Granular Fuzzy Inference System (FIS) Design by Lattice Computing

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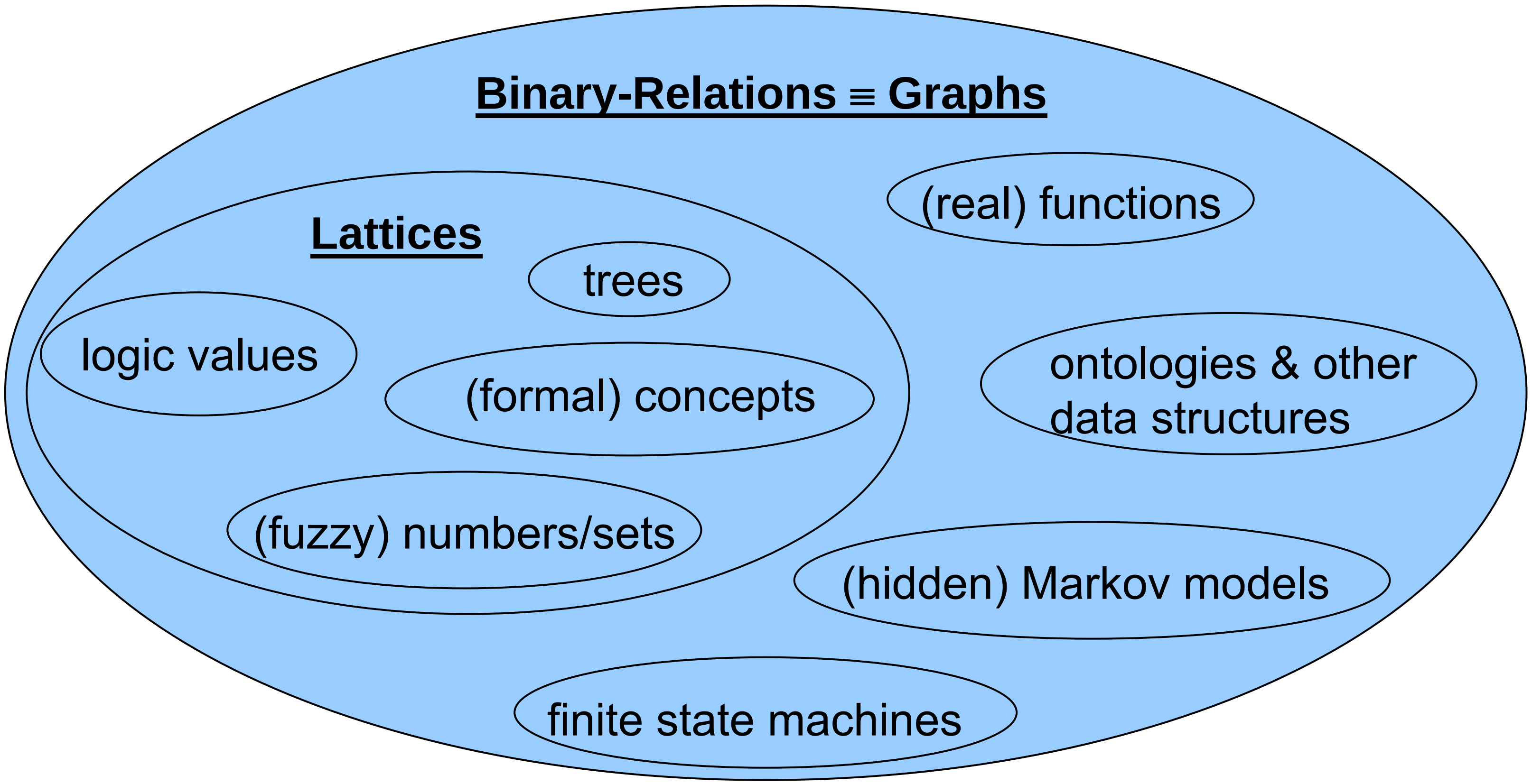
The BIG picture & A conjecture



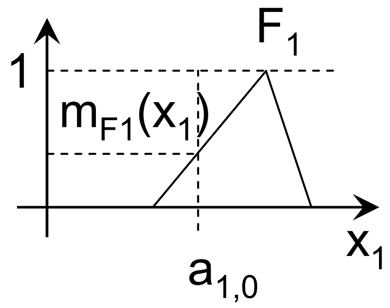
- Lattices could be an instrument for the study of binary-relations.

