Advantages of using Lattice Theory in Computational Intelligence

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1. Introductory Material

The Context

Mathematical Modeling is the art of describing mathematically a world aspect.

Mathematical Modeling is instrumental for

- Control,
- Decision-support,
- Knowledge extraction,
- Information enhancement, etc.

- Due to the conventional measurement practice of “successive comparisons”, *Mathematical Modeling* is, typically, pursued in R^N.

- With the advent of computers, non-numeric data have proliferated in applications.

- One way for modeling based on non-numeric data is to *transform* them to numeric data in R^N.

$\rightarrow$ However, critical *content* may be lost.

- Another way for modeling based on non-numeric data is to treat them beyond R^N.

$\rightarrow$ However, an enabling (mathematical) framework is currently missing.

Fact

Popular types of data in applications are *partially(lattice)-ordered*.

For example,

- 0-D,1-D, 2-D,... Arrays of Real Numbers
- Logic Values
- A *Set** Partitions
- *Sets** in a Power-Set
- (Strings of) Symbols

*A *Set* may be a *Relation* $R \subseteq A \times B$, e.g. a *graph*.

Hypothesis

- *Order-Theory* (or, equivalently, *Lattice Theory*) is an enabling framework for unified data modeling.

Two different ways of employing Lattice Theory:

1. “order-based” (it emphasizes *semantics*)

2. “algebra-based” (it emphasizes *operations*)

State-of-the-Art

Computing in Lattices is employed, by "isolated" research communities, in applications of

- Logic and Reasoning
- Mathematical Morphology
- Formal Concept Analysis
- Computational Intelligence

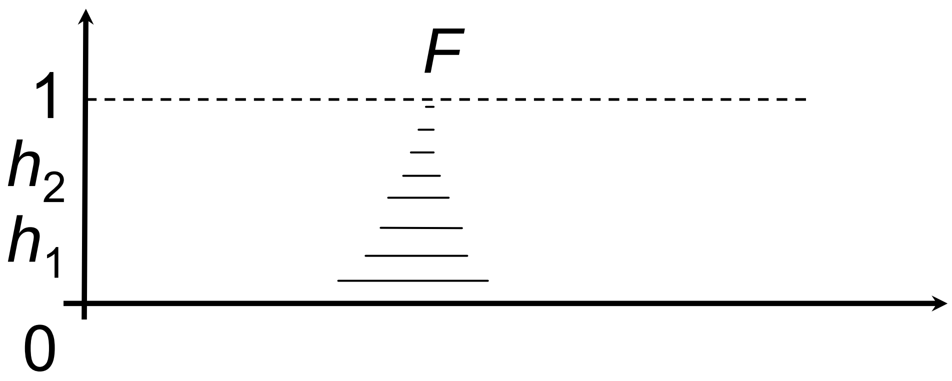
Efforts to cross-fertilize *Lattice Computing* practices:

- Kaburlasos VG, Ritter GX, Eds. (2007) Computational Intelligence Based on Lattice Theory. Springer, series: Studies in Computational Intelligence, 67.
- Kaburlasos V, Priss U, Graña M, Eds. (2008) Proc. Lattice-Based Modeling Workshop (LBM 2008), Olomouc, The Czech Republic: Palacký Univ.
- Kaburlasos VG, Guest Editor, Information Sciences, planned 2010 Special Issue entitled "Information Engineering Applications Based on Lattices".

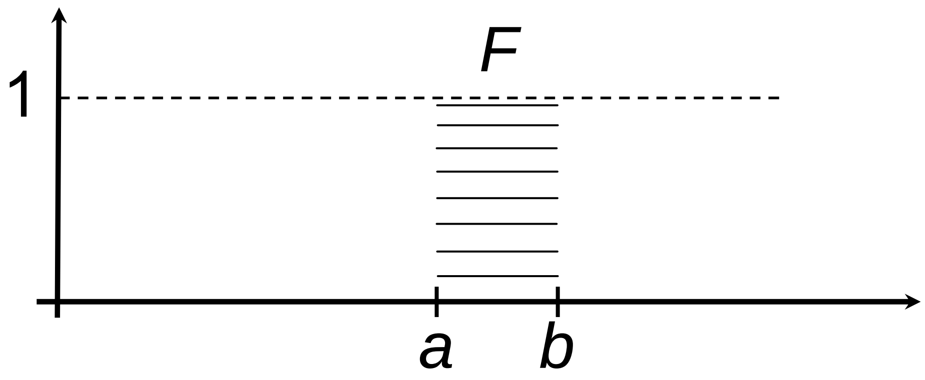
- In *Lattice Computing*, **Intervals' Numbers** (**IN**s) have emerged with a promising potential.

