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The supercore for normal-form games

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Abstract

This paper analyzes the supercore of a system derived from a normal-form game. For the case of a finite game with pure strategies, we define a sequence of games and show that the supercore coincides with the set of Nash equilibria of the last game in that sequence. This result is illustrated with the characterization of the supercore for the *n*-person prisoner's dilemma. With regard to the mixed extension of a normal-form game, we show that the set of Nash equilibrium profiles coincides with the supercore for games with a finite number of Nash equilibria.

JEL classification: C70; C71; C72

Keywords: Individual contingent threat situation; Nash equilibrium; Subsolution; Von Neumann and Morgenstern stable set

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1. Introduction

This paper studies the relation between the supercore of games in what Greenberg [2] calls individual contingent threat situations and the Nash equilibria (see [3]) of the corresponding normal-form games. It is organized as follows. Section 2 contains the preliminaries. In Section 3, starting from the definition of a system associated to a normal-form game with pure strategies, we present a procedure that allows the determination of the supercore for that system. The procedure defines a sequence of games and shows that the supercore

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coincides with the set of Nash equilibria (NE) of the last game in that sequence. In addition, we characterize the supercore for the n-person prisoner's dilemma. Lastly, Section 4 studies the supercore of the system associated to the mixed extension of a normal-form game, and shows conditions under which the supercore coincides with the set of NE.

2. Preliminaries

An *abstract system* [6] is a pair (X, R), where X is a set and R is an irreflexive binary relation where for $x, y \in X$, xRy means that x dominates y.

For any $x \in X$, let $\mathcal{D}(x)$ denote the dominion of x, i.e., $\mathcal{D}(x) = \{x' \in X : xRx'\}$. Given any subset A of X, we define the following sets: $\mathcal{D}(A) = \bigcup_{x \in A} \mathcal{D}(x)$, $\mathcal{U}(A) = X - \mathcal{D}(A)$, and $\mathcal{P}(A) = \mathcal{U}(A) - A$.[1] A *subsolution* of (X, R) [5] is a subset A of X such that $A \subseteq \mathcal{U}(A)$ and $A = \mathcal{U}^2(A)$, where $\mathcal{U}^2(A) = \mathcal{U}(\mathcal{U}(A))$. Condition $A \subseteq \mathcal{U}(A)$, known as the internal stability condition, implies that no point y in A is dominated by some other point x in A. Condition $A = \mathcal{U}^2(A)$ contains two implications: (i) $A \subseteq \mathcal{U}^2(A)$, which means that if y dominates some point x in A then y is itself dominated by some point in A (Roth calls this self-protection), and (ii) $\mathcal{U}^2(A) \subseteq A$, which implies that $\mathcal{P}(A) \subseteq \mathcal{D}(\mathcal{P}(A))$, or to quote Roth [5, p. 44], that: "… every point in $\mathcal{U}(A) - A$ is dominated by some other point in the same set and the entire set, thus 'overrules' itself leaving only the set as 'sound'." The intersection of all subsolutions of (X, R) is also a subsolution which is known as the *supercore*.

A subset A of X is a *von Neumann and Morgenstern* (*vN&M*) *stable set* of (X, R) if $A = \mathcal{U}(A)$. Thus, a *vN&M* stable set is characterized by the internal stability condition $A \subseteq \mathcal{U}(A)$, and by $\mathcal{U}(A) \subseteq A$, which is known as the external stability condition. Clearly, a *vN&M* stable set is a subsolution that satisfies $\mathcal{P}(A) = \emptyset$.

A subset A of X is the *core* of (X, R) if $A = \mathcal{U}(X)$.

A finite *normal form game* Γ^N is a triple $\langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, where $N = \{1, \ldots, n\}$ is a finite set of players, S_i is a finite set of strategies for player i, and $u_i : S = \times_{i \in N} S_i \longrightarrow \mathbb{R}$ is player i's payoff function.

A strategy of player i, $\widehat{s_i}$, is a best response to s_{-i} if for all $s_i \in S_i$, $u_i(\widehat{s_i}, s_{-i}) \geqslant u_i(s_i, s_{-i})$, where $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Let $s = (s_1, \ldots, s_n)$ denote a strategy profile. Then, $s^* = (s_1^*, \ldots, s_n^*)$ is a *Nash equilibrium* in Γ^N if s_i^* is a best response to s_{-i}^* for all $i \in N$.