- Computing in lattices is called **Lattice Computing** (LC), that is a term introduced by Manuel Graña.
- LC is not esoteric /mystic, as shown next.

Our objective here: Fuzzy Inference System (FIS) improvements

<u>Reminder</u>

Rule #i: IF



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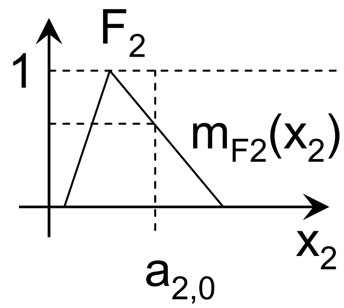

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- Activation of $F_1 = m_{F1}(a_{1,0})$.
- Activation of $F_2 = m_{F2}(a_{2,0})$.
- Activation of Rule #i = either $m_{F1}(a_{1,0}) \wedge m_{F2}(a_{2,0})$ or $m_{F1}(a_{1,0}) m_{F2}(a_{2,0})$.

Preliminary Material

- Our proposed Lattice Computing approach is based on two functions in a lattice $(L,\leq)$:

A) A *positive valuation* function $v: L\rightarrow R$, and

B) A (bijective) *dual isomorphic* function $\theta: L\rightarrow L$.

A) A *positive valuation* function v: L→R introduces two useful functions, namely

(i) A *metric distance* function d: L×L→$R^{\geq 0}$ given by d(x,y) = v(x∨y) – v(x∧y), and

(ii) An *inclusion measure* σ: L×L→[0,1] given by

either $\sigma_{\vee}(x,y) = \sigma_{\vee}(x \leq y) = \frac{v(y)}{v(x \vee y)}$

or $\sigma_{\wedge}(x,y) = \sigma_{\wedge}(x \leq y) = \frac{v(x \wedge y)}{v(x)}$

where …

An *inclusion measure* function σ: $L \times L \rightarrow [0,1]$, in a complete lattice $(L, \leq)$ with minimum element O, by definition, satisfies conditions

1) $\sigma(x,O) = 0$, $x \neq O$.
2) $\sigma(x,x) = 1, \forall x \in L$.
3) $u \leq w \Rightarrow \sigma(x,u) \leq \sigma(x,w)$.
4) $x \wedge y < x \Rightarrow \sigma(x,y) < 1$.

B) A *dual isomorphic* function θ: $L \rightarrow L$ extends both $d(x,y)$ and $\sigma(x,y)$ to the complete lattice $(\tau(L),\leq)$ of intervals in $(L,\leq)$.

A Hierarchy of Complete Lattices

Hierarchy Level-0: Real Numbers

- The *totally-ordered* lattice of real numbers.

$\rightarrow$ For any two real numbers a and b it is either $a \leq b$ or $a > b$.

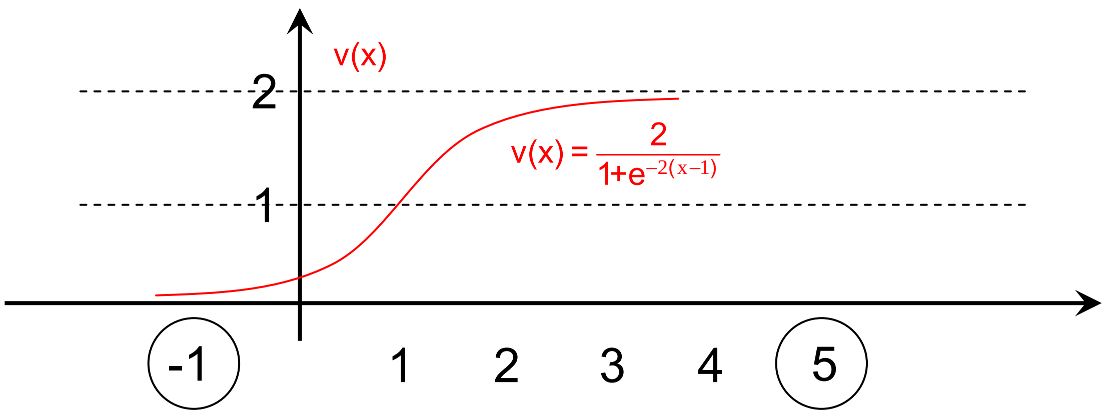
- Consider the set $\bar{R}$ of real numbers also including both symbols $-\infty$ and $+\infty$; i.e. $\bar{R} = R \cup \{-\infty, +\infty\}$.

