S26. PDE methods and challenges in control and inverse problems

Organizers:

- Francesca Bucci (Università degli Studi di Firenze, Italy)
- Manuel González Burgos (Universidad de Sevilla, Spain)
- Enrique Fernández-Cara (Universidad de Sevilla, Spain)

Speakers:

1. Karine Beauchard (École Polytechnique, France)
   Control of Schrödinger-Poisson system
2. Francesca Bucci (Università degli Studi di Firenze, Italy)
   Frequency domain analysis and decay rate estimates for a fluid-structure dynamics
3. Piermarco Cannarsa (Università degli Studi di Roma "Tor Vergata", Italy)
   Inverse problems for some classes of degenerate parabolic operators
4. Pedro Caro (University of Helsinki, Finland)
   On global uniqueness for an IBVP for the time-harmonic Maxwell equations
5. Carlos Castro (Universidad Politécnica de Madrid, Spain)
   Numerical approximation of the inverse scattering problem
6. Michel Cristofol (Université d'Aix-Marseille, France)
   New Kind of Observations in an Inverse Parabolic Problem
7. Diego Araujo de Souza (Universidad de Sevilla, Spain)
   On the uniform control of some α-models
8. Luz de Teresa (Universidad Nacional Autónoma de México, Mexico)
   Minimal time of controllability for some parabolic systems
9. Giuseppe Floridia (Università degli Studi di Roma "Tor Vergata", Italy)
   Bilinear control of nonlinear degenerate parabolic problems

10. Genni Fragnelli (Università degli Studi di Bari, Italy)
Identification problems in strongly degenerate parabolic systems

11. Elisa Francini (Università degli Studi di Firenze, Italy)
On the determination of finitely many parameters in some elliptic equations and systems from boundary measurements

12. Elsa Maria Marchini (Politecnico di Milano, Italy)
Some relaxation results for state constrained inclusions in infinite dimension, with applications to PDE control problems

13. Francisco Periago (Universidad Politécnica de Cartagena, Spain)
Robust Shape Optimization for Stochastic Elliptic PDEs

14. Dario Prandi (Université de Toulon, France)
Spectral properties and Aharonov-Bohm effect on Grushin-like structures

15. Fabio Priuli (IAC-CNR, Italy)
On the controllability for Temple class systems with characteristic boundary

Control of Schrödinger-Poisson system

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We consider the Schrödinger equation, in a bounded $2D$-domain, coupled to the Poisson equation. The control acts on this system trough a boundary condition on the potential. We investigate the exact controllability of the wave function. This study is motivated by applications in electronic devices at nanometric scales (single electron transistor).

Frequency domain analysis and decay rate estimates for a fluid-structure dynamics

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The problem of establishing rates of uniform decay for the solutions to linear and nonlinear evolution equations is a classical topic within the analysis of Partial Differential Equations (PDE). The energy/multipliers method underlies the fundamental work carried out by various authors on stabilization of wave and plate equations, under the possible challenges of nonlinear localized or boundary damping, as well as nonlinear forces. The most significant ideas and techniques developed during the eighties and nineties have been further refined and subsequently adapted to treat various (physically relevant) *composite systems of PDE*, such as thermoelastic equations, magnetoelastic systems, structural acoustics models, etc. It should be observed that the analysis of acoustic-structure and fluid-structure interactions (FSI) encounters significantly higher difficulties, these partly owing to the coupling which takes place via suitable boundary traces on an interface. Indeed, the topic of uniform stability of FSI is still rich with open questions, inasmuch as the subject of well-posedness itself of renowned FSI has been an open problem until very recently (even in the case of linearized versions like the Stokes-Lamé system).

In this talk we will discuss recently obtained results pertaining to the uniform decay of solutions to a (linear) PDE system which describes a certain FSI; this model has appeared repeatedly in the literature, in one form or another (see, e.g., [1]). We will discuss, in particular, two distinct stability results whose method of proof hinges upon the invocation of appropriate resolvent criteria for exponential or rational decay of strongly continuous semigroups, thereby confirming how the "frequency domain" approach proves effective in obtaining sharp decay rates of solutions to FSI.

The talk is based on joint work with George Avalos (University of Nebraska-Lincoln, USA).

[1] Chueshov, I., Ryzhkova, I., A global attractor for a fluid-plate interaction model, *Comm. Pure Appl. Anal.* **12** (2013), no. 4, 1635–1656.

