Recent progress on the combinatorics of polyhedra and simplicial complexes

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We describe recent progress on two questions about the combinatorics of polytopes and simplicial complexes:

- *How many simplicial spheres are there, of a given dimension and number of vertices?* The upper bound theorem on spheres (Stanley, 1975) implies that the number of combinatorially different d-spheres with n vertices is in $\exp(O(n^{\lceil d/2 \rceil} \log n))$. On the other hand, Kalai (1988) showed how to construct $\exp(\Omega(n^{\lfloor d/2 \rfloor}))$ of them. In even dimension the gap is not that significant, but in odd dimension it is. In recent work with Nevo and Wilson we have closed this gap, showing how to construct $\exp(\Omega(n^{\lceil d/2 \rceil}))$. As a by-product, we also construct 4-polytopes with $\Omega(n^{d/3})$ facets that are not simplices, thus answering a question of G. M. Ziegler.

- *What is the maximum diameter of a polytope or sphere, of a given dimension and number of vertices?* The Hirsch Conjecture stated that the graph-diameter of a d-polytope with n facets cannot exceed $n - d$. Equivalently, that the adjacency-diameter of a polytopal simplicial sphere of dimension $d - 1$ on n vertices cannot exceed $n - d$. Although the conjecture itself has been disproved (Santos 2012) the underlying question is wide open; the counter-examples to Hirsch still have a linear bound, while no polynomial upper bound for the diameter is known.

  Some work towards answering this question goes by generalizing it to more general simplicial complexes. We report on recent results in this direction: Adiprasito and Benedetti (2013) have proved the Hirsch-conjecture under certain combinatorial assumption on the complex (flagness). On the other end, the author (2013) has proved that with no topological assumption whatsoever, one can construct pure d-complexes on n vertices and with diameter $\Omega(n^{2d/3})$.