S1. Algebraic Entropy and Topological Entropy

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- Luigi Salce (Universitá di Padova, Italy)
- Manuel Sanchis López (Universitat Jaume I de Castelló, Spain)

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Volume entropy for minimal presentations of surface groups

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We study the volume entropy of certain presentations of surface groups (which include the classical ones) introduced by J. Los [2], called *minimal geometric*. This study uses a dynamical system construction based on an idea due to Bowen and Series [1] and extended to the case of geometric presentations by J. Los.

We obtain a surprising explicit formula for the volume entropy of classical presentations in all genus, showing a polynomial dependence in g. More precisely we prove that if Γ_g is a surface group of genus $g \geq 2$ with a minimal geometric presentation P_g, then the volume entropy of Γ_g with respect to the presentation P_g is $\log(\lambda_{2g})$ where λ_{2g} is the unique real root larger than one of the polynomial

$$x^{2g} - 2(2g-1)\sum_{j=1}^{2g-1} x^j + 1.$$

Moreover,

$$4g - 1 - \frac{1}{(4g-1)^{2(g-1)}} < \lambda_{2g} < 4g - 1.$$

[1] Rufus Bowen and Caroline Series, Markov maps associated with Fuchsian groups, *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 153–170.

[2] Jérôme Los, Volume entropy for surface groups via Bowen-Series-like maps, *J. Topology* (2013); doi: 10.1112/jtopol/jtt032.

Topological entropy and related notions

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Let (X, f) be a topological dynamical system, where X is a non-empty compact Hausdorff space and $f : X \to X$ a continuous map. Topological entropy is a non-negative number which measures the complexity of the system. Roughly, it measures the exponential growth rate of the number of distinguishable orbits as time advances.

As a short history, the original definition was introduced in 1965 by Adler, Konheim and McAndrew. Their idea of assigning a number to an open cover to measure its size was inspired by a paper of 1961 of Kolmogorov and Thomirov. Then to introduce the definition of topological entropy for continuous maps, they strictly imitated the definition from Kolmogorov-Sinai entropy of a measure preserving transformation in ergodic theory. In metric spaces, a different definition was introduced by Bowen in 1971 and independently by Dinaburg in 1970. It uses the notion of ϵ-separated points.

Equivalence between the above two notions was proved by Bowen in 1971. The most important characterization of topological entropy in terms of Kolmogorov-Sinai entropy, called the Variational Principle was proved arounf 1970 by Dinaburg, Goodman and Goodwyn.

In this talk we will concentrate in such Variational Principle and in the relation between the topological entropy and the Kolmorov-Sinay entropy. We also will give an interpretation of the topological entropy when it is generate by an important structure in Topological Dynamics called *horseshoe*. Some formulae for the calculation of topological entropy will be applied to some examples of piecewise monotone interval maps and we will introduced the new notion of topological entropy for non-autonomous systems.

Scale function and topological entropy

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For a long period, the structure theory of locally compact totally disconnected groups (lctd groups) has been considered intractable. Only recently, in 1993, it was proved that for every topological automorphism of a lctd group there exists a so-called *tidy* subgroup. This allows the definition of a function on the topological automorphisms of a lctd group, the *scale function*.

We intend to discuss the relationship between the scale function and the topological entropy of a topological automorphism of a lctd group. Furthermore, a Bridge Theorem for the scale function will be presented.

Based on a joint work with Anna Giordano Bruno and Dikran Dikranjan.

On entropy of fuzzy extensions of continuous maps

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Let (X, d) be a compact metric space and consider a continuous map $f : X \to X$. It is well–know that topological entropy is a non–negative number which is a measure of the dynamic complexity of f. In this talk we consider the Zadeh extension of f to the set of fuzzy sets, denoted by Φ_f with the aim of solving two questions. Have f and Φ_f similar dynamical behavior? and more precisely, have they the same entropy? If not, how the fuzzy sets which are essentially fuzzy contribute to increase the complexity of Φ_f?

This is a joint work with Jiri Kupka (University of Ostrava, Czech Republic).

Bridge Theorems

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A *flow* in a category $\mathfrak{X}$ is a pair (X, ϕ), where X is an object and $\phi : X \to X$ is an endomorphism of X in $\mathfrak{X}$. A morphism between two flows (X, ϕ) and (Y, ψ) in $\mathfrak{X}$ is a morphism $\alpha : X \to Y$ in $\mathfrak{X}$ such that $\alpha \circ \phi = \psi \circ \alpha$. This defines the category $\mathbf{Flow}_{\mathfrak{X}}$ of flows in $\mathfrak{X}$.

We call *entropy* of $\mathfrak{X}$ a function $h : \mathbf{Flow}_{\mathfrak{X}} \to \mathbb{R}_+ = \mathbb{R}_{\geq 0} \cup \{\infty\}$ taking the same values on isomorphic flows.