Inverse problems for some classes of degenerate parabolic operators

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The Carleman estimates strategy introduced by Imanuvilov and Yamamoto in 1998 to prove Lipschitz stability results for uniformly parabolic equations, seems hard to apply to parabolic equations of Grushin-type. Indeed, Carleman estimates are still missing for operators that degenerate in the interior of the space domain such as the Grushin operator. Nevertheless, we will discuss how to recover Lipschitz stability for the determination of the source term and degenerate coefficients of such equations from locally distributed measurements of the solution. Our approach combines a method due to Lebeau and Robbiano, which relies on Fourier decomposition, with classical Carleman inequalities for the heat equations with nonsmooth coefficients that are solved by the Fourier modes.

This is joint work with K. Beauchard and M. Yamamoto.

On global uniqueness for an IBVP for the time-harmonic Maxwell equations

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In this talk I will show a uniqueness result for an inverse boundary value problem (IBVP) arising in electrodynamics. We assume that the electromagnetic properties of the medium, namely the magnetic permeability, the electric permittivity and the conductivity, are described by continuously differentiable functions.

The result to be presented in this talk is a joint work with Ting Zhou [1] and extends a result by Haberman and Tataru for the Calderón problem [2].

[1] Caro, P., Zhou, T., On global uniqueness for an IBVP for the time-harmonic Maxwell equations, *Anal. PDE*, to appear.

[2] Haberman, B., Tataru, H., Uniqueness in Calderón's problem with Lipschitz conductivities, *Duke Math. J.* **162** (2013), no. 3, 496–516.

Numerical approximation of the inverse scattering problem

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The Hemholtz equation in $\mathbb{R}^2$ with unknown compactly supported potential $V(x)$ is considered. This work is aimed at recovering the potential $V(x)$ from scattering data . One of the most successful approaches in this field is to recover the Born approximation $V_B(x)$ that is obtained from a suitable inversion of the *far field pattern*. It is well known that in many cases the Born approximation shares the same discontinuities as the potential $V(x)$. Thus, if the potential is the characteristic function of a bounded, open set the Born approximation allows to recover $V(x)$ completely from the discontinuities of $V_B(x)$.

In this talk, error estimates of the numerical approximation of the Born approximation are presented. The following situations are considered: a) fixed energy, b) backscattering and c) fixed incident wave direction. The numerical simulations for the different Born approximations are compared.

New Kind of Observations in an Inverse Parabolic Problem

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In this talk, I analyze the inverse problem of determining the reaction term $f(x, u)$ in reaction-diffusion equations of the form $\partial_t u - D\partial_{xx} u = f(x, u)$, where f is assumed to be periodic with respect to $x \in \mathbb{R}$. Starting from a family of exponentially decaying initial conditions $u_{0,\lambda}$, I will show that the solutions u_λ of this equation propagate with constant asymptotic spreading speeds w_λ. The main result shows that the linearization of f around the steady state 0, $\partial_u f(x, 0)$, is uniquely determined (up to a symmetry) among a subset of piecewise linear functions, by the observation of the asymptotic spreading speeds w_λ.

On the uniform control of some α-models

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This talk deals with the internal and boundary controllability results of the so called Leray-α model of turbulence. This is a regularized variant of the Navier-Stokes system (α is a small positive parameter), the usual transport term is regularized with an operator that depends on α. In the limit α tending to 0^+, we find the classical Navier-Stokes system. The main aim of the talk is to prove that the Leray-α systems are locally null controllable, with controls uniformly bounded with respect to α. We also prove that, if the initial data are sufficiently small, the controls converge as $\alpha \to 0^+$ to a null control of the Navier-Stokes equations. We also discuss some additional results and open questions.

1. Cheskidov, A., Holm, D. D., Olson, E., and Titi, E. S., On a Leray-α model of turbulence, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461** (2005), no. 2055, 629–649.
2. Fernández-Cara, E., Guerrero, S., Imanuvilov, O. Y., and Puel, J.-P., Local exact controllability of the Navier-Stokes system, *J. Math. Pures Appl. (9)* **83** (2004), no. 12, 1501–1542.

Minimal time of controllability for some parabolic systems

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In this talk we analyze the controllability properties of two different coupled parabolic systems. In the first problem the control is distributed, that is, the control is exerted on a small open set ω of the domain. In the second problem we present a boundary control, that is, the control is set on a part of the boundary. In both cases we will show that even if the system is parabolic, an explicit minimal time of control $T_0 \in [0, \infty]$ arises. We will have that system is not null controllable at time T if $T < T_0$ and system is null controllable for $T > T_0$.

