Monotone normality, quasi-metrizable spaces and the role of the T_1 axiom

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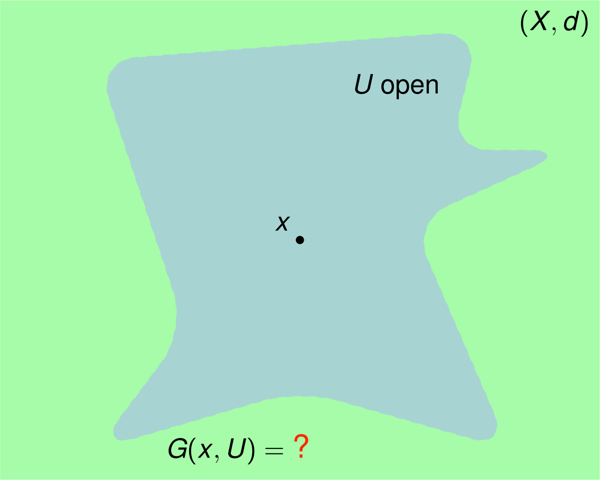
Why monotone normality?

Separation axioms for metric spaces



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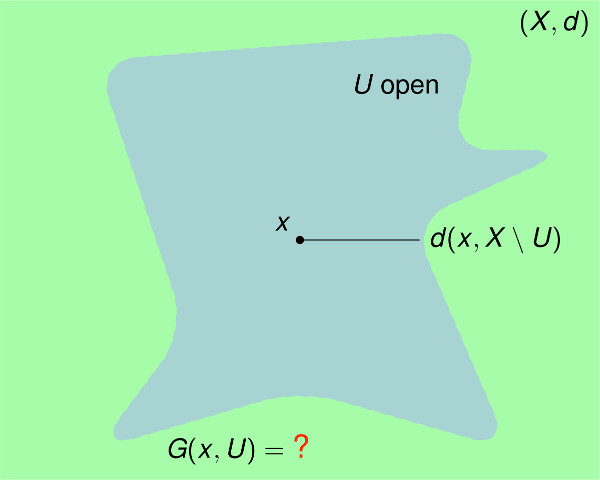

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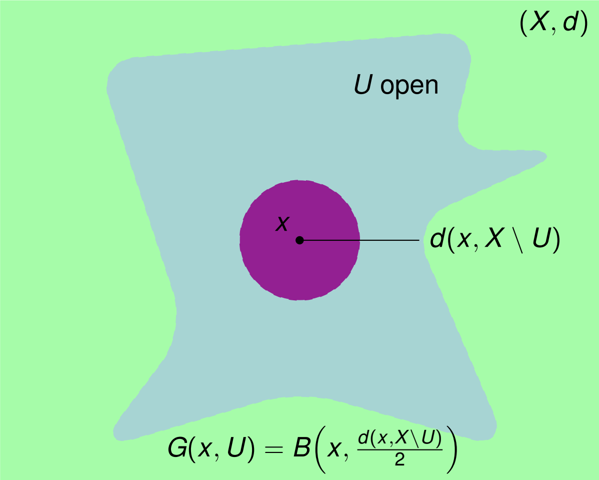

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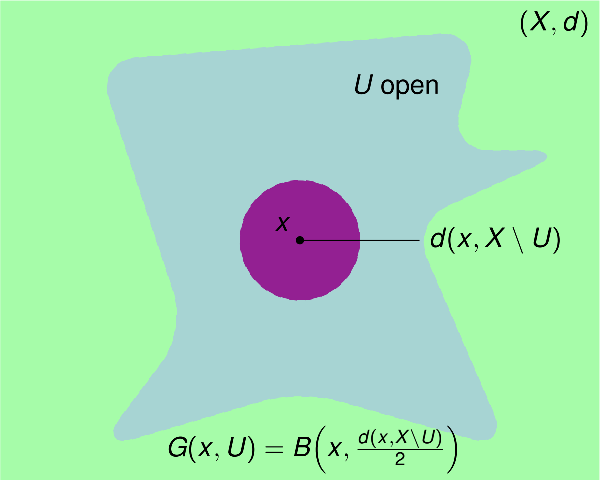

