S6. Control of PDE: Theory, Numerics, and Applications

Organizers:

- Eduardo Casas (University of Cantabria, Spain)
- Paola Loreti (University of Rome I, Italy)

Speakers:

1. Fatiha Alabau-Boussouira (University of Lorraine, France)
   Control and observation of under-actionned systems of PDE's

2. Karine Beauchard (École Polytechnique, France)
   Null controllability of degenerate parabolic equations of hypoelliptic type

3. Eduardo Casas (University of Cantabria, Spain)
   Optimal control of the 3D evolutionary Navier-Stokes equations

4. Konstantinos Chrysafinos (National Technical University, Athens, Greece)
   Error estimates for parabolic boundary control problems

5. Paola Loreti (University of Rome I, Italy)
   Observability of membranes and plates in small sets

6. Mariano Mateos (University of Oviedo, Spain)
   On Dirichlet control problems governed by elliptic equations

7. Ira Neitzel (Technical University of Munich, Germany)
   Finite element error estimates for the state in elliptic optimal control problems with inequality state constraints

8. Cristina Pignotti (University of L'Aquila, Italy)
   Exponential stability for a viscoelastic wave equation with anti-damping/time delay

9. Christopher Ryll (Technical University of Berlin, Germany)
   Second-order analysis for sparse control of FitzHugh-Nagumo equations and application

10. Daniela Sforza (University of Rome I, Italy)
    Coupled integro-differential equations

Control and observation of under-actionned systems of PDE's

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This talk is concerned with controllability and observability issues for under-actioned coupled systems. Such systems possess less controls (or observations) than the number of equations, and occur for instance in insensitizing control or simultaneous control of devices in parallel. We shall present recent sharp advances on these issues, and discuss geometric conditions and coupling effects.

Control of degenerate parabolic equations: minimal time and geometric control condition

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We consider a Kolmogorov-type equation on a rectangle domain, coupling diffusion in variable v with transport in variable x at speed v^m (m is a positive integer). We study the null controllability of this equation; the control is a source term localized on an open subset ω of the rectangle. This study is motivated by the control of boundary layers in fluids (Prandtl and Crocco equations).

We will see that, depending on the form of ω and the value of m, this controllability property may hold or not. In particular, a geometric control condition seems to be needed when $m \in \{1, 2\}$ and a positive minimal time is required when $m = 2$.

These facts contrast with the classical results proved in the uniformly parabolic case (heat equation).

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Optimal control of the 3D evolutionary Navier-Stokes equations

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The velocity tracking problem for the evolutionary Navier-Stokes equations in 3d is studied. The controls are of distributed type and they are submitted to bound constraints. The classical cost functional is modified in such a manner that a full analysis of the control problem is possible. First and second order necessary and sufficient optimality conditions are proved. A fully-discrete scheme based on discontinuous (in time) Galerkin approach combined with conforming finite element subspaces in space, is proposed and analyzed. Provided that the time and space discretization parameters, τ and h respectively, satisfy $\tau \leq Ch^2$, then L^2 error estimates of order $O(h)$ are proved for the difference between the locally optimal controls and their discrete approximations.

Error estimates for parabolic boundary control problems

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We consider boundary optimal control problems for evolutionary parabolic partial differential equations, under low regularity assumptions on the given data. In particular, optimal control problems related to the minimization of the tracking or gradient type functionals subject to parabolic equations are discussed. The controls act on the boundary and are of Robin type. The main goal is to present stability, and error estimates for fully-discrete schemes under suitable regularity assumptions. The schemes under consideration are based on a discontinuous time-stepping approach combined with standard conforming finite elements for the spacial discretization. Special emphasis will be placed to the role of duality arguments in developing error estimates, and to the role of the corresponding adjoint equations. A similar problem related to the evolutionary Stokes system will be also discussed. Finally, we will discuss possible extensions to the Dirichlet boundary control case.

Observability of membranes and plates in small sets

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In this talk we discuss observability estimates for the solutions of wave or plate equations in a rectangle on some small observation subsets, as the union of two orthogonal segments for membranes and a finite union of small segments for plates.