2. Intervals' Numbers (INs)

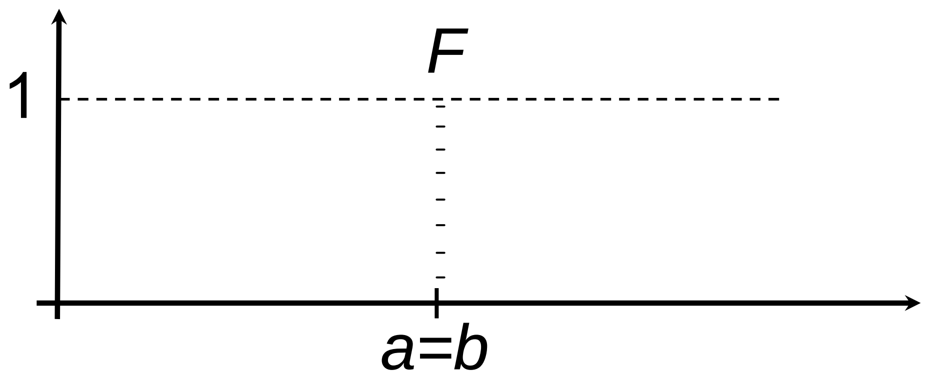
- IN $F = \bigcup_{h \in (0,1]} \{[a_h, b_h]\}$, also denoted $F = \bigcup_{h \in (0,1]} \{F(h)\}$, is a set of intervals: $0 < h_1 \leq h_2 \leq 1 \Rightarrow F(h_1) \geq F(h_2)$.



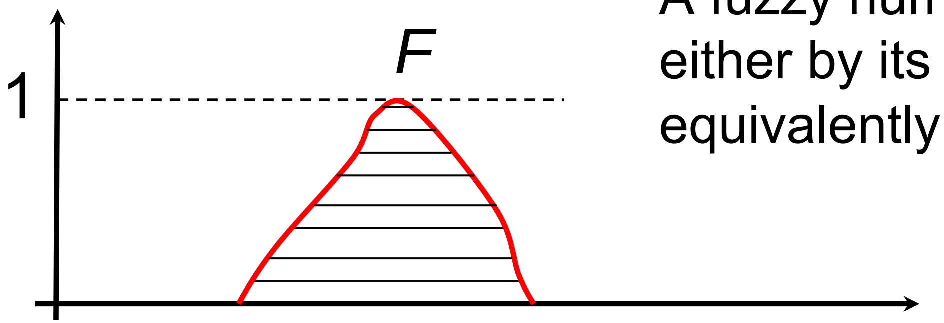
- IN $F = \bigcup_{h \in (0,1]} \{[a,b]\}$ represents a conventional **interval** $[a,b]$.



- For $a = b$, IN $F = \bigcup_{h \in (0,1]} \{[a,b]\}$ represents a **real number**.