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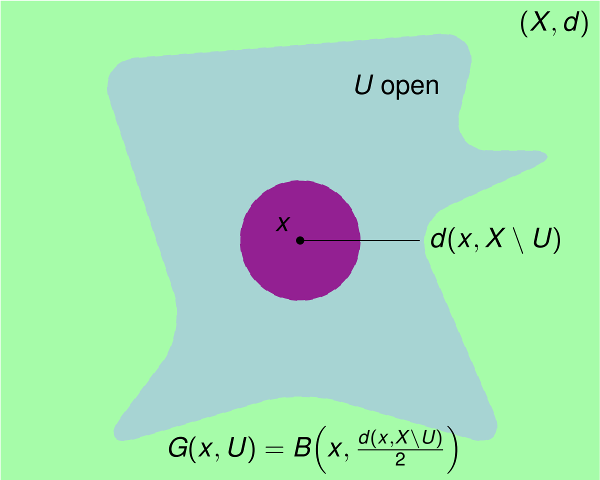

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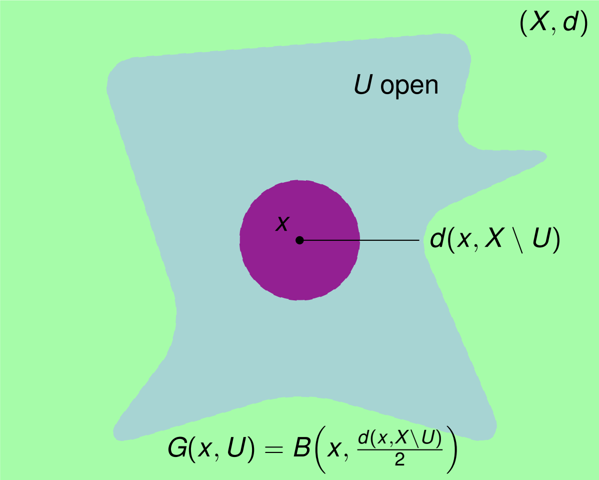

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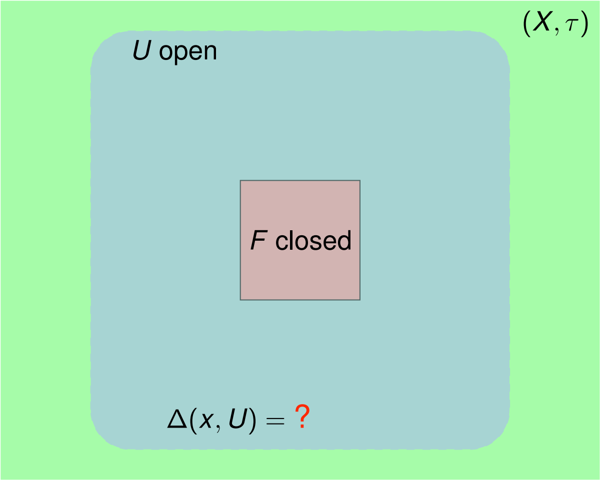

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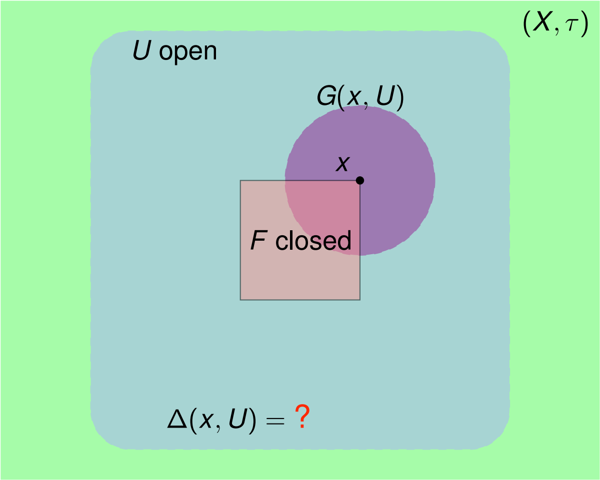

