establish a category to play the same role with respect to a given notion of fuzzy topology as that locales play for topological spaces.*"

D. Zhang, Y. Liu, *L-fuzzy version of Stone's representation theorem for distributive lattices*, Fuzzy Sets and Systems 76 (1995) 259-270.

Chain-valued frames

- Motivation: the iota functor ι_L
- Definition

The ι_L functor

The iota functor ι_L was originally introduced by Lowen with $L = [0, 1]$ and later on extended by Kubiak to an arbitrary complete lattice.

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Let L be a complete lattice and X be a set. For a fixed $\alpha \in L \setminus \{1\}$ and let $a \in L^X$, we denote

$$[a \not\leq \alpha] = \{x \in X : a(x) \not\leq \alpha\}.$$

This defines a map $\iota_\alpha : L^X \to \mathbf{2}^X$ by $\iota_\alpha(a) = [a \not\leq \alpha]$.

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Now, given an L-topology τ on X, we consider the topology

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This defines a functor ι_L : L-Top $\to$ Top by

$$\iota_L(X,\tau) = (X, \iota_L(\tau)), \qquad \iota_L(h) = h.$$

The ι_L functor — chain valued frames

Let L be a complete chain. Then $\iota_\alpha(a) = [a > \alpha]$.

The ι_L functor

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(F1) $\iota_L(\tau) = \langle \bigcup_{\alpha \in L \setminus \{1\}} \iota_\alpha(\tau) \rangle$. (collectionwise extremally epimorphic)

(F2) If $a \neq b$ in τ, then $\iota_\alpha(a) \neq \iota_\alpha(b)$ for some $\alpha \in L \setminus \{1\}$. (collectionwise monomorphic)

Chain valued frames

"*The notion of chain-valued frame, is introduced to be an abstraction of the distinctive properties of the system of level mappings from an L-topology τ into $\iota_L(\tau)$. These conditions, when L is a complete chain, were taken as axioms* (F0), (F1) *and* (F2) *in order to define L-frames and the associated category L*-Frm."

- A. Pultr, S.E. Rodabaugh, *Lattice-valued frames, functor categories, and classes of sober spaces*, in: Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets,
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Chain valued frames

Let L be a complete chain. An L-frame A is a system

$$\left(\varphi_\alpha^A : A^{\mathfrak{u}} \to A^{\mathfrak{l}} \mid \alpha \in L \setminus \{1\}\right)$$

of frame morphisms – $A^{\mathfrak{u}}$ is the upper frame and $A^{\mathfrak{l}}$ is the lower frame – satisfying each of these conditions:

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(F0) $\varphi_\alpha^A = \bigvee_{\beta \in L\setminus\{1\};\alpha<\beta} \varphi_\beta^A$ for every $\alpha \in L \setminus \{1\}$.

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(F0) $\varphi_\alpha^A = \bigvee_{\beta \in L \setminus \{1\}; \alpha < \beta} \varphi_\beta^A$ for every $\alpha \in L \setminus \{1\}$.

(F1) $A^{\mathfrak{l}} = \langle \bigcup_{\alpha \in L \setminus \{1\}} \varphi_\alpha^A(A^{\mathfrak{u}}) \rangle$. (collectionwise extremally epimorphic)

A. Pultr, S.E. Rodabaugh, *Category theoretic aspects of chain-valued frames: Part I: Categorical and presheaf theoretic foundations, Part II: Applications to lattice-valued topology*, Fuzzy Sets and Systems 159 (2008) 501–528 and 529–558.

Chain valued frames

Let L be a complete chain. An L-frame A is a system

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The resulting category, with composition and identities component-wise in Frm, is denoted by *L-Frm*.

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"*During the preparation of the Volume, U. Höhle communicated to the authors of Chapter 6 that a complete chain is really only needed for its meet-irreducibles, and that for spatial L one also has meet-irreducibles which suffice for the constructions of Chapter 6.*"

U. Höhle and S.E. Rodabaugh, *Weakening the requirement that L be a complete chain*, in: Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, 2003, pp. 189–197, (Chapter 7).

The completely distributive case

- Completely distributive lattices
- Lattice-valued frames for CD lattices

Completely distributive lattices

Completely distributive lattices

Given $\alpha, \beta \in L$, we say that α is way below β, in symbols $\alpha \ll \beta$, if and only if

$$\left.\begin{array}{l} S \subseteq L \text{ and} \\ \beta \leq \bigvee S \end{array}\right\} \Longrightarrow \text{ there exist } \gamma_1, \ldots \gamma_n \in S \text{ such that } \alpha \leq \vee_{i=1}^{n} \gamma_i.$$

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Recall that L is continuous if and only if the way-below relation is approximating, i.e., if and only if

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Let $\alpha, \beta, \gamma, \delta \in L$, then:

(1) $\alpha \ll \beta$ implies $\alpha \leq \beta$.

(2) $\alpha \leq \beta \ll \gamma \leq \delta$ implies $\alpha \ll \delta$.

(3) If L is continuous, then $\alpha \ll \beta$ implies $\alpha \ll \gamma \ll \beta$ for some $\gamma \in L$

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Then we have that L^{op} is alertcontinuous if and only if it satisfies

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The following properties of the binary relation $\lll$ will be needed:

(1) $\alpha \lll \beta$ implies $\alpha \leq \beta$.

(2) $\alpha \leq \beta \lll \gamma \leq \delta$ implies $\alpha \lll \delta$.

