The Dedekind order completion of $\mathrm{C}(L)$

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[1]Joint work with I. Mozo Carollo and J. Picado

Introduction

Motivation

A partially ordered set $(P, \leq)$ is called Dedekind ordered complete if every non–void subset A of P which is bounded from above has a supremum and, dually, every non–void subset B of P which is bounded from below has a infimum.

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A (Dedekind) order completion of a poset $(P, \leq)$ is a pair $(P^{\#}, \Phi)$ where

- $P^{\#}$ is a Dedekind order complete poset,
- $\Phi\colon P \to P^{\#}$ is an order embedding (usually $P \subseteq P^{\#}$) that preserve all suprema and infima that exists in P and satisfies

$$\hat{p} = \bigvee^{P^{\#}} \{\Phi(p) \in \Phi(P) \mid \Phi(p) \leq \widehat{p}\} = \bigwedge^{P^{\#}} \{\Phi(p) \in \Phi(P) \mid \Phi(p) \geq \widehat{p}\}$$

for every $\widehat{p} \in P^{\#}$.

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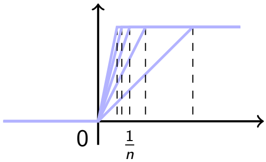
$$\mathrm{C}(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$$

$\mathrm{C}(X)^{\#}_{D} =$

$\{f : X \to \mathbb{R} \mid f$ is normal, lower semicont.$\}$

(Dilworth et al.)



$$f_n(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ n\,x, & \text{if } 0 \leq x \leq \frac{1}{n}; \\ 1, & \text{if } x \geq 1. \end{cases}$$

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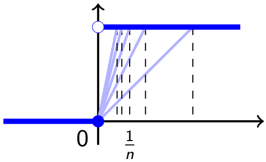

$$\bigvee f_n(x) = \begin{cases} 0, & \text{if } x \le 0; \\ 1, & \text{if } x > 0. \end{cases}$$

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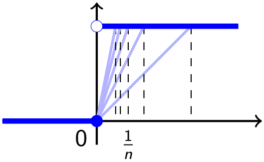

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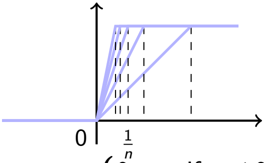

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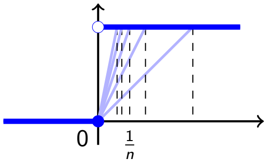

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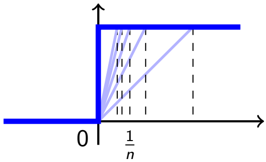

$$\bigvee f_n(x) = \begin{cases} [0,0], & \text{if } x < 0; \\ [0,1], & \text{if } x = 0; \\ [1,1], & \text{if } x > 0. \end{cases}$$

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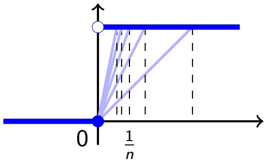

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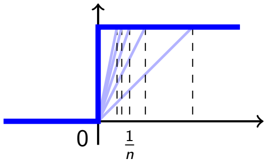

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- Order completeness of $C(L)$

Banaschewski and Hong (2003):

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- Order completeness of $\mathrm{C}(L)$

Banaschewski and Hong (2003):

– $\mathrm{C}(L)$ is order complete iff L is extremally disconnected.

– Is it possible to describe the Dedekind order completion of $\mathrm{C}(L)$?