A *mixed extension of the game* Γ^N is a triple $< N, \{\Delta(S_i)\}_{i \in N}, \{U_i\}_{i \in N} >$, where $\Delta(S_i)$ is the simplex of the mixed strategies for player i, and $U_i : \Delta(S) = \times_{i \in N} \Delta(S_i) \longrightarrow \mathbb{R}$ assigns to $\sigma \in \Delta(S)$ the expected value under u_i of the lottery over S that is induced by σ, so that $U_i(\sigma) = \sum_{s \in S} (\prod_{j \in N} \sigma_j(s_j)) u_i(s)$.

A mixed strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* in the mixed extension of the game Γ^N if σ_i^* is a best response to $\sigma_{-i}^* = (\sigma_1^*, \ldots, \sigma_{i-1}^*, \sigma_{i+1}^*, \ldots, \sigma_n^*)$ for all $i \in N$.

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[1] The symbol " $-$" stands for the difference binary relation.

3. The supercore for $(S, \succ)$

In order to associate an abstract system (X, R) to the normal-form game $\langle N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$, first let the set of elements equal the set of strategy profiles, $X = S$. Following Greenberg [2], an *individual contingent threat situation of a game* Γ^N is a 4-tuple $\langle N, S, (u_i)_{i\in N}, (\gamma_i)_{i\in N}\rangle$ where γ_i is the correspondence from S into itself defined by:

$$\gamma_i(s) = \{(s_i', s_{-i}) \text{ for some } s_i' \in S_i\}.$$

Definition 1. The system associated to an individual contingent threat situation of a game Γ^N is the pair $(S, \succ)$, where $\succ$ is the binary relation defined on S as follows:

$$s' \succ s \text{ if there exists } i \in N \text{ such that } s' \in \gamma_i(s) \text{ and } u_i(s') > u_i(s).$$

This means that s' dominates s if s' is derived from s via a deviation of player i who is better off under s' than under s.

3.1. *A procedure to compute the supercore for $(S, \succ)$*

In this subsection, we first give an example that illustrates the notions introduced in the preliminaries and the procedure to compute the supercore. We then formalize such a procedure.

Consider the following game Γ_0^N:

	b_1	b_2	b_3	b_4
a_1	6,6	5,5	1,3	2,2
a_2	3,4	4,4	7,2	1,3
a_3	6,2	2,3	8,8	6,2
a_4	2,3	2,5	9,4	2,5

The set of NE of game Γ_0^N is $S_0^* = \{(a_1, b_1)\}$. Let $(S, \succ_0)$ be the system associated to Γ_0^N. Note that $\mathcal{U}(S) = \{(a_1, b_1)\}$. Therefore, the set of NE of game Γ_0^N is the core for the system $(S, \succ_0)$ included in every subsolution. In this system there are three subsolutions: $A_1 = \{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4)\}$, $A_2 = \{(a_1, b_1), (a_2, b_2), (a_4, b_3), (a_3, b_4)\}$ and $A_3 = \{(a_1, b_1), (a_2, b_2)\}$. The first two are vN&M stable sets since $A_1 = \mathcal{U}(A_1)$ and $A_2 = \mathcal{U}(A_2)$. With regard to the third one, we have $\mathcal{U}(A_3) = \{(a_1, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_3), (a_3, b_4), (a_4, b_3), (a_4, b_4)\}$ and $\mathcal{U}^2(A_3) = \{(a_1, b_1),(a_2, b_2)\}$. Therefore, $A_3 \subseteq \mathcal{U}(A_3)$ and $\mathcal{U}^2(A_3) = A_3$. Since $A_3 = A_1 \cap A_2 \cap A_3$, then A_3 is the supercore.

Now, we describe the procedure to compute the supercore. Starting from $S_0^* = \{(a_1, b_1)\}$, compute $\mathcal{D}(S_0^*) = \{(a_1, b_2), (a_1, b_3), (a_1, b_4), (a_2, b_1), (a_4, b_1)\}$. Then, replace the payoffs to the profiles in $\mathcal{D}(S_0^*)$ by the corresponding players' lowest payoffs in game Γ_0^N, that is by (1,2), so that game Γ_1^N is obtained:

	b_1	b_2	b_3	b_4
a_1	6,6	1,2	1,2	1,2
a_2	1,2	4,4	7,2	1,3
a_3	6,2	2,3	8,8	6,2
a_4	1,2	2,5	9,4	2,5