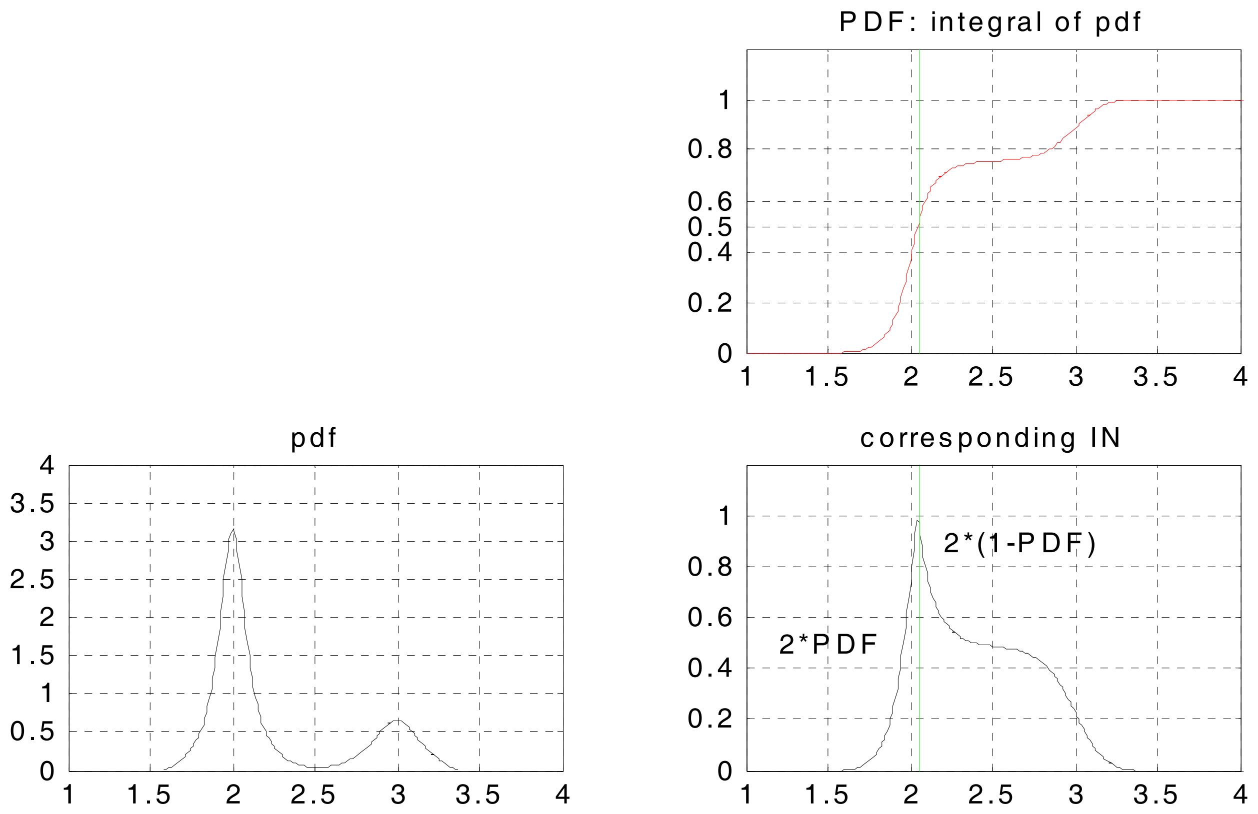


- Based on the “resolution identity theorem” a IN may represent a **fuzzy number**.

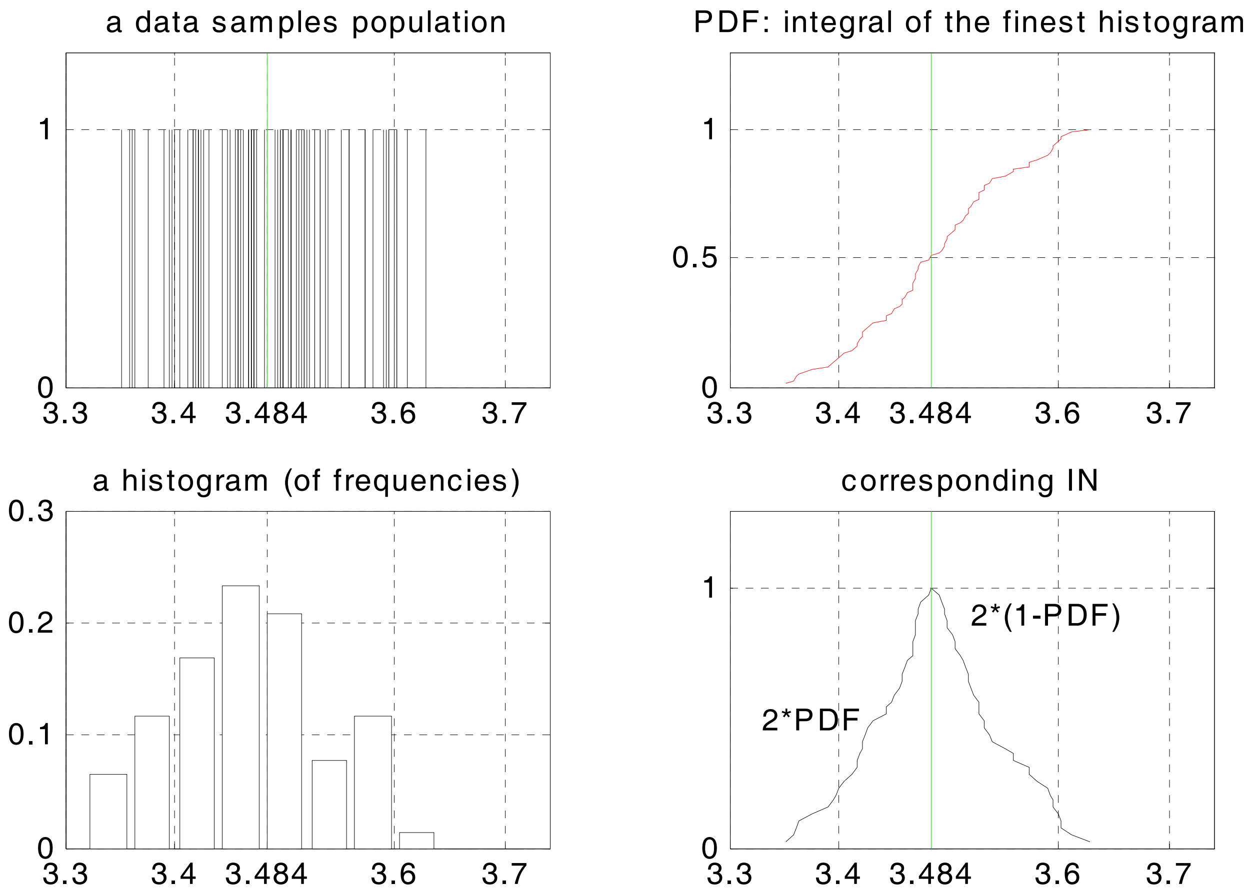


A fuzzy number *F* can be represented, either by its *membership function* or, equivalently, by its (interval) α–cuts.