The talk is based on a joint work with V. Komornik [KL].

[KL] Observability of rectangular membranes and plates on small sets, submitted to *Evol. Equ. Control Theory* (EECT).

On Dirichlet control problems governed by elliptic equations

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Dirichlet control problems are challenging due to the low regularity of the optimal solutions, when compared with distributed or Neumann control problems. This implies poor convergence rates of the numerical approximations. In [2] it is proved that the the error in both control and state of the finite element approximation –using quasi-uniform meshes– of a control-constrained problem posed on a convex polygon is of order $O(h^{1-1/p})$, where p depends on the biggest angle of Ω. The error estimate for the optimal state is improved in [3] in the case of having a linear equation and no constraints.

In this work, we investigate problems posed on non-convex domains. We prove some new regularity results in both non-weighted and weighted Sobolev spaces. For control constrained problems, we prove that the regularity of the control no longer depends on the biggest angle of the domain, but on the biggest convex angle. Next we consider the finite element approximation of these problems. We obtain error estimates for both quasi-uniform and graded meshes. This allows us to obtain the optimal grading parameter in terms of the data of the problem. We also show how to perform the optimization in practice, with a version of the active set strategy introduced in [1] that uses the concept of discrete normal derivative introduced in [2]. Finally, we show some results for state constrained problems.

The presented results have been obtained with the collaboration of Thomas Apel, Ira Neitzel, Johannes Pfefferer and Arnd Rösch.

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Finite Element Error Estimates for the State in Elliptic Optimal Control Problems with Inequality State Constraints

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We derive a priori error estimates for the finite element discretization of an elliptic optimal control problem with finitely many pointwise inequality constraints on the state. In particular, following ideas of Leykekhman, Meidner, and Vexler, who considered a similar problem with equality constraints, we obtain an optimal rate of convergence for the state variable. In contrast to earlier works, the number of inequality constraints may vary on refined meshes.

In a second step, we use these results to derive an optimal rate of convergence for the state variable for problems with infinitely many state constraints.

Exponential stability for a viscoelastic wave equation with anti–damping/time delay

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We will analyze the asymptotic behaviour of two viscoelastic models: the first combining viscoelastic damping and frictional anti–damping, the second one combining viscoelastic damping and time–delayed damping. We will show that even if anti–damping terms or time delays generate instability ([2],[3], [4]), the viscoelastic damping may counterbalance them. Indeed, we will prove that both models are exponentially stable if the amplitudes of the anti–damping term and the time–delayed damping are small enough. Moreover, we will give precise estimates on the smallness conditions. To give our stability results we will use a perturbative approach first introduced in [5], and arguments developed in [1] for wave type equations with viscoelastic damping.

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Second-order analysis for sparse control of FitzHugh-Nagumo equations and application

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We investigate the problem of sparse optimal controls for the so-called FitzHugh-Nagumo system. The solution of this reaction-diffusion equation forms, for certain settings of parameters, patterns of spiral waves. Sparsity is the consequence of the presence of the L^1-norm of the control in the objective functional. We refer to [1], where the first order necessary optimality condition for the mentioned problem is derived. Moreover, some numerical examples are given. In [1], a Tykhonov regularization term is used. This term is usually weighted with a small constant $\kappa > 0$. We observed in our examples that even a very small $\kappa > 0$ does not influence the numerical stability. Therefore we investigate the convergence analysis for $\kappa \to 0$. To this aim, following [2], we derive a first order necessary optimality condition for $\kappa = 0$ and second order sufficient optimality conditions for the optimal control of the FitzHugh-Nagumo system not only for $\kappa > 0$, but also for $\kappa = 0$. Finally, also the convergences of the control and state variables are proved for passing to the limit $\kappa \to 0$.

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Coupled integro-differential equations

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The primary aim of the talk is to show some control results about systems of coupled evolution equations in the presence of nonlocal terms of integral convolution type. For the solution of the adjoint system, given as a Fourier series, we provide observability estimates of Ingham type [1]. Then, the application of the so-called Hilbert Uniqueness Method yields reachability results for sufficiently large times, as expected for hyperbolic problems. Those results are presented in papers [2, 3].

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