The control time is related with the geometric position of the support of the coupling term when it does not intersect the control region (distributed control) and with the condensation index of the sequence of eigenvalues corresponding to the elliptic coupling operator (boundary control).

Bilinear control of nonlinear degenerate parabolic problems

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In this talk we study the degenerate Cauchy-Neumann problem

$$\begin{cases} u_t - (a(x)u_x)_x = \alpha(t,x)u + f(t,x,u) & \text{in } Q_T := (0,T)\times(-1,1) \\ a(x)u_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\ u(0,x) = u_0(x) & x \in (-1,1)\ , \end{cases}$$

where $u_0(x) \in L^2(-1,1)$ and $a(x)$ is a nonnegative function on $[-1,1]$ such that $a \in C^1([-1,1]),\ a(x) > 0\ \forall x \in (-1,1),\ a(\pm 1) = 0$. The linear part of the above equation reduces to the one of the Budyko-Sellers climate model for $a(x) = 1 - x^2$. This model is an energy balance model which attempts to study the evolution of the temperature on the Planet Earth, as a result of the interaction between large ice masses and solar radiation, see, for instance, [2]. We are interested in studying the controllability properties of this strongly degenerate problem, where $\alpha(t,x) \in L^\infty(Q_T)$ represents a *bilinear control.* Some embedding results for weighted Sobolev spaces, obtained in [3], have proved decisive in reaching the desired well-posedness of the above problem. For this nonlinear problem (specific polynomial growth bounds are assumed for f) we will discuss recent results guaranteeing global nonnegative approximate controllability in large time (see [1, 3]), that is, we will show that the above system can be steered in $L^2(-1,1)$ from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear piecewise static controls. Moreover, we extend the above result relaxing the sign constraint on the initial state.

[1] P. Cannarsa, G. Floridia, Approximate controllability for linear degenerate parabolic problems with bilinear control, *Proc. Evolution Equations and Materials with Memory 2010*, Casa Editrice Università La Sapienza Roma, 2011, 19–36.

[2] J.I. Diaz, G. Hetzer, L. Tello, An Energy Balance Climate Model with Hysteresis, *Nonlinear Anal.* **64** (2006), 2053–2074.

[3] G. Floridia, Approximate controllability for nonlinear degenerate parabolic problems with bilinear control, preprint.

Identification problems in strongly degenerate parabolic systems

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We will present identification problems associated with strongly degenerate parabolic evolution equations of the type

$$y_t - Ay = f(t,x), \quad (t,x) \in (0,T) \times (0,L),$$

where $T > 0, L > 0$ and f is in a suitable L^2 space. The operator A has both the forms $A_1 y = (uy_x)_x$ and $A_2 y = uy_{xx}$; *strong degeneracy* means that the diffusion coefficient u satisfies $u(x) > 0$ except for an interior point of $(0,L)$ and $\frac{1}{u} \not\in L^1(0,L)$.

On the determination of finitely many parameters in some elliptic equations and systems from boundary measurements

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I will describe some nonlinear severely ill-posed inverse boundary value problems involving elliptic equations and elliptic systems with applications to medical imaging, non destructive testing of materials and seismology. In these settings one wants to determine a coefficient appearing in an elliptic equation or system in a bounded domain from observations of solutions and of their derivatives on the boundary of the domain. I will focus my attention on the conductivity problem, the Gelfand-Calderon problem and the elasticity inverse problem.

In particular, I will talk about the issue of continuous dependence, crucial for effective reconstruction, describing some recent results where Lipschitz continuous estimates have been derived for unknown coefficient depending on a finite number of parameters.

The results I will present have been obtained in collaboration with E. Beretta, M. de Hoop and S. Vessella.

Some relaxation results for state constrained inclusions in infinite dimension, with applications to PDE control problems

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We discuss some relaxation results dealing with state constrained differential inclusions, as well as some approximation results of feasible trajectories with trajectories lying in the interior of the constraints. The setting is an infinite dimensional Banach space and the main assumption is a Soner type condition on the constraints. Some applications to a class of PDE control problems will be given.