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the frame of reals

Two alternative descriptions of the frame of reals $\mathfrak{L}(\mathbb{R})$

Generators:	$(p,q), \quad p,q \in \mathbb{Q}$
Relations:	
(R1)	$(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,
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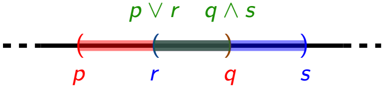
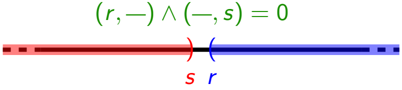
(R1) $(p,q) \wedge (r,s)$



(r1) $r \geq s$



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(R2) $p \leq r < q \leq s$

$(p,q) \vee (r,s) = (p,s)$

(r2) $r < s$

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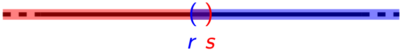

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$$(p,q)\longmapsto \begin{cases}(r,\text{—})=\bigvee_{r<p<q}(p,q)\\ (\text{—},s)=\bigvee_{p<q<s}(p,q)\end{cases}$$

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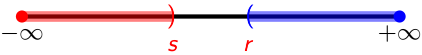
$$\overline{\mathbb{R}} = [-\infty, +\infty]$$



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(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

~~(R4) $\bigvee_{p,q \in \mathbb{Q}} (p, q) = 1$.~~

Generators: $(r, —), (—, s)$, $r, s \in \mathbb{Q}$

Relations:

(r1) $(r, —) \wedge (—, s) = 0$ whenever $r \geq s$,

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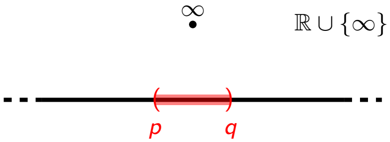
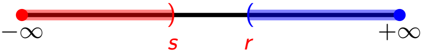


$\overline{\mathbb{R}} = [-\infty, +\infty]$



Introduction

the frame of reals

We will use the second description:

Generators: $(r, —), (—, s), \quad r, s \in \mathbb{Q}$

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Introduction the ring of continuous real functions on a frame: $C(L)$

Since continuous real functions on a space X may be represented as frame homomorphisms $h\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{O}X$, we have the equivalence:

$$C(X) = \mathbf{Top}(X, \mathbb{R}) \xrightarrow{\sim} \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X)$$

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Regarding the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, for a general frame L, as the continuous real functions on L provides a natural extension of the classical notion. They form a lattice-ordered ring that we denote

$$\mathrm{C}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), L)$$

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$$\mathrm{C}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), L)$$

$\mathrm{C}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{R}), L)$ is partially ordered by

$$f \leq g \ \text{ iff } \ f(r,—) \leq g(r,—) \quad \text{for all } r \in \mathbb{Q} \ \text{ iff } \ g(—,r) \leq f(—,r) \quad \text{for all } r \in \mathbb{Q}.$$

Introduction the ring of continuous real functions on a frame: $\mathrm{C}(L)$

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$\mathrm{C}(X)$ is isomorphic, as a lattice-ordered ring, to the function ring $\mathrm{C}(\mathcal{O}X)$.

Introduction the ring of continuous real functions on a frame: $\mathrm{C}(L)$

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Introduction the ring of continuous real functions on a frame: $\mathrm{C}(L)$

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Introduction

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Introduction

the ring of continuous real functions on a frame: $\mathrm{C}(L)$

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We cannot ensure that $h \in \mathrm{C}(L)$ because of (r2).

$\mathrm{C}(L)$ fails to be Dedekind order complete because of (r2)!

The frame of partial reals

$\mathfrak{L}(\mathbb{IR})$

Generators: $(r, —), (—, s), \quad r, s \in \mathbb{Q}$

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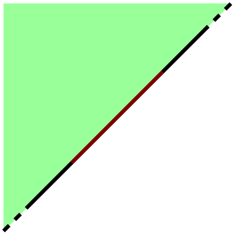

$$\mathbb{IR} = \{\mathbf{a} := [\underline{a}, \overline{a}] \subset \mathbb{R} \mid \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\}$$

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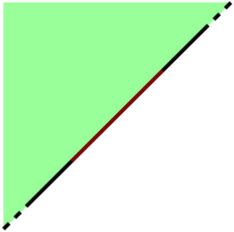

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$\mathbf{a} \sqsubseteq \mathbf{b}$ iff $[\underline{a}, \overline{a}] \supseteq [\underline{b}, \overline{b}]$

