# S24. Number Theory

**Organizers:**

- Fernando Chamizo (Universidad Autónoma de Madrid and ICMAT).
- Alberto Perelli (Università di Genova)

**Program:**

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2. Andrej Dujella (University of Zagreb)
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# Iwasawa Main Conjecture for cyclotomic extensions of function fields

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Let $F := \mathbb{F}_q(\theta)$ be the rational function field of characteristic $p$ and fix $\frac{1}{\theta}$ as the prime at $\infty$. Let $\Phi$ be the Carlitz module associated with $A := \mathbb{F}_q[\theta]$ and fix a prime $\mathfrak{p}$ of $A$. For any $n \geqslant 0$ let $\Phi[\mathfrak{p}^n]$ be the $\mathfrak{p}^n$-torsion of the Carlitz module and $F_n := F(\Phi[\mathfrak{p}^{n+1}])$. We define the $\mathfrak{p}$-*cyclotomic extension* of $F$ as $\mathcal{F} := \cup F_n$: it is a Galois extension with $\mathrm{Gal}(\mathcal{F}/F_0) := \Gamma \simeq \mathbb{Z}_p^\infty$. Let $\mathcal{C}\ell_n$ be the class group of $F_n$ and put $\mathcal{C}\ell(\mathcal{F})$ as the inverse limit (on the natural norm maps) of the $\mathcal{C}\ell_n$. We study $\mathcal{C}\ell(\mathcal{F})$ as a module over the non-noetherian Iwasawa algebra $\Lambda := \mathbb{Z}_p[[\Gamma]]$.

In [1] we define a characteristic ideal in $\Lambda$ for $\mathcal{C}\ell(\mathcal{F})$ using a filtration of $\mathbb{Z}_p^d$-extensions for $\mathcal{F}/F$, while in [2] we use (for the same purpose) the natural filtration of the $F_n$. Both approaches lead to ideals generated by Stickelberger elements and in [2] we are able to prove a Main Conjecture relating such element to a $\mathfrak{p}$-adic $L$-function. This result can be used to provide informations on the values of the Goss-Carlitz $\zeta$-function at odd integers (i.e., $\zeta_A(-j)$ for $j \not\equiv 0 \pmod{q-1}$).

[1] A. Bandini, F. Bars, I. Longhi, Characteristic ideals and Iwasawa theory, 2013, preprint; http://arxiv.org/abs/1310.0680.

[2] B. Anglès, A. Bandini, F. Bars, I. Longhi, Iwasawa Main Conjecture for the Carlitz cyclotomic extension and applications, in progress.

# Mordell-Weil groups of elliptic curves induced by Diophantine triples

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We study the possible structure of the groups of rational points on elliptic curves of the form $y^2 = (x + ab)(x + ac)(x + bc)$, where $a$, $b$, $c$ are non-zero rationals such that the product of any two of them is one less than a square. Such a triple $\{a, b, c\}$ is called a rational Diophantine triple. There are exactly four types of possible torsion groups for elliptic curves of this form. In each case, we construct examples and parametric families of elliptic curves with relatively high rank. In particular, we describe a joint work with Juan Carlos Peral, with construction of an elliptic curve over the field of rational functions $\mathbb{Q}(t)$ with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and generic rank equal to 4, and an elliptic curve over $\mathbb{Q}$ with the same torsion group and rank 9. Both results improve previous records for ranks of curves with this torsion group.

# Effective Sato-Tate for generic abelian varieties with an application

Francesc Fité

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Let $A_1$ and $A_2$ be two abelian varieties defined over $\mathbb{Q}$. Let $V_\ell(A_i)$ denote the rational Tate module of $A_i$, for $i = 1, 2$. If $A_1$ and $A_2$ are non-isogenous, then Faltings Theorem implies that there exists a prime $p$ of good reduction for both $A_1$ and $A_2$ such that the traces $a_p(A_1)$ and $a_p(A_2)$ of the Frobenius element at $p$ acting on $V_\ell(A_1)$ and $V_\ell(A_2)$ are distinct. However, Faltings result is non effective, in the sense that it does not say anything about the size of the minimal prime $p$ with this property. We say that an abelian variety (of dimension $g$) is generic if the Zariski closure of the image of its attached $\ell$-adic representation is maximal, that is, equal to $\mathrm{GSp}_{2g}/\mathbb{Q}_\ell$. In this talk, from the GRH, we obtain an effective version of the Sato-Tate Conjecture for a generic abelian variety (or rather for the product of two generic abelian varieties) over $\mathbb{Q}$, from which we deduce an upper bound for the smallest prime $p$ at which $a_p(A_1)$ and $a_p(A_2)$ are distinct in case both $A_1$ and $A_2$ are generic.

This is a work in progress with Alina Bucur and Kiran Kedlaya.

## Modular forms over cubic fields and algebraic points in elliptic curves

Xavier Guitart

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In this talk I will describe a conjectural construction of algebraic points on modular elliptic curves defined over cubic number fields of mixed signature. The points are defined as integrals of the corresponding modular form, in a way that resembles Darmon's ATR points for curves over real quadratic fields. I will also present some numerical evidence in support of the conjectured rationality.

This is joint work with Marc Masdeu and Haluk Sengun.