IN representation of a *pdf*



IN representation of a *data samples* population

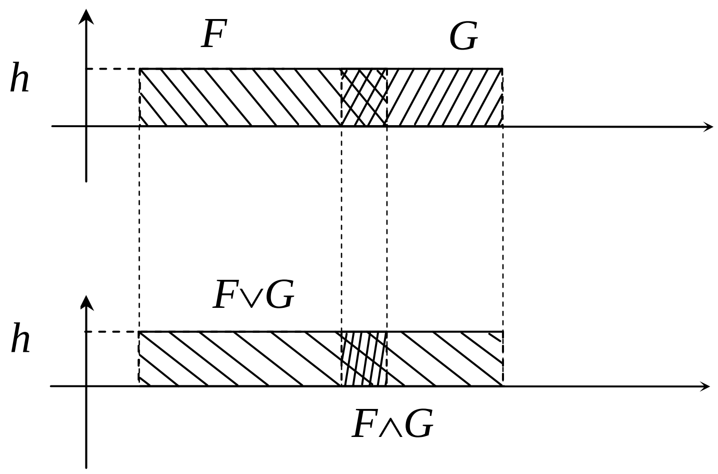


- A IN is a mathematical object (a number) — Different information processing paradigms may interpret it differently.

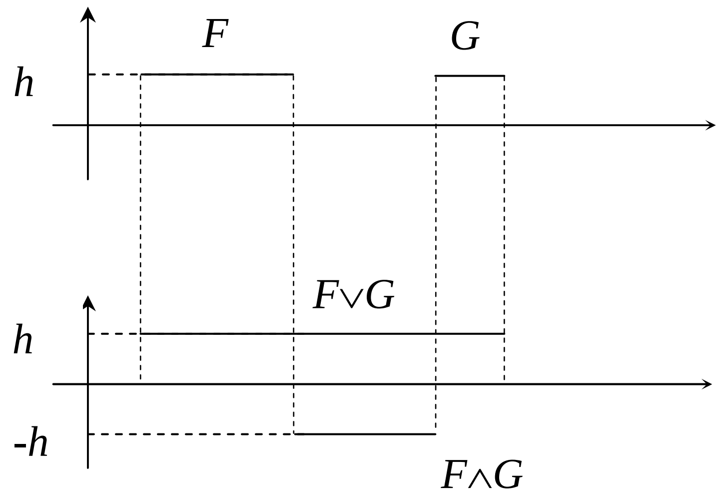
- The lattice (F,≤) of INs emerges as the Cartesian Product of lattices (Δ,≤) of **generalized intervals**.

The lattice (Δ,≤) of generalized intervals

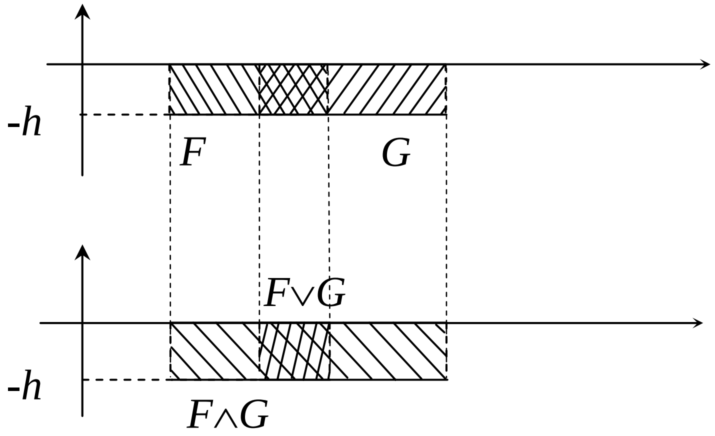
- **Generalized Intervals** [a,b], at height $h \in (0,1]$, can be either *positive* ($a \leq b$) or *negative* ($a > b$)



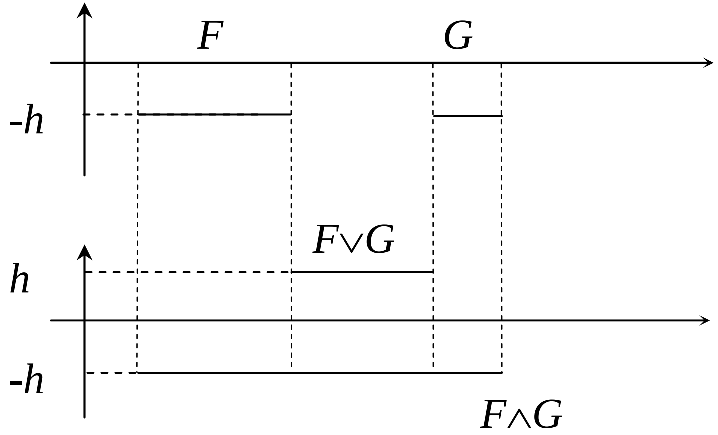
“Overlapping” positive generalized intervals.