The frame of partial reals

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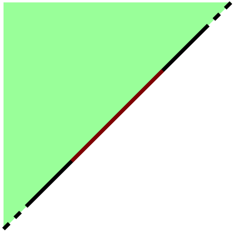

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$(\mathbb{IR},\sqsubseteq)$ is partial real line (or interval-domain)

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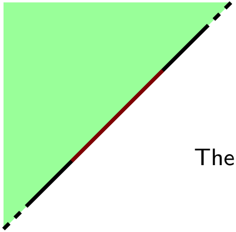

$\mathbb{IR} = \{\mathbf{a} := [\underline{a}, \overline{a}] \subset \mathbb{R} \mid \underline{a}, \overline{a} \in \mathbb{R}$ and $\underline{a} \leq \overline{a}\}$

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$(\mathbb{IR}, \sqsubseteq)$ is partial real line (or interval-domain)

The Scott topology on $(\mathbb{IR}, \sqsubseteq)$ is isomorphic with $\mathfrak{L}(\mathbb{IR})$

The frame of partial reals

$\mathfrak{L}(\mathbb{IR})$

Generators: $(r, —), (—, s), \quad r, s \in \mathbb{Q}$

Relations:

(r1) $(r, —) \wedge (—, s) = 0$ whenever $r \geq s$,

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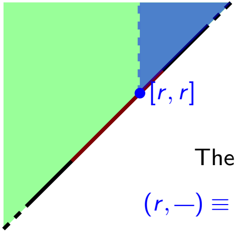

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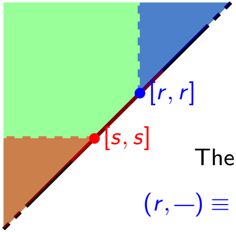

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continuous partial real functions

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$$\boxed{\mathrm{IC}(L) = \mathbf{Frm}(\mathfrak{L}(\mathbb{IR}), L)}$$

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Using the obvious surjective frame homomorphism $\varrho\colon \mathfrak{L}(\mathbb{IR}) \to \mathfrak{L}(\mathbb{R})$, continuous real maps $h \in \mathrm{C}(L)$ are in a one-to-one correspondence with the $\hat{h} = \varrho \cdot h \in \mathrm{IC}(L)$ such that $\hat{h}(\text{—}, r) \vee \hat{h}(s, \text{—}) = 1$ whenever $s < r$.

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Dedekind order completion of $\mathrm{C}(L)$

$\mathrm{IC}(L)$

Given a map $h\colon \mathfrak{L}(\mathbb{IR}) \to L$, in order to check that $h \in \mathrm{IC}(L)$ it is enough to prove that it turns the defining relations of $\mathfrak{L}(\mathbb{IR})$ into identities in L, i.e.

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Let $\{h_i\}_{i\in I} \subset \mathrm{C}(L)$ be such that there exists an $f \in \mathrm{C}(L)$ such that $h_i \le h$ for all $i \in I$.
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Consequently we have the following:

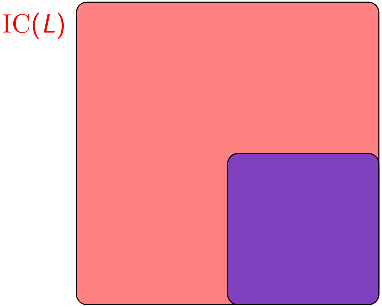
Proposition

$\mathrm{IC}(L)$ is Dedekind order complete.

Dedekind order completion of $\mathrm{C}(L)$

$\mathrm{IC}(L)$

Recall that we can consider $\mathrm{C}(L)$ as a subset of $\mathrm{IC}(L)$.



$$\mathrm{C}(L) = \{h \in \mathrm{IC}(L) \mid h(-, r) \vee h(s, -) = 1 \text{ for each } s < r\}$$