- $(\bar{R}, \leq)$ is a complete lattice with $O = -\infty$, $I = +\infty$.

- Any strictly increasing function $v: \bar{R} \rightarrow R$ is a *positive valuation*.

- A metric distance d: $\bar{R}\times\bar{R}\rightarrow R^{\geq 0}$ is given by d(x,y) = v(x∨y) – v(x∧y).

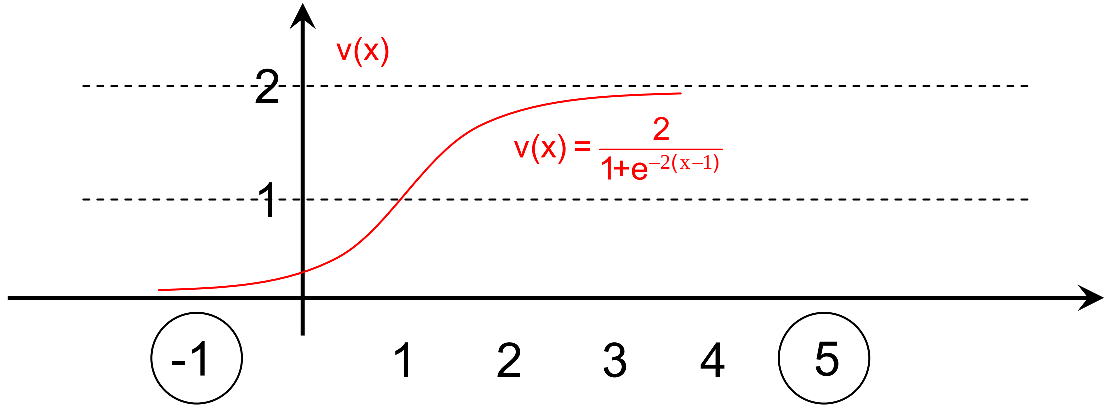
Example



d(-1,5) = v(5) – v(-1) = 1.9993 - 0.0360 = 1.9633

- Inclusion measures σ: $\bar{\mathbb{R}}\times\bar{\mathbb{R}}\rightarrow[0,1]$ are given by $\sigma_{\vee}(x,y)=v(y)/v(x\vee y)$ and $\sigma_{\wedge}(x,y)=v(x\wedge y)/v(x)$.

Example



$$\sigma_{\wedge}(-1\le 5)=\frac{v(-1\wedge 5)}{v(-1)}\cong 1 \qquad \sigma_{\wedge}(5\le -1)=\frac{v(5\wedge -1)}{v(5)}\cong 0.0180$$

$$\sigma_{\vee}(-1\le 5)=\frac{v(5)}{v(-1\vee 5)}\cong 1 \qquad \sigma_{\vee}(5\le -1)=\frac{v(-1)}{v(5\vee -1)}=0.0180$$

Hierarchy Level-1: Intervals

- Consider the complete lattice $(\bar{\mathbb{R}},\leq^{\partial})\times(\bar{\mathbb{R}},\leq)$ = $(\bar{\mathbb{R}}\times\bar{\mathbb{R}},\leq^{\partial}\times\leq)$ = $(\bar{\mathbb{R}}\times\bar{\mathbb{R}},\geq\times\leq)$ = $(\Delta,\leq)$.

- An element, namely *generalized interval*, of $(\Delta,\leq)$ will be denoted within square brackets.

$\rightarrow$ $[a,b] \leq [c,d] \Leftrightarrow$ both $c \leq a$ and $b \leq d$.

- Any strictly decreasing (*bijective*, usually) function θ: $\bar{\mathbb{R}} \to \bar{\mathbb{R}}$ is *dual isomorphic* in the complete lattice $(\bar{\mathbb{R}}, \leq)$.

- In conclusion, function v_Δ: $\Delta \to \mathbb{R}$ given by $v_\Delta([a,b]) = v(\theta(a)) + v(b)$ is a positive valuation in the complete lattice $(\Delta, \leq)$.