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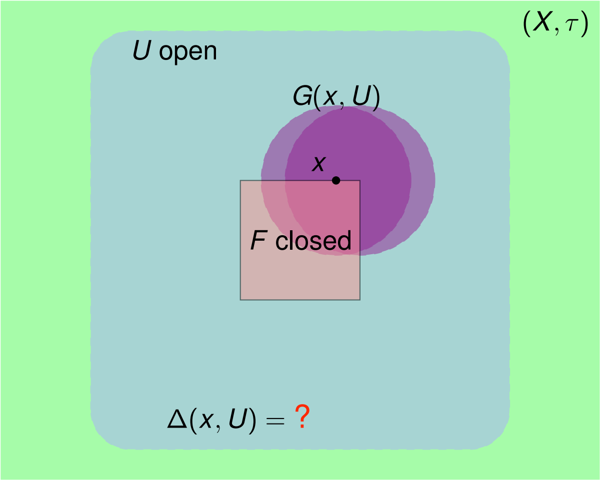

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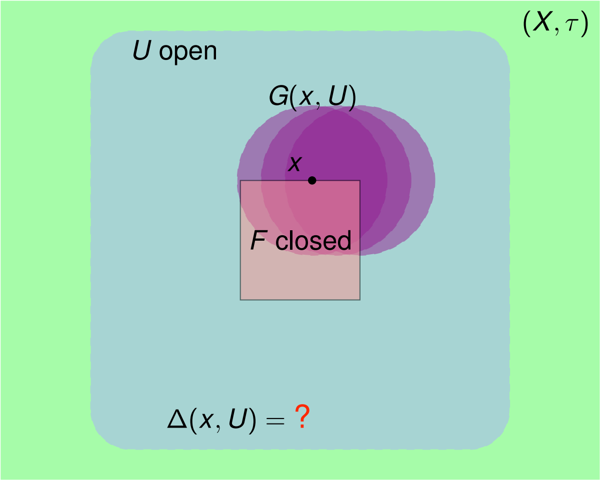

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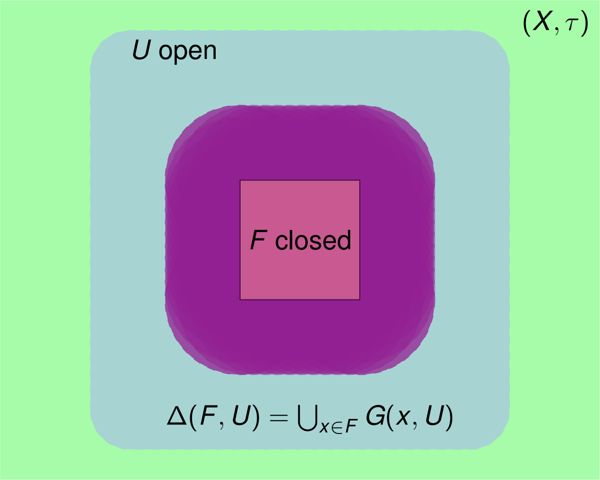

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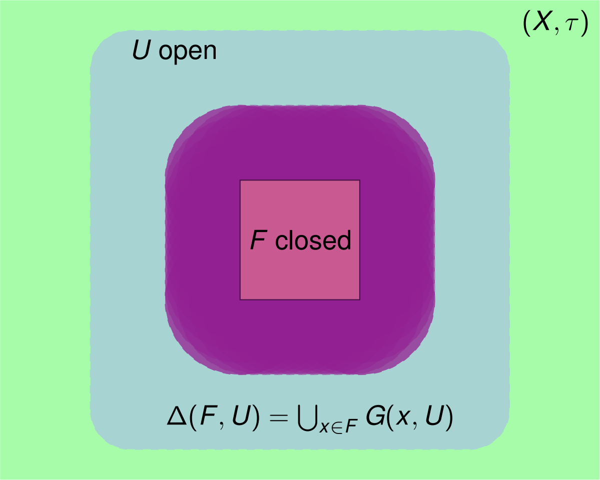

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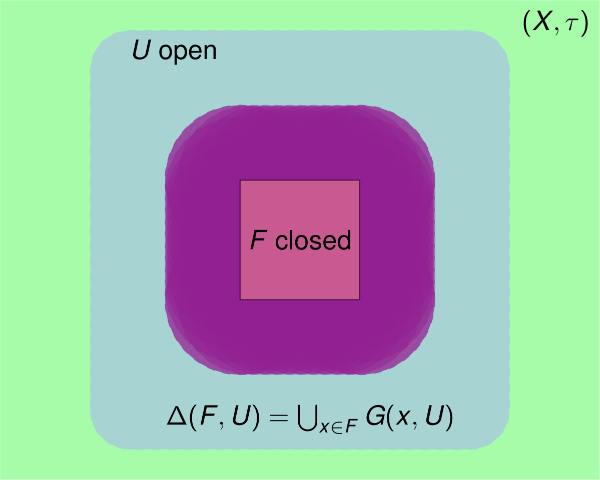