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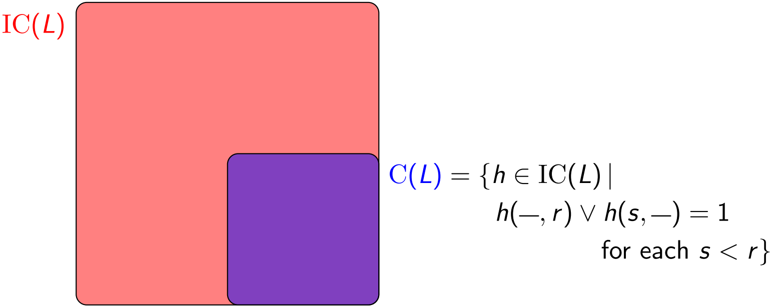

Now, since $\mathrm{IC}(L)$ is Dedekind order complete it follows that it contains the Dedekind order completion of all its subsets, in particular $\mathrm{C}(L)$.

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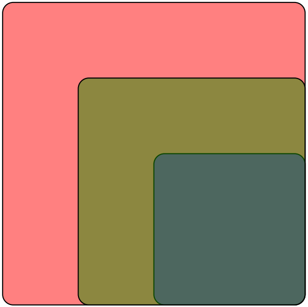
$\mathrm{IC}(L)$



$$\mathrm{C}(L)^{\#} = \{ h \in \mathrm{IC}(L) \mid \ ??? \ \}$$

$$\mathrm{C}(L) = \{ h \in \mathrm{IC}(L) \mid h(-, r) \vee h(s, -) = 1 \text{ for each } s < r \}$$

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$$\text{(r2)}\ f(\text{—}, r) \vee f(s, \text{—}) = 1 \quad \forall s < r \implies \text{(r2)'} \begin{cases} f(s, \text{—})^* \leq f(\text{—}, r) \\ f(\text{—}, r)^* \leq f(s, \text{—}) \end{cases} \quad \forall s < r$$

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Moreover, if L is extremally disconnected then the converse is also true.

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Let us denote

$$\mathrm{C}(L)^{\rotatebox{90}{\ensuremath{\bowtie}}} = \{h \in \mathrm{IC}(L) \mid (1)\ \exists f, g \in \mathrm{C}(L) : f \leq h \leq g$$
$$(2)\ h(s,\text{—})^* \leq h(\text{—}, r) \text{ and } h(\text{—}, r)^* \leq h(s, \text{—}) \text{ if } s < r\}$$

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It follows that

$$\mathrm{C}(L) \subset \mathrm{C}(L)^{\text{※}} \subset \mathrm{IC}(L)$$

If L is extremally disconnected then $\mathrm{C}(L) = \mathrm{C}(L)^{\text{※}}$.

Dedekind order completion of $\mathrm{C}(L)$

$$\mathrm{C}(L)^{\bowtie} = \{h \in \mathrm{IC}(L) \mid (1)\ \exists f, g \in \mathrm{C}(L) : f \leq h \leq g$$
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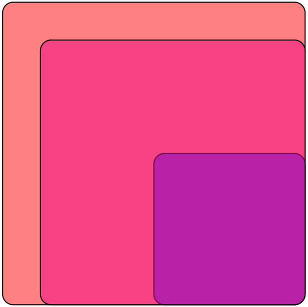

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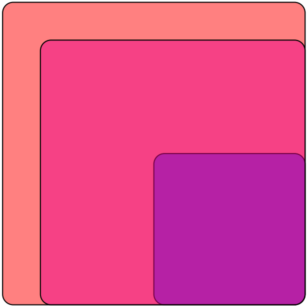
Dedekind order completion of $\mathrm{C}(L)$

Lemma

$\mathrm{C}(L)^{\between}$ is Dedekind order complete.

Dedekind order completion of $\mathrm{C}(L)$

$\mathrm{IC}(L)$



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Dedekind order completion of $\mathrm{C}(L)$

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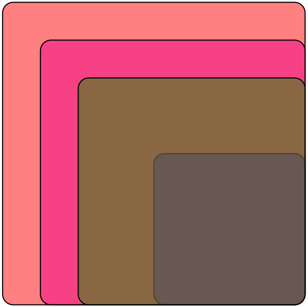

$\mathrm{C}(L)^{\between}$ is Dedekind order complete

$\mathrm{C}(L)^{\#}$

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Dedekind order completion of $\mathrm{C}(L)$

Lemma

$\mathrm{C}(L)^{\mathbb{X}}$ is Dedekind order complete.