The set of NE of Γ_1^N is $S_1^* = \{(a_1, b_1), (a_2, b_2)\}$. Note that (a_2, b_2) is a profile dominated by (a_1, b_2) in the system $(S, \succ_0)$. The dominion of S_1^* is $\mathcal{D}(S_1^*) = \mathcal{D}(S_0^*) \cup \{(a_2, b_3), (a_2, b_4), (a_3, b_2), (a_4, b_2)\}$. Once again, replacing the payoffs to the profiles in $\mathcal{D}(S_1^*)$ by (1,2), game Γ_2^N is obtained:

	b_1	b_2	b_3	b_4
a_1	6,6	1,2	1,2	1,2
a_2	1,2	4,4	1,2	1,2
a_3	6,2	1,2	8,8	6,2
a_4	1,2	1,2	9,4	2,5

The set of NE profiles of game Γ_2^N is $S_2^* = \{(a_1, b_1), (a_2, b_2)\}$, which coincides with the supercore for the system $(S, \succ_0)$. Since $S_2^* = S_1^*$ the procedure concludes.

Summing up, this procedure generates a sequence of games $\langle \Gamma_0^N, \Gamma_1^N, \Gamma_2^N \rangle$ such that the set of NE profiles of the last game in this sequence is the supercore for the system associated to Γ_0^N.

Now, we formalize this procedure. Let us consider a game Γ^N with at least one NE strategy profile. (This is not a restriction since the supercore of $(S, \succ)$ for a game Γ^N with no NE strategy profile is the empty set [5]. Let S^* be the set of NE strategy profiles of game Γ^N and let $\upsilon_i(\Gamma^N)$ be the lowest payoff for player i in game Γ^N, that is: $\upsilon_i(\Gamma^N) = \min\{u_i(s)\colon s \in S\}$. Next, we define inductively a sequence of games $\langle \Gamma_t^N \rangle_{t=0}^{\infty}$ and a sequence of systems $\langle (S, \succ_t) \rangle_{t=0}^{\infty}$ as follows:

(i) $\Gamma_0^N = \Gamma^N$ and $(S, \succ_0) = (S, \succ)$.
(ii) For $t \geqslant 1$, $\Gamma_t^N = \langle N, \{S_i\}_{i \in N}, \{u_i^t\}_{i \in N} \rangle$, with

$$u_i^t(s) = \begin{cases} \upsilon_i(\Gamma^N) & \text{if } s \in \mathcal{D}(S_{t-1}^*) \text{ in } (S, \succ_{t-1}) \\ u_i^{t-1}(s) & \text{otherwise,} \end{cases}$$

where S_{t-1}^* denotes the set of NE strategy profiles of Γ_{t-1}^N, and $(S, \succ_t)$ is the system associated to Γ_t^N. Formally, this procedure can be summarized as follows:

Step 0: Let $\Gamma_0^N = \Gamma^N$. Compute S_0^*, define $(S, \succ_0)$ and determine $\mathcal{D}(S_0^*)$. Using the players' payoff functions $\{u_i^1\}_{i \in N}$, derive game Γ_1^N and the associated system $(S, \succ_1)$.

Step t: Consider game Γ_t^N. Compute S_t^*. If $S_t^* = S_{t-1}^*$, then the procedure concludes. If $S_{t-1}^* \subset S_t^*$, define $(S, \succ_t)$ and compute $\mathcal{D}(S_t^*)$. Using the players' payoff functions $\left\{u_i^{t+1}\right\}_{i\in N}$, derive game Γ_{t+1}^N and the associated system $(S, \succ_{t+1})$.

Given that S is finite, there exists a $k \in \mathbb{N}$ such that $S_t^* \neq S_{t+1}^*$ for all $t = 0, \ldots, k-2$, and $S_k^* = S_{k-1}^*$.

Proposition 1. *Let S_k^* be the set of NE strategy profiles of game Γ_k^N. Then S_k^* is the supercore for $(S, \succ)$.*

Proof. We will prove that the following two conditions hold:

(i) S_k^* is a subsolution for $(S, \succ)$. That is, $S_k^* \subseteq \mathcal{U}(S_k^*)$ and $S_k^* = \mathcal{U}^2(S_k^*)$;

(ii) Any other subsolution $\overline{S}$ for $(S, \succ)$ contains S_k^*.

