# The Baker–Campbell–Hausdorff formula up to terms of degree 20 written in a Hall basis or a Lyndon basis

We write the BCH formula

$$\begin{aligned}\log(\exp(X)\exp(Y)) &= X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]\\ &\quad -\frac{1}{24}[Y,[X,[X,Y]]]+\cdots\end{aligned}$$

in the form

$$\log(\exp(X)\exp(Y)) = \sum_{i\geq 1} z_i\, E_i, \tag{1}$$

where $z_i \in \mathbb{Q}$ $(i \geq 1)$ and $\{E_i \ : \ i = 1, 2, 3, \ldots\}$ is a basis of the free Lie algebra $\mathcal{L}(X, Y)$ generated by the symbols $X$ and $Y$, with elements $E_i$ of the form

$$E_1 = X, \quad E_2 = Y, \quad \text{and} \quad E_i = [E_{i'}, E_{i''}] \quad \text{for each} \quad i \geq 3 \tag{2}$$

for appropriate values of the integers $i', i'' < i$ (for $i = 3, 4, \ldots$). We denote as $|i|$ the homogeneous degree of the element $E_i$ of the basis, that is, $|1| = |2| = 1$, and $|i| = |i'| + |i''|$ for $i \geq 3$.

In the file 'bchHall20.dat', there is a table with five columns with integer entries that can be used to obtain the BCH formula up to terms of degree 20 in a classical Hall basis. The first three columns of the table determines the 111013 elements of a Hall basis $E_i$ with homogeneous degree $|i| \leq 20$. The first column is simply $i = 1, 2, 3 \ldots, 111013$. The second and third column give the values (for each $i$) of $i'$ and $i''$ respectively. The fourth and fifth columns contains integers $p_i$ and $q_i$ such that (1) holds with $z_i = p_i/q_i$.

In the file 'bchLyndon20.dat', a table with five columns with integer entries is provided that similarly can be used to obtain the BCH formula up to terms of degree 20 in the Lyndon basis of the free Lie algebra $\mathcal{L}(X, Y)$ generated by the symbols $X$ and $Y$.

In both cases, the elements of the basis are ordered (in the table) in such a way that $i < j$ if $|i| < |j|$. Thus, in order to identify the terms of homogeneous degree $n \leq 20$, it is enough to know the number $l_n$ of elements $E_i$ in the basis with homegeneous degree $|i| = n$ $(1 \leq n \leq 20)$. The values of $l_n$ for $n = 1, 2 \ldots, 20$ are: 2, 1, 2, 3, 6, 9, 18, 30, 56, 99, 186, 335, 630, 1161, 2182, 4080, 7710, 14532, 27594, 52377.