Robust Shape Optimization for Stochastic Elliptic PDEs

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2010 Mathematics Subject Classification. 35J20, 49J20, 49M20, 65K10

The presentation shall review on recent results concerning robust shape optimization for stochastic elliptic PDEs. As a toy model, a shape optimization problem for an elliptic partial differential equation with uncertainty in its input data (force term, diffusion coefficient and boundary conditions) is considered [1]. The design variable enters the lower-order term of the state equation and is modelled through the characteristic function of a measurable subset of the spatial domain. As usual, a measure constraint is imposed on the design variable. In order to compute a robust optimal shape, the objective function involves a weighted sum of both the mean and the variance of the compliance. Since the optimization problem is not convex, a full relaxation of it is first obtained. The relaxed problem is then solved numerically by using a gradient-based optimization algorithm. To this end, the adjoint method is used to compute the continuous gradient of the cost function. Since the variance enters the cost function, the underlying adjoint equation is non-local in the probabilistic space. Both, the direct and adjoint equations are solved numerically by using a sparse grid stochastic collocation method. A number of numerical experiments illustrate the theoretical results and show the computational issues which arise when uncertainty is quantified through random fields.

[1] Martínez-Frutos, J., Kessler, M., Periago, F., Robust optimal shape design for an elliptic PDE with uncertainty in its input data, submitted.

Spectral properties and Aharonov-Bohm effect on Grushin-like structures

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We study spectral properties of the Laplace-Beltrami operator on a class of relevant almost-Riemannian manifolds, namely the Grushin structures on the cylinder and on the sphere. As for general almost-Riemannian structures [1], the singular set acts as a barrier for the evolution of the heat and of a quantum particle, although geodesics can cross it. This is a consequence of the self-adjointness of the Laplace-Beltrami operator on each connected component of the manifold without the singular set.

We will present explicit descriptions of the spectrum, of the eigenfunctions and their properties. In particular in both cases we obtain a Weyl law with dominant term $E \log E$. We will also discuss the drastic effect that an Aharonov-Bohm magnetic potential has on the spectral properties.

Finally, in the last part of the talk we will consider some other generalized Riemannian structures including conic and anti-conic type manifolds [2]. In this case, the Aharonov-Bohm magnetic potential can affect the self-adjointness of the Laplace-Beltrami operator, altering the nature of the communication between the two sides of the singularity.

This is a joint work with U. Boscain and M. Seri.

[1] Boscain, U., Laurent, C., The Laplace-Beltrami operator in almost-Riemannian Geometry, *Ann. Inst. Fourier*, to appear.

[2] Boscain, U., Prandi, D., The heat and Schrödinger equations on conic and anticonic-type surfaces, preprint.

On the controllability for Temple class systems with characteristic boundary

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We study the exact controllability problem for entropy weak solutions to strictly hyperbolic, genuinely nonlinear, Temple class systems of balance laws

$$u_t + f(u)_x = g(t) \qquad 0 \le x \le 1\,, \qquad u \in \mathbb{R}^N, \tag{1}$$

with possibly characteristic boundaries. Namely, we consider the initial–boundary value problem for (1), with initial data $u(0,x) = \overline{u}(x)$, and we regard both the source term g and the boundary data $\alpha^0(t), \alpha^1(t)$ at $x = 0, x = 1$, as control functions.

We show that, for every given profile $\Phi \in \mathrm{BV}(0,1)$, whose components Φ_i in Riemann coordinates satisfy $D\Phi_i \le C$ (in the sense of measures) for some $C > 0$, one can choose a source term g and boundary controls α^0, α^1, so that the corresponding solution to (1) attains the value Φ, at a sufficiently large time time $T > 0$.

This result in particular extends exact boundary controllability properties previously obtained (with no source term acting as a control) in [1] for Temple class systems with non characteristic boundary, and in [2] for quasilinear hyperbolic systems with characteristic boundary when the initial and terminal data $\overline{u}, \Phi$ are smooth and sufficiently close in $\mathcal{C}^1$-norm to an equilibrium state.

The talk is based on joint work with Fabio Ancona (University of Padua, Italy) and Khai T. Nguyen (Penn State University, USA).

[1] F. Ancona, G.M. Coclite, On the attainable set for Temple class systems with boundary controls, *SIAM J. Control Optim.* **43** (6) (2005), 2166–2190.

[2] J.-M. Coron, O. Glass, Z.Q. Wang, Exact boundary controllability for 1-D quasilinear hyperbolic systems with a vanishing characteristic speed, *SIAM J. Control Optim.* **48** (5) (2010), 3105–3122.