## The ternary Goldbach conjecture

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The ternary Goldbach conjecture (1742) asserts that every odd number greater than 5 can be written as the sum of three prime numbers. Following the pioneering work of Hardy and Littlewood, Vinogradov proved (1937) that every odd number larger than a constant $C$ satisfies the conjecture. In the years since then, there has been a succession of results reducing $C$, but only to levels much too high for a verification by computer up to $C$ to be possible ($C > 10^{1300}$). (Works by Ramare and Tao have solved the corresponding problems for six and five prime numbers instead of three.) My recent work proves the conjecture. We will go over the main ideas in the proof.

# Darmon points in mixed signature

Marc Masdeu

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Let $F$ be a number field, and let $E/F$ be a (modular) elliptic curve defined over $F$. I will present a conjectural $p$-adic analytic construction of algebraic points on $E$, which properly contains as special cases previous constructions of Darmon, Greenberg and Trifkovic. If time permits I will illustrate with examples.

This is joint work with Xavier Guitart and Haluk Sengun.

# Iwasawa theory of Heegner cycles

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Iwasawa theory of Heegner points on (modular) abelian varieties has been studied by, among other authors, Perrin-Riou ([6]), Bertolini ([1]) and Howard ([2]). The purpose of this talk is to describe an extension of these results (partly "work in progress", see [3] and [4]) in which abelian varieties are replaced by the Galois cohomology of Deligne's $p$-adic representation attached to a modular form $f$ of even weight $> 2$.

In this more general setting, the role of Heegner points is played by higher-dimensional Heegner cycles in the sense of Nekovář ([5]). In particular, we show that the Pontryagin dual of a certain Bloch–Kato Selmer group associated with $f$ has rank 1 over a suitable Iwasawa algebra.

1. M. Bertolini, Selmer groups and Heegner points in anticyclotomic $\mathbf{Z}_p$-extensions, *Compos. Math.* **99** (1995), no. 2, 153–182.
2. B. Howard, Iwasawa theory of Heegner points on abelian varieties of $\mathrm{GL}_2$ type, *Duke Math. J.* **124** (2004), no. 1, 1–45.
3. M. Longo, S. Vigni, Iwasawa theory of Heegner cycles, I. Rank over the Iwasawa algebra, 2014, preprint.
4. M. Longo, S. Vigni, Iwasawa theory of Heegner cycles, II. Annihilators and Main Conjecture, in preparation.
5. J. Nekovář, Kolyvagin's method for Chow groups of Kuga–Sato varieties, *Invent. Math.* **107** (1992), no. 1, 99–125.
6. B. Perrin-Riou, Fonctions $L$ $p$-adiques, théorie d'Iwasawa et points de Heegner, *Bull. Soc. Math. France* **115** (1987), no. 4, 399–456.

# Linear independence of logarithmic and dilogarithmic values

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We describe recent advances concerning irrationality results of values of the logarithm and of the dilogarithm, defined by the Taylor series

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad (|x| < 1),$$

as well as linear independence results over $\mathbb{Q}$ of the set

$$1,\ \mathrm{Li}_1(x) := \sum_{n=1}^{\infty} x^n/n = -\log(1-x),\ \mathrm{Li}_2(x) \text{ and } \mathrm{Li}_2(x/(x-1)),$$

for suitable $x \in \mathbb{Q}$. These results were obtained by the author, jointly with other mathematicians (F. Amoroso, G. Rhin, R. Marcovecchio, W. Zudilin), and are based upon the permutation group method introduced in 1996 by G. Rhin and C. Viola.

# Representation of integers as sums of primes

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2010 Mathematics Subject Classification. 11P

We present recent results, obtained by the speaker in a series of joint papers with Alessandro Languasco (Padova), concerning averages of the number of representations of a positive integers as a sum of several primes. Let $R_j(n)$ denote

$$\sum_{n_1+\cdots+n_j=n} \Lambda(n_1)\cdots\Lambda(n_j),$$

where $\Lambda$ is the von Mangoldt function. In [3] we improved a paper by Bhowmik and Schlage-Puchta [1] concerning the asymptotic expansion with main term and "secondary main term" of

$$\sum_{n\le N} R_2(n)$$

assuming that the Riemann Hypothesis is true. In [4] we gave similar improvements on earlier results due to Friedlander and Goldston [2] for the value of $R_j(n)$ when $j \ge 3$ (assuming the Generalised Riemann Hypothesis), and in [5] we tackled the problem of giving the corresponding expansions with several terms for

$$\sum_{n\le N} \Bigl(1-\frac{n}{N}\Bigr)^k R_2(n),$$

where $k > 1$ is a fixed real number, assuming the Riemann Hypothesis.

[1] G. Bhowmik and J.-C. Schlage-Puchta, Mean representation number of integers as the sum of primes, *Nagoya Math. J.* **200** (2010), 27–33.

[2] J. B. Friedlander and D. A. Goldston, Sums of three or more primes, *Trans. Amer. Math. Soc.* **349** (1997), no. 1, 287–310.

[3] A. Languasco and A. Zaccagnini, The number of Goldbach representations of an integer, *Proc. Amer. Math. Soc.* **140** (2012), 795–804.

[4] A. Languasco and A. Zaccagnini, Sums of many primes, *J. Number Theory* **132** (2012), 1265–1283.

[5] A. Languasco and A. Zaccagnini, A Cesàro average of Goldbach numbers, *Forum Math.* (2014), to appear.