Let $\varepsilon : \mathfrak{X}_1 \to \mathfrak{X}_2$ be a functor between two categories $\mathfrak{X}_1$ and $\mathfrak{X}_2$, and let $h_1 : \mathbf{Flow}_{\mathfrak{X}_1} \to \mathbb{R}_+$ and $h_2 : \mathbf{Flow}_{\mathfrak{X}_2} \to \mathbb{R}_+$ be entropies of $\mathfrak{X}_1$ and $\mathfrak{X}_2$ respectively, such that $h_1 = h_2 \circ \varepsilon$:

$$\begin{array}{ccc} \mathbf{Flow}_{\mathfrak{X}_1} & \xrightarrow{\;h_1\;} & \\ \downarrow{\scriptstyle \varepsilon} & & \mathbb{R}_+ \\ \mathbf{Flow}_{\mathfrak{X}_2} & \xrightarrow{\;h_2\;} & \end{array}$$

We say that the pair (h_1, h_2) satisfies the *Bridge Theorem* with respect to ε if there exists a positive constant C_ε, such that for every (X, ϕ) in $\mathbf{Flow}_{\mathfrak{X}_1}$

$$h_2(\varepsilon(\phi)) = C_\varepsilon h_1(\phi).$$

We discuss in this general scheme many known so-called Bridge Theorems, that is, specific entropies and functors on pairs of suitable categories satisfying the condition given above. Our inspiring example is the case of the topological and the algebraic entropy, satisfying the Bridge Theorem with respect to the Pontryagin duality functor, on suitable categories. Furthermore, we consider in this scheme the pair composed by the algebraic entropy and the adjoint algebraic entropy, and we describe the relation of the topological entropy with the measure entropy, but also with the frame entropy and the set-theoretic entropy.

Algebraic entropy vs Topological entropy

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We consider both the algebraic entropy and the topological entropy for continuous endomorphisms of locally compact abelian groups, and in this setting we discuss the connection between these two entropies using Pontryagin duality.

Indeed, Weiss proved that the topological entropy of a continuous endomorphism of a totally disconnected compact abelian group coincides with the algebraic entropy of the dual endomorphism of the Pontryagin dual group, that is a discrete torsion abelian group. We see how to extend this result in two directions, first to all compact abelian groups and then to all totally disconnected locally compact abelian groups.

Can negative Schwarzian derivative be used to extract order from chaos?

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In 1992, Pyragas made a remarkable discovering: it is possible to transform a chaotic system into a non-chaotic one using delayed feedback. A very natural and simple way to do this, introduced by Buchner and Zebrowski in 2000, is to modify a one-dimensional discrete dynamical system $x_{n+1} = h(x_n)$ via an "echo type" feedback loop,

$$x_{n+1} = (1 - \alpha)h(x_n) + \alpha x_{n-k},$$

with $0 < \alpha < 1$ being the feedback amplitude and k being the delay time. The point is that (when the delay time k is even) a repelling fixed point of the original system may become attracting for the delayed one. This may happen even if the map h has positive entropy and exhibits observable chaos.

In this regard, maps belonging to the so-called class S are of special interest because, as it is well known, they have a unique metric attractor (that is, a compact set attracting the orbit of almost all points —in the sense of Lebesgue measure) and, in the particular case when its fixed point u is locally attracting (which is equivalent to $|h'(u)| \leq 1$), u is this metric attractor and, indeed, it attracts the orbits of *all* points. (We say that a map C^3 map $h : I \to I$, with I being a subinterval of $\mathbb{R}$, *belongs to the class* S if, roughly speaking, it is unimodal, has a unique fixed point u and the Schwarzian derivative of h, $Sh(x) = h'''(x)/h'(x) - (3/2)(h''(x)/h'(x))^2$, is negative.)

Thus this natural question arises: if the map h belongs to the class S, is local attraction for the delayed system always *global*? In this work we answer this question in the negative: a counterexample is provided (for $k = 2$) by the Ricker function $h(x) = pxe^{-qx}$. Yet we give some arguments supporting the validity of the statement when k is large enough.

The hierarchy of algebraic entropies

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The algebraic entropy originally defined in 1965 for Abelian groups by Adler, Konheim and McAndrew (see [1] and [3]), its generalization due to Peters in 1979 (see [2]), and the more recent intrinsic algebraic entropy (see [4]), are illustrated and compared. Analogies and differences of the three algebraic entropies are focused, with special emphasis on the Addition Theorem and the Uniqueness Theorem, and their connections with length functions are discussed (see [5]). Some directions for further developments are outlined.

[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, Topological entropy, *Trans. Amer. Math. Soc.* **114** (1965), 309–319.