Completely distributive lattices

A lattice is called completely distributive iff it is complete and for any familiy $\{x_{j,k} : j \in J, k \in K(j)\}$ in L the identity

$$\bigwedge_{j\in J} \bigvee_{k\in K(j)} x_{j,k} = \bigvee_{f\in M} \bigwedge_{j\in J} x_{j,f(j)} \tag{CD}$$

holds, where M is the set of choice functions defined on J with values $f(j) \in K(j)$.

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We recall now the following result:

Let L be a complete lattice. Then the following are equivalent:

(1) L is completely distributive.

(2) L is a spatial frame and L^{op} is continuous.

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(ii) $p = \bigwedge (\twoheaduparrow p \cap \mathrm{Spec}\, L)$ for each $p \in \mathrm{Spec}\, L$.

Lattice-valued frames for CD lattices

The ι_L functor completely distributive lattices

Let L be a completely distributive lattice. The mapping $\iota_p : \tau \to \iota_L(\tau)$ is a frame morphism for each $p \in \operatorname{Spec} L$ (this is not true in general if p fails to be prime). Consider the system of frame morphisms

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(F0) Since $p = \bigwedge \left(\twoheaduparrow p \cap \mathrm{Spec}\, L\right)$ for each $p \in \mathrm{Spec}\, L$, we have that

$$[f \not\leq p] = \bigcup_{q \in \twoheaduparrow p \cap \mathrm{Spec}\, L} [f \not\leq q] \qquad \text{for each } p \in \mathrm{Spec}\, L \text{ and } f \in L^X.$$

Consequently, for each $p \in \mathrm{Spec}\, L$,

$$\iota_p = \bigvee_{q \in \twoheaduparrow p \cap \mathrm{Spec}\, L} \iota_q.$$

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(F1) Since L is a spatial frame then $\{\iota_p(f) : f \in \tau, p \in \mathrm{Spec}\, L\}$ is a subbase of $\iota_L(\tau)$. Indeed, for each $\alpha \in L$ we have $\alpha = \bigwedge(\uparrow\alpha \cap \mathrm{Spec}\, L)$ and so

$$\iota_\alpha(f) = [f \not\leq \alpha] = \bigcup_{p \in \uparrow\alpha \cap \mathrm{Spec}\, L} [f \not\leq p] = \bigcup_{p \in \uparrow\alpha \cap \mathrm{Spec}\, L} \iota_p(f).$$

Hence,

$$\iota_L(\tau) = \Big\langle \bigcup_{p \in \mathrm{Spec}\, L} \iota_p(\tau) \Big\rangle.$$

The ι_L functor — completely distributive lattices

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(F2) Since L is a spatial frame, for each distinct $a, b \in \tau$ there exists $x \in X$ such that $a(x) \neq b(x)$, hence there exists $p \in \mathrm{Spec}\, L$ such that either $f(x) \leq p$ and $g(x) \not\leq p$ or $a(x) \not\leq p$ and $b(x) \leq p$ and so $[a \not\leq p] \neq [b \not\leq p]$. It follows that

$$\text{if } a \neq b \in \tau \text{ then } \iota_p(a) \neq \iota_p(b) \text{ for some } p \in \mathrm{Spec}\, L.$$

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Let L be a completely distributive lattice. An L-frame A is a system

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of frame morphisms – $A^{\mathfrak{u}}$ is the upper frame and $A^{\mathfrak{l}}$ is the lower frame – satisfying each of these conditions:

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- $\mathfrak{F}_1$ is complete and cocomplete.
- L-Frm is complete and cocomplete.

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We have already specified an answer by proving that the condition of a complete chain can be relaxed to a completely distributive lattice and that the completeness and cocompleteness of $\mathfrak{F}_0$, $\mathfrak{F}_1$ and L-Frm are still satisfied.

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In this context the natural question arises whether weakening of complete distributivity is still possible. As an answer to this question we show that complete distributivity is necessary for the property that for every L-topological space (X, τ) the system

$$\big(\iota_p : \tau \to \iota_L(\tau) \mid p \in \mathrm{Spec}\, L\big)$$

of frame homomorphisms ι_p satisfies (F0) and (F2).

L-valued frames possible extensions

Let L be a frame, (X, τ) an L-topological space and

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the system of frame morphisms determined by the ι-functor. Then:

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- If $(\iota_p)_{p \in \mathrm{Spec}\, L}$ satisfies axiom (F0) for each (X, τ), then

$$p = \bigwedge \left(\twoheaduparrow p \cap \mathrm{Spec}\, L \right) \quad \text{for each } p \in \mathrm{Spec}\, L.$$

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- If $(\iota_p)_{p \in \mathrm{Spec}\, L}$ satisfies axiom (F2) for each (X, τ), then L is spatial.

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- If $(\iota_p)_{p\in\operatorname{Spec} L}$ satisfies axiom $(\mathrm{F0})$ for each (X,τ), then

$$p = \bigwedge \left(\twoheaduparrow p \cap \operatorname{Spec} L\right) \quad \text{for each } p \in \operatorname{Spec} L.$$

- If $(\iota_p)_{p\in\operatorname{Spec} L}$ satisfies axiom $(\mathrm{F2})$ for each (X,τ), then L is spatial.
- If $(\iota_p)_{p\in\operatorname{Spec} L}$ is an L-frame for each (X,τ), then L is a completely distributive lattice.

Eskerrik asko! – ¡Muchas gracias!

Paldies!