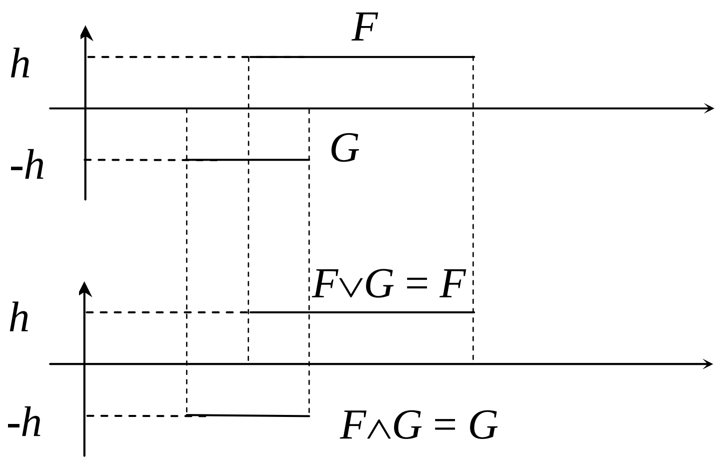
“Non- overlapping” positive generalized intervals.



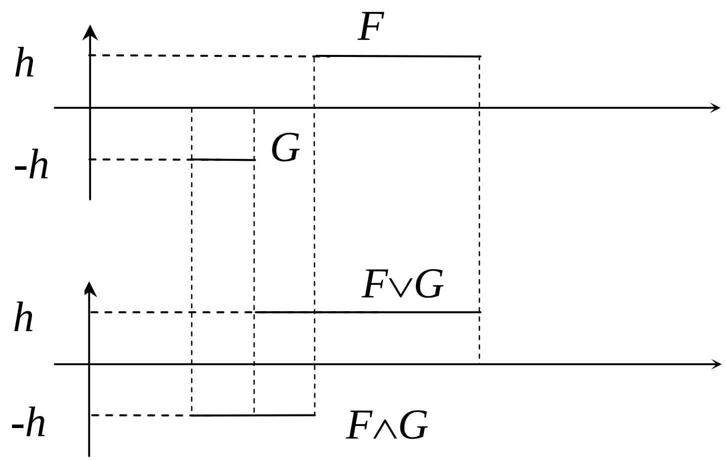
"Overlapping" negative generalized intervals.



"Non-overlapping" negative generalized intervals.



"Overlapping" positive and negative gen. ints.

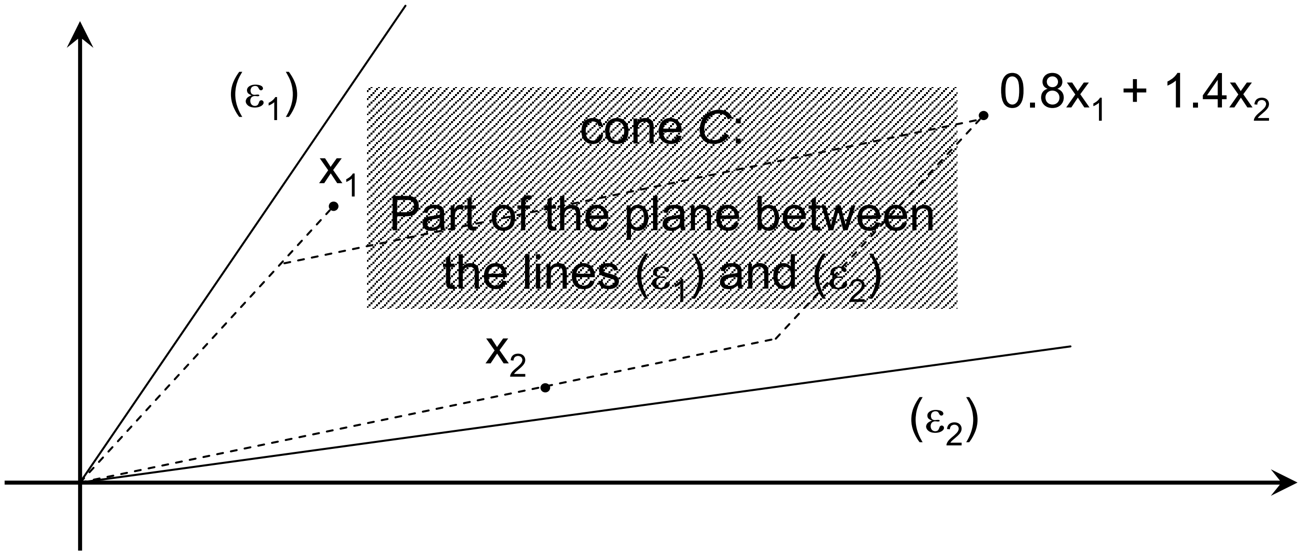


"Non-overlapping" positive and negative gen. ints.

- Interest focuses on **positive** generalized intervals, which give rise to INs.
- The set Δ_+ of positive generalized intervals is a **cone** in the linear space of generalized intervals.