[2] J. Peters, Entropy on discrete Abelian groups, *Adv. Math.* **33** (1979), 113.

[3] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, Algebraic entropy of endomorphisms of Abelian groups, *Trans. Amer. Math. Soc.* **361** (2009), 3401–3434.

[4] D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili, Intrinsic algebraic entropy, preprint.

[5] L. Salce, P. Vámos, S. Virili, Length functions, multiplicities and algebraic entropy, *Forum Math.* **25**(2) (2013), 255–282.

A Notion of Entropy in the Realm of Fuzzy Metric Spaces

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We introduce an entropy function h_F for continuous self-maps on fuzzy metric spaces (in the sense of Kramosil and Michalek). We prove that this function enjoys the usual properties of the topological entropy. In addition, given a metric space (X, d) and a continuous map $f\colon X \to X$, we study the relationship between the Bowen entropy of f and $h_F(f)$ when we consider f as a self-map on the fuzzy metric space $(X, M_d, \star)$ where M_d stands for the standard fuzzy metric induced by d and $\star$ is any continuous t-norm on X. We also analyze the completion theorem (in the sense of Kimura) for this new function.

Polyentropy, Poincaré Series and Multiplicity

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For a ring R, Mod R will denote the category of left R-modules. A function $L\colon \text{Mod } R \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a length function if it satisfies

(i) *additive:* if $L(A) = L(A') + L(A'')$ for every exact sequence $0 \to A' \to A \to A'' \to 0$;

(ii) *upper continuous*: if for every $M \in \text{Mod } R$, $L(M) = \sup_{F \in \mathcal{F}(M)} L(F)$, where $\mathcal{F}(M)$ denotes the set of the finitely generated submodules of M.

(iii) *discrete* if for all M the set $\{L(F) \mid F \subseteq M,\ L(F) < \infty\}$ is order-isomorphic to a subset of $\mathbb{N}$.

Let $L\colon \text{Mod } R \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a length function, $M \in \text{Mod } R$, and let $\Phi = \{\phi_1, \ldots, \phi_d\}$ be a set of d pairwise commuting endomorphisms of M. We will define the the *L-(poly)entropy of* Φ *on* M, $\text{ent}_L(\Phi, M)$. This generalizes the case of the entropy of a single endomorphism ($d = 1$) as defined in [1]. Then, analogously to the single endomorphism case, the polyentropy $\text{ent}_L(\Phi, M)$ will now be interpreted as a function on the modules over the polynomial ring $R[X] = R[x_1, \ldots, x_d]$.

We then introduce the Hilbert function and Poincaré-Hilbert series $P_{M,F}(t)$ associated to the entropy $\text{ent}_L(\Phi, M)$. Our main result is that (under suitable conditions on M) the Poincaré-Hilbert series $P_{M,F}(t)$ is a rational function of t of the form $f(t)/(1-t)^d$ where $f(t)$ is a polynomial. Moreover, $P_{M,F}(t)$ is additive over short exact sequences of (Noetherian) graded $R[X]$-modules. This will allow us to conclude that $\text{ent}_L(\Phi, M)$ is again a length function on Mod $R[X]$ and is equal to the multiplicity of X on M whenever M_Φ is a finitely generated $R[X]$-module. Also, polyentropy is then an operator from length functions on Mod R to length functions on Mod $R[X]$

[1] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, Algebraic entropy for abelian groups, *Trans. Amer. Math. Soc.* **361** (2009), 3401–3434.

Algebraic entropy of Amenable group actions

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Consider a set X and a self-map $T : X \to X$. Denote by (X, T) the discrete-time dynamical system whose evolution law $\mathbb{N} \times X \to X$ is given by $(n, x) \mapsto T^n(x)$. Depending on the possible structures on (X, T), one can introduce different real-valued invariants to measure the "disorder" or "mixing" produced by the action of T on X.

In recent years, many papers were published studying a specific invariant, called *algebraic entropy*, of dynamical systems consisting of an Abelian group X and an endomorphism T. In turn, such dynamical systems can be naturally considered as $\mathbb{Z}[X]$-modules.

Let now R be an arbitrary ring, M a left R-module and Γ a group. In the present talk we will be concerned with dynamical systems of the form (M, λ), where $\lambda : \Gamma \to \mathrm{Aut}_R(M)$ is a left action. After recalling that such systems can be identified with left $R[\Gamma]$-modules, we explain how to generalize the techniques of algebraic entropy to these group actions, and more generally to modules over crossed group rings, provided Γ is an amenable group.

In the final part of the talk we present applications of the general theory of algebraic entropy to classical problems in (crossed) group algebras such as (strong versions of) the stable finiteness and the zero-divisor conjectures.