Corollary

$\mathrm{C}(L)$ is Dedekind order complete if and only if L is extremally disconnected.

Dedekind order completion of $\mathrm{C}(L)$

Lemma

$\mathrm{C}(L)^{\between}$ is Dedekind order complete.

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$\mathrm{C}(L)$ is Dedekind order complete if and only if L is extremally disconnected.

Lemma

Let L be a completely regular frame and let $h \in \mathrm{C}(L)^{\between}$. Then

$$h = \bigvee^{\mathrm{IC}(L)} \{f \in \mathrm{C}(L) \mid f \leq h\} = \bigvee^{\mathrm{IC}(L)} \{g \in \mathrm{C}(L) \mid h \leq g\}.$$

Dedekind order completion of $\mathrm{C}(L)$

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Theorem

Let L be a completely regular frame. Let L be a frame. Then the Dedekind order completion $\mathrm{C}(L)^{\#}$ of $\mathrm{C}(L)$ coincides with $\mathrm{C}(L)^{\between}$, i.e. the set of continuous partial functions, $h \in \mathrm{IC}(L)$ such that:

(1) there exist $f, g \in \mathrm{C}(L)$ such that $f \leq h \leq g$

(2) $h(s, —)^* \leq h(—, r)$ and $h(—, r)^* \leq h(s, —)$ for any $s < r$.

Dedekind order completion of $\mathrm{C}^*(L)$

The bounded case follows similarly:

An $h \in \mathrm{IC}(L)$ is said to be bounded if there exists $r \in \mathbb{Q}$ such that $h(-r, r) = 1$. We denote

$$\begin{aligned} \mathrm{IC}^*(L) &= \{h \in \mathrm{IC}(L) \mid h \text{ is bounded}\}; \\ \mathrm{C}^*(L)^{\bowtie} &= \mathrm{C}(L)^{\bowtie} \cap \mathrm{IC}^*(L); \\ \mathrm{C}^*(L) &= \mathrm{C}(L) \cap \mathrm{IC}^*(L). \end{aligned}$$

Proposition

For any completely regular frame L, $\mathrm{C}^*(L)^{\bowtie}$ is the Dedekind order completion of $\mathrm{C}^*(L)$.

Proposition

For any completely regular frame L, $\mathrm{C}^*(L)$ is Dedekind order complete if and only if L is extremally disconnected.

Dedekind order completion of $\mathfrak{Z}L$

The integer-valued case also follows similarly:

An $h \in \mathrm{IC}(L)$ is said to be integer-valued if $f(r,s) = f(\lfloor r \rfloor, \lceil s \rceil)$ for all $r, s \in \mathbb{Q}$, (where $\lfloor r \rfloor$ denotes the biggest integer $\leq r$ and $\lceil s \rceil$ the smallest integer $\geq s$).
We denote

$$\begin{aligned} \mathrm{IC}(L,\mathbb{Z}) &= \{h \in \mathrm{IC}(L) \mid h \text{ is integer-valued}\}; \\ \mathrm{C}(L,\mathbb{Z})^{\Join} &= \mathrm{C}(L)^{\Join} \cap \mathrm{IC}(L,\mathbb{Z}); \\ \mathfrak{Z}L \simeq \mathrm{C}(L,\mathbb{Z}) &= \mathrm{C}(L) \cap \mathrm{IC}(L,\mathbb{Z}). \end{aligned}$$

Proposition

For any zero-dimensional frame L, $\mathrm{C}(L,\mathbb{Z})^{\Join}$ is the Dedekind order completion of $\mathrm{C}(L,\mathbb{Z})$.

Proposition

For any zero-dimensional frame L, $\mathrm{C}(L,\mathbb{Z})$ is Dedekind order complete if and only if L is extremally disconnected.

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Muchas gracias

Thank you