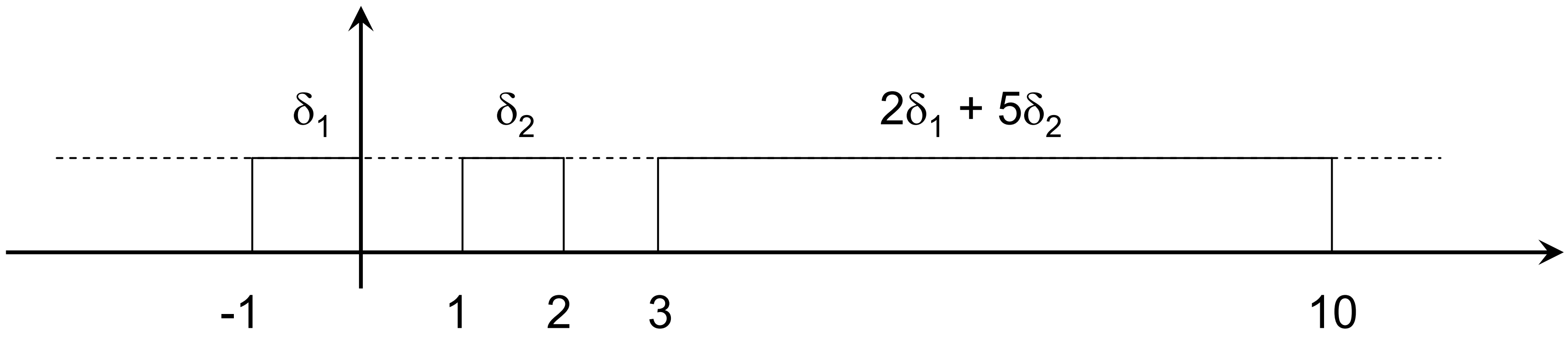
Definition (reminder)

Cone is a subspace *C* of a linear space such that for $x_1, x_2 \in C$ and $c_1, c_2 \geq 0$ it follows $(c_1 x_1 + c_2 x_2) \in C$.



- Let $\delta_1,\delta_2 \in \Delta_+$ and $c_1,c_2 \geq 0$. Then $(c_1\delta_1+c_2\delta_2) \in \Delta_+$.

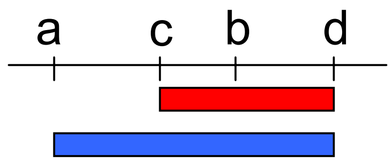
<u>Example</u> $2[-1,0] + 5[1,2] = [3,10]$



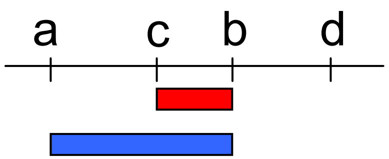
Inclusion Measure $\sigma: \Delta_+ \times \Delta_+ \rightarrow [0,1]$

- A *degree of inclusion* of an interval $[a,b] \in \Delta_+$ in another interval $[c,d] \in \Delta_+$ can be defined by

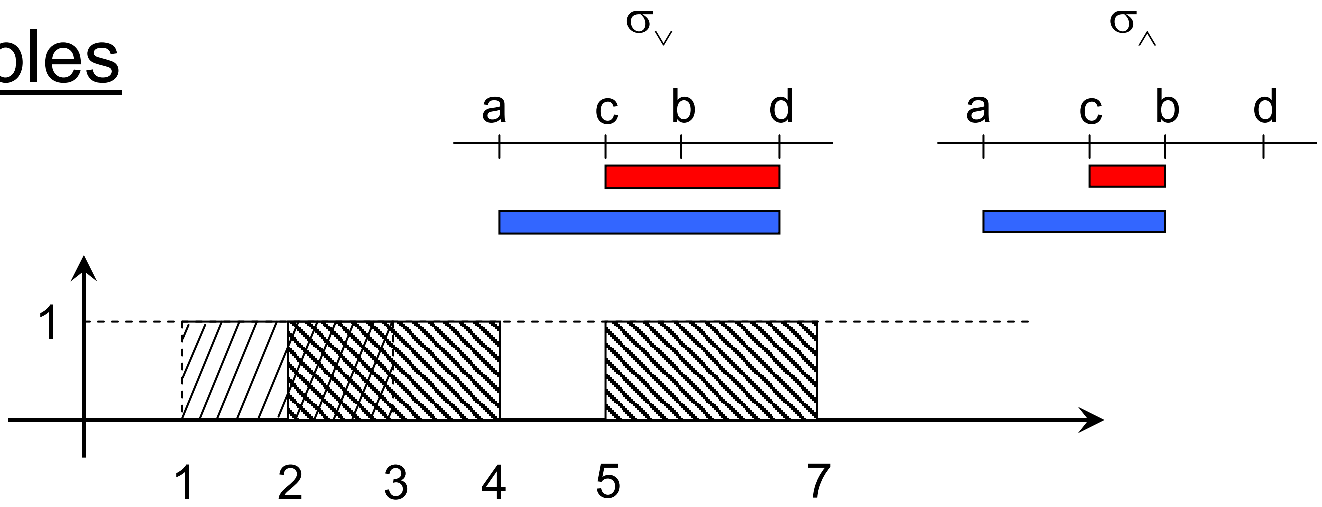
either $$\sigma_\vee([a,b] \le [c,d]) = \frac{|d - c|}{|b \vee d - a \wedge c|}$$



or $$\sigma_\wedge([a,b] \le [c,d]) = \begin{cases} \dfrac{|b \wedge d - a \vee c|}{|b - a|}, & a \vee c \le b \wedge d \\ 0, & \text{otherwise} \end{cases}$$