Monotone Normality

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Here again we have more!:

If $F_1 \subseteq F_2$ and $U_1 \subseteq U_2$ then $\Delta(F_1, U_1) \leq \Delta(F_2, U_2)$

Monotonization of normality

Let X be a topological space with topology $\mathcal{O}X$ (and $\mathcal{C}X$ being the family of all closed sets of X), let

$$P = \{(F, U) \in \mathcal{C}X \times \mathcal{O}X : F \subseteq U\} \quad \text{and} \quad Q = \mathcal{O}X.$$

Both P and Q carry natural orderings. Namely, $\leq_Q$ is the usual inclusion and P is ordered by componentwise inclusion $\leq_P$, i.e.,

$$(F_1, U_1) \leq_P (F_2, U_2) \iff F_1 \subseteq F_2 \text{ and } U_1 \subseteq U_2.$$

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Definition

A space X is monotonically normal if there exists a monotone $\Delta : P \to Q$ such that

(Δ1) $F \subseteq \Delta(F, U) \subseteq \overline{\Delta(F, U)} \subseteq U$ for all $(F, U) \in P$;

(Δ2) if $(F_1, U_1) \leq_P (F_2, U_2)$ then $\Delta(F_1, U_1) \leq_Q \Delta(F_2, U_2)$.

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Δ is called a monotone normality operator.

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Theorem (Borges, Heath, Lutzer, Zenor $\simeq$ 1970)

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Equivalent formulation of monotone normality

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Let X be a T_1 topological space. The following are equivalent:

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Example

Let (X, τ) an arbitrary space and $Y = X \cup \{\infty\}$ with $\infty \notin X$.

Define on Y the topology $\tau^\star = \tau \cup \{Y\}$.

X is an open, dense subspace of the monotonically normal non T_1 compact space Y.

If (X, τ) fails to be monotonically normal, we have the desired counterexample.

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What about the monotonically normal analogue of the Tietze-Urysohn theorem?

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However, the Sorgenfrey plane is also quasi-metrizable but not even normal.

Hence it is natural to try to find which quasi-metrizable spaces are monotonically normal.

Monotone normality without T_1

Every topological X induces, in a natural way, a partial order $\leq$ on X (called the specialization order) defined by $y \leq x \iff y \in \overline{\{x\}}$.

For each $x \in X$ we shall also denote $\downarrow x = \{y \in X : y \leq x\} = \overline{\{x\}}$.

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  (iii) *if ${\downarrow}x \cap {\downarrow}y = \varnothing$, then $H(x, X \setminus {\downarrow}y) \cap H(y, X{\downarrow}x) = \varnothing$.*

J.G.G., I. Mardones-Pérez and M.A. de Prada Vicente, *Monotone normality free of T_1 axiom*, Acta Math. Hungar., (2009).

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Consequences: Heritability

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(1) Monotone normality is a weakly hereditary property (any closed subspace of a monotonically normal space is monotonically normal), but not hereditary.

(2) Monotone normality is hereditary under the assumption of the T_1 axiom.

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As a corollary of the previous characterization, and in connection with hereditary monotone normality we have the following:

Facts

(1) Monotone normality is a weakly hereditary property (any closed subspace of a monotonically normal space is monotonically normal), but not hereditary.

(2) Monotone normality is hereditary under the assumption of the T_1 axiom.

(3) A space X is hereditarily monotonically normal if and only if every open subspace of X is monotonically normal.

Monotone normality without T_1 Consequences: Tietze-type theorem

As a second corollary of the characterization, we can conclude that the monotone version of the Tietze's result is still valid for monotone normality in the T_1-free context.