There follow functions

1) **metric** d_Δ: $\Delta \to R^{\geq 0}$ given by

$$d_\Delta([a,b],[c,d]) = v(\theta(a \wedge c)) - v(\theta(a \vee c)) + v(b \vee d) - v(b \wedge d)$$

2) **inclusion measure** $\sigma_\vee$: $\Delta \times \Delta \to [0,1]$ given by

$$\sigma_\vee([a,b] \leq [c,d]) = \frac{v(\theta(c)) + v(d)}{v(\theta(a \wedge c)) + v(b \vee d)}$$

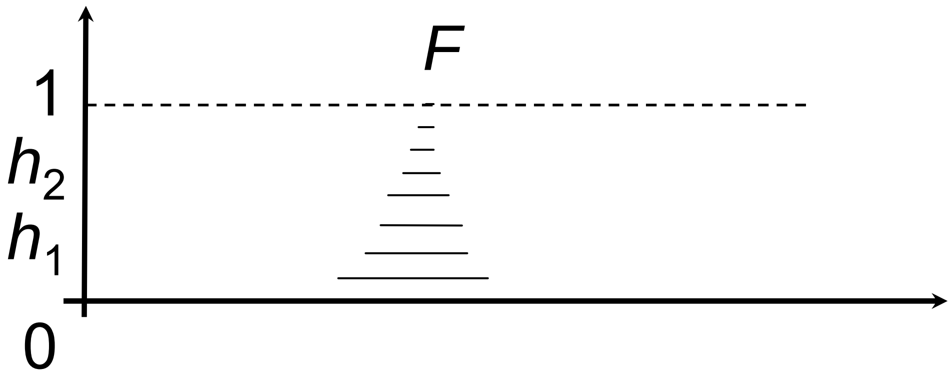
3) **inclusion measure** $\sigma_\wedge$: $\Delta \times \Delta \to [0,1]$ given by

$$\sigma_\wedge([a,b] \leq [c,d]) = \frac{v(\theta(a \vee c)) + v(b \wedge d)}{v(\theta(a)) + v(b)}$$

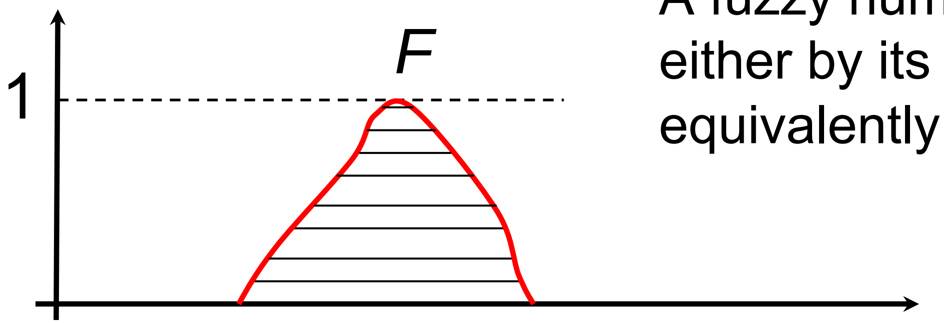
- Our interest focuses on the complete lattice $(\tau_O(\bar{R}),\leq)$ of intervals (sets) also including the empty interval $O=[+\infty,-\infty]$.

Hierarchy Level-2: Intervals' Numbers (INs)

- A IN is a function F: $(0,1] \rightarrow \tau_O(\bar{\mathsf{R}})$ such that $0<h_1\leq h_2\leq 1 \Rightarrow F(h_1) \geq F(h_2)$.



- Based on Zadeh's "resolution identity theorem" a IN may represent a *fuzzy number*.



A fuzzy number *F* can be represented, either by its *membership function* or, equivalently, by its (interval) α–cuts.