Examples



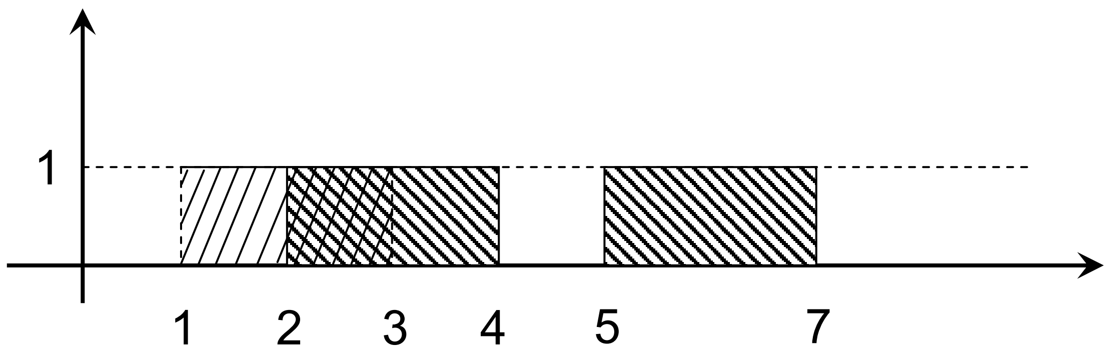
$$\sigma_{\wedge}([1,3] \leq [2,4]) = \frac{1}{2} \cong 0.5 \qquad \sigma_{\wedge}([1,3] \leq [5,7]) = 0$$

$$\sigma_{\vee}([1,3] \leq [2,4]) = \frac{2}{3} \cong 0.667 \qquad \sigma_{\vee}([1,3] \leq [5,7]) = \frac{2}{6} \cong 0.334$$

Metric Distance d_Δ: $\Delta \times \Delta \rightarrow R^{\geq 0}$

- A *metric* between generalized intervals can be defined by $d_\Delta([a,b],[c,d]) = |a-c| + |b-d|$.

Examples



$d_\Delta([1,3],[2,4]) = |1-2| + |3-4| = 2$

$d_\Delta([1,3],[5,7]) = |1-5| + |3-7| = 8$

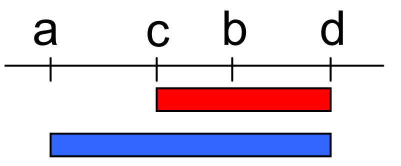
- The previous analysis has (implicitly) assumed

(1) **positive valuation** (*strictly increasing*) function $v(x)=x$, and

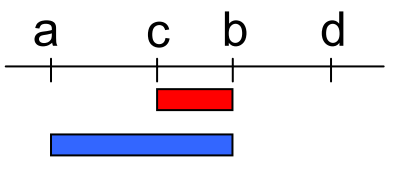
(2) **dual isomophic** (*strictly decreasing*) function $\theta(x)=-x$.

- Tunable non-linearities can be introduced, for alternative functions $v(x)$ and $\theta(x)$, as follows.

- $$\sigma_{\vee}([a,b] \le [c,d]) = \frac{v(\theta(c)) + v(d)}{v(\theta(a \wedge c)) + v(b \vee d)}$$



- $$\sigma_{\wedge}([a,b] \le [c,d]) = \begin{cases} \dfrac{v(\theta(a \vee c)) + v(b \wedge d)}{v(\theta(a)) + v(b)}, & a \vee c \le b \wedge d \\ 0, & \text{otherwise} \end{cases}$$

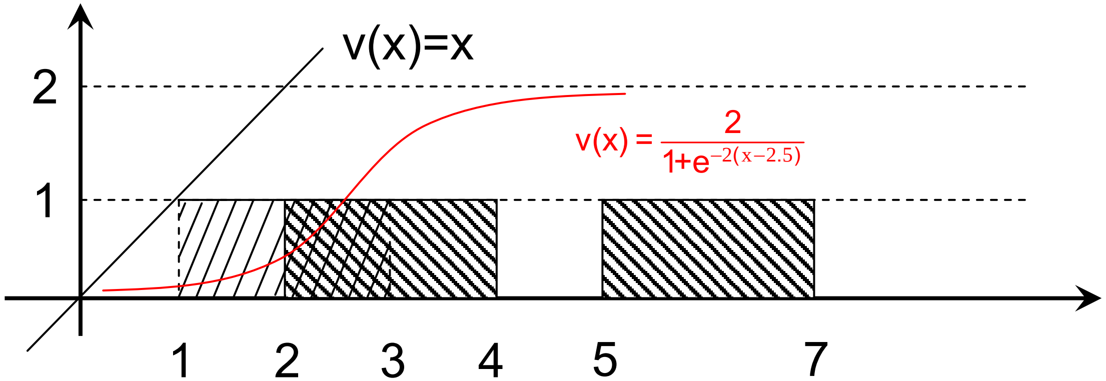


Moreover,

- $d_{\Delta}([a,b],[c,d]) = [v(\theta(a \wedge c)) - v(\theta(a \vee c))] + [v(b \vee d) - v(b \wedge d)]$