Theorem

Let X be a monotonically normal space. Then for each closed $A \subseteq X$ there exists a function $\Phi_A \colon C(A, [0,1]) \to C(X, [0,1])$ such that:

(1) *for each $f \in C(A, [0,1])$, $\Phi_A(f)$ extends f;*

(2) *if $f, g \in C(A, [0,1])$ and $f \leq g$ in A, then $\Phi_A(f) \leq \Phi_A(g)$ in X.*

Monotone normality without T_1

Consequences: Tietze-type theorem

Even more, the following characterization proved in:
I.S. Stares, Monotone normality and extension of functions, (1995)
remain valid in the T_1-free context.

Theorem

A space X is monotonically normal iff for each closed $A \subseteq X$ there exists a function $\Phi_A : C(A, [0,1]) \to C(X, [0,1])$ such that:

1. *for each $f \in C(A, [0,1])$, $\Phi_A(f)$ extends f;*
2. *if $f, g \in C(A, [0,1])$ and $f \leq g$ in A, then $\Phi_A(f) \leq \Phi_A(g)$ in X.*
3. *If $A_1 \subseteq A_2$ are closed and $f_i \in C(A_i, [0,1])$ are such that $f_2|_{A_1} \geq f_1$ and $f_2(x) = 1$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \geq \Phi_{A_1}(f_1)$.*
4. *If $A_1 \subseteq A_2$ are closed and $f_i \in C(A_i, [0,1])$ are such that $f_2|_{A_1} \leq f_1$ and $f_2(x) = 0$ for any $x \in A_2 \setminus A_1$, then $\Phi_{A_2}(f_2) \leq \Phi_{A_1}(f_1)$.*

Quasi-metrizable spaces

Let X be a non-empty set. A map $d\colon X \times X \to [0, +\infty)$ is a quasi-metric if the following two conditions hold for all $x, y, z \in X$:

(QM1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

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A topological space (X, τ) is said to be quasi-metrizable if there exists a quasi-metric d on X such that $\tau = \tau_d$.

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The specialization order $\leq_d$ on X is given by

$$y \leq_d x \iff d(y, x) = 0 \iff y \in \overline{\{x\}}.$$

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It is natural to think then about the question of which quasi-metrizable spaces are normal, or perhaps monotonically normal.
In this sense it could be mentioned, citing from:
P.M. Gartside, Cardinal invariants of monotonically normal spaces, (1997)

> *"Whenever a space can be explicitly and constructively shown to be normal, then it is probably monotonically normal."*

Quasi-metrizable spaces

Characterization for T_1 spaces

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Theorem

Let (X, d) be a T_1 quasi-metric space. The following are equivalent:

(1) (X, τ_d) *is monotonically normal;*

(2) *There exists a map* $h\colon X \times (0, +\infty) \to (0, +\infty)$ *such that:*

- (h1) $0 < h(x, \varepsilon) \leq \varepsilon$;
- (h2) *if* $\varepsilon_1 < \varepsilon_2$, *then* $h(x, \varepsilon_1) \leq h(x, \varepsilon_2)$;
- (h3) *if* $x \neq y$, *then* $B_d(x, h(x, d(x, y))) \cap B_d(y, h(y, d(y, x))) = \varnothing$.

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Corollary

Let (X, d) be a T_1 quasi-metric space satisfying:

$$x \neq y \quad \Longrightarrow \quad B_d(x, k \cdot d(x, y)) \cap B_d(y, k \cdot d(y, x)) = \varnothing \qquad (*)$$

for some $k \in (0, 1]$. Then (X, τ_d) is monotonically normal.

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- If d is a metric, then condition $(*)$ is satisfied with $k = \frac{1}{2}$.

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- If d is a metric, then condition $(*)$ is satisfied with $k = \frac{1}{2}$.
- If d is a the Sorgenfrey quasi-metric on $\mathbb{R}$ ($d(x, y) = \min\{y - x, 1\}$ if $x \leq y$ and $d^*(x, y) = 1$ otherwise), then condition $(*)$ is satisfied with $k = 1$.

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Finally, we can also provide a sufficient condition for a quasi-metric space to be monotonically normal:

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Then (X, τ_d) *is monotonically normal.*

Note that if d is indeed a metric, the condition $(*)$ above is obviously satisfied. In fact, this is precisely the Hausdorff condition.
In this case the previous proposition is, once again, nothing but the well known fact that metrizable spaces are monotonically normal.

Quasi-metrizable spaces

Examples (non T_1)

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