There follow

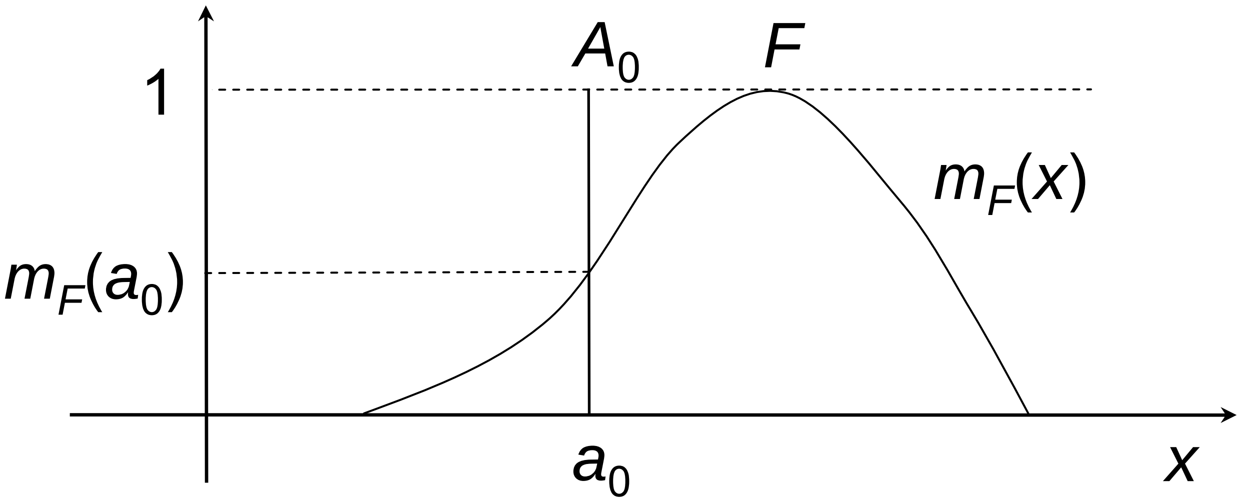
- $$\sigma_{\wedge}(F_1 \leq F_2) = \int_0^1 \sigma_{\wedge}(F_1(h) \leq F_2(h))\,\mathrm{d}h$$

- $$\sigma_{\vee}(F_1 \leq F_2) = \int_0^1 \sigma_{\vee}(F_1(h) \leq F_2(h))\,\mathrm{d}h$$

- $$d_F(F_1, F_2) = \int_0^1 d_{\Delta}(F_1(h), F_2(h))\,\mathrm{d}h$$

Proposition #1

- $\sigma_\wedge(A_0 \leq F) = m_F(a_0)$



→Proposition #1 couples a IN's *interval-representation* and its *membership-function-representation*.

Proposition #2

- Consider complete lattices $(L_i,\leq)$ each equipped with an inclusion measure function σ_i, $i\in\{1,\ldots,N\}$. Let $x=(x_1,\ldots,x_N),y=(y_1,\ldots,y_N)\in L=L_1\times\ldots\times L_N$. Then, both functions

$$\sigma_{\wedge}(\mathbf{x}\leq\mathbf{y})=\min_{i}\{\sigma_i(x_i\leq y_i)\}$$ and

$$\sigma_{\prod}(\mathbf{x}\leq\mathbf{y})=\prod_{i}\sigma_i(x_i\leq y_i)$$

are inclusion measures.

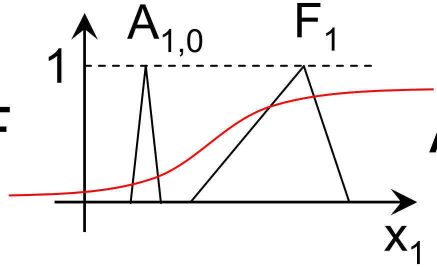
Conclusion

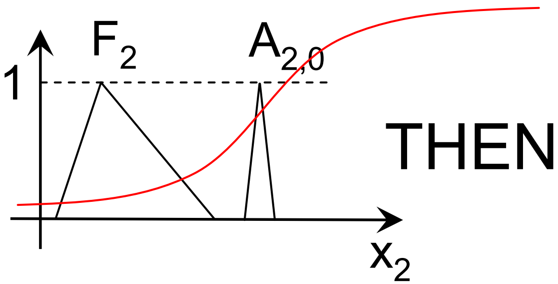
- A conventional FIS (implicitly) employs $\sigma_{\wedge}(.,.)$ with trivial inputs, within rule supports, and positive valuation $v(x)=x$.

Improvements

- Both $\sigma_{\wedge}(.,.)$ and $\sigma_{\vee}(.,.)$ can deal with non-trivial (granular) input INs.
- Only $\sigma_{\vee}(.,.)$ can deal with input INs beyond rule support.
- Both $\sigma_{\wedge}(.,.)$ and $\sigma_{\vee}(.,.)$ can use parametric functions $v(.)$ and $\theta(.)$ towards introducing tunable nonlinearities.

In summary,

Rule #i: IF 

AND 

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