Example

$\theta(x) = -x$



$$\sigma_{\wedge}([1,3] \le [2,4]) = \begin{cases} 0.5 \\ 0.9989 \end{cases} \qquad \sigma_{\wedge}([1,3] \le [5,7]) = \begin{cases} 0 \\ 0 \end{cases}$$

$$\sigma_{\vee}([1,3] \le [2,4]) = \begin{cases} 0.667 \\ 0.9992 \end{cases} \qquad \sigma_{\vee}([1,3] \le [5,7]) = \begin{cases} 0.334 \\ 0.9991 \end{cases}$$

$$d_{\Delta}([1,3],[2,4]) = \begin{cases} 2 \\ 0.4446 \end{cases} \qquad d_{\Delta}([1,3],[5,7]) = \begin{cases} 8 \\ 0.5395 \end{cases}$$

Extensions to the lattice (F,≤) of INs

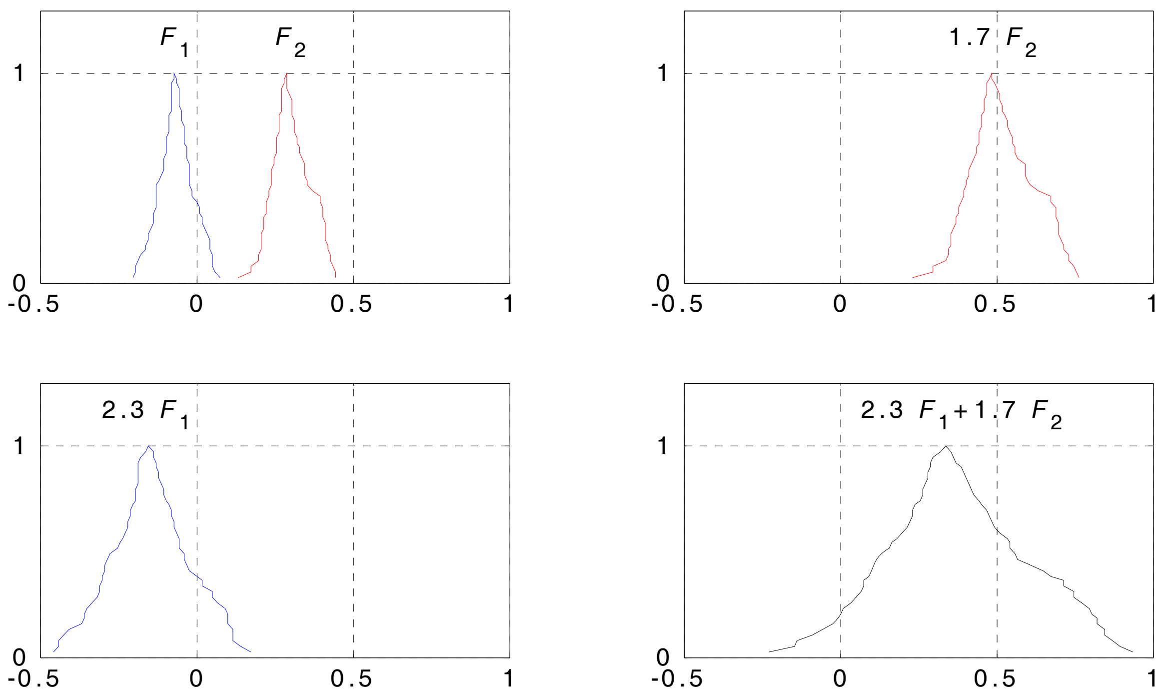
- $\sigma_{\wedge}(F_1 \le F_2) = \int_0^1 \sigma_{\wedge}(F_1(h) \le F_2(h))dh$

- $\sigma_{\vee}(F_1 \le F_2) = \int_0^1 \sigma_{\vee}(F_1(h) \le F_2(h))dh$

- $d_F(F_1, F_2) = \int_0^1 d_{\Delta}(F_1(h), F_2(h))dh$

Metric, fuzzy lattice (F,≤) is a cone

Example



3. Conclusion

1. The presented tools are “unifying”. For instance, graphs can be processed by IN-computing on shortest paths.

2. Popular *Computational Intelligence* algorithms can be extended from the Euclidean space R^N to the *metric, fuzzy-lattice, cone* $(F,\leq)$ so as to rigorously deal with “non-crisp” (input, etc.) data.

3. The proposed technology may have a far-reaching potential for Human-Computer Interaction (HCI) based on disparate types of (non)numeric data.