(i) Given the way the sequence of games $\langle \Gamma_0^N, \ldots, \Gamma_k^N \rangle$ is constructed, the payoff of every player i in game Γ_k^N may be written as

$$u_i^k(s) = \begin{cases} \upsilon_i(\Gamma^N) & \text{if } s \in \mathcal{D}(S_k^*) \text{ in } (S, \succ), \\ u_i(s) & \text{otherwise.} \end{cases} \tag{1}$$

Clearly, the NE strategy profiles of Γ_k^N in the system $(S, \succ)$ do not dominate each other and can only be dominated by the strategy profiles in $\mathcal{D}(S_k^*)$. Hence, $S_k^* \subseteq \mathcal{U}(S_k^*)$ and $S_k^* \subseteq \mathcal{U}(\mathcal{U}(S_k^*))$. To show that $\mathcal{U}(\mathcal{U}(S_k^*)) \subseteq S_k^*$, assume that there is a strategy profile $s \in \mathcal{U}(\mathcal{U}(S_k^*))$ such that $s \notin S_k^*$. Then $s' \succ_k s$ in $(S, \succ_k)$, and from (1) it follows that $s' \notin \mathcal{D}(S_k^*)$ in $(S, \succ)$. Since $s, s' \notin \mathcal{D}(S_k^*)$, the players' payoffs to profiles s and s' are the same in games Γ_k^N and Γ^N. Therefore, it follows that $s' \succ s$ and $s \in \mathcal{D}(\mathcal{U}(S_k^*))$, which contradicts $s \in \mathcal{U}(\mathcal{U}(S_k^*))$. Consequently, $S_k^* = \mathcal{U}(\mathcal{U}(S_k^*))$.

(ii) We argue by contradiction. Suppose there is a subsolution $\overline{S}$ for $(S, \succ)$ such that $S_k^* \not\subset \overline{S}$. Consider $S_0^* \subseteq \cdots \subseteq S_k^*$ and define $l = \min\{t : S_t^* \not\subset \overline{S}, t = 0, \ldots, k\}$. Note that the core S_0^* is included in any subsolution. Therefore, $l \neq 0$. Let $s \in S_l^*$ be such that $s \notin \overline{S}$. Then, either $s \in \mathcal{D}(\overline{S})$ or $s \in \mathcal{P}(\overline{S})$ in $(S, \succ)$. Given that s is a Nash equilibrium in Γ_l^N, it can only be dominated by some strategy profiles in $\mathcal{D}(S_{l-1}^*)$ and, by the definition of l, we have that $\overline{S}$ is a subsolution for which $S_{l-1}^* \subseteq \overline{S}$ so that $s \notin \mathcal{D}(\overline{S})$. Hence, $s \in \mathcal{P}(\overline{S})$. Given that $\mathcal{P}(\overline{S}) \subseteq \mathcal{D}(\mathcal{P}(\overline{S}))$, there exists $s' \in \mathcal{P}(\overline{S})$ such that $s' \succ s$. Since $s' \in \mathcal{D}(S_{l-1}^*)$ and $S_{l-1}^* \subseteq \overline{S}$, then $s' \notin \mathcal{P}(\overline{S})$. Thus, we have reached a contradiction. ☐

The following observation is helpful to understand why we derive the iterated games defined in the process replacing some entries by the lowest payoffs in the original game.[2]

Observation. *A is the supercore of (X, R) if and only if A is the supercore of (Y, R) where $Y = X - D(Core(X, R))$ in (X, R).*

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[2] We are indebted to an anonymous referee for this observation.

Proof. First, notice that for any subset B of Y such that $Core(X, R) \subseteq B$ the set $\mathcal{U}(B)$ is the same in both abstract systems since $D(Core(X, R)) \subseteq D(B)$ in (X, R).

Now we show that $\mathcal{S}$ is a subsolution of (X, R) if and only if $\mathcal{S}$ is a subsolution of (Y, R). If $\mathcal{S}$ is a subsolution of (Y, R) then we have that $\mathcal{S} \subseteq \mathcal{U}(\mathcal{S})$ and $\mathcal{S} = \mathcal{U}^2(\mathcal{S})$ in (Y, R). Since $Core(X, R) \subseteq \mathcal{S}$ and $Core(X, R) \subseteq \mathcal{U}(\mathcal{S})$ then the sets $\mathcal{U}(\mathcal{S})$ and $\mathcal{U}^2(\mathcal{S})$ in (X, R) coincide with $\mathcal{U}(\mathcal{S})$ and $\mathcal{U}^2(\mathcal{S})$ in (Y, R). Hence, $\mathcal{S}$ is a subsolution of (X, R). To show the converse implication, notice that if $\mathcal{S}$ is a subsolution of (X, R) then $\mathcal{S} \subseteq Y$. Therefore, proceeding as above, we can conclude that $\mathcal{S}$ is a subsolution of (Y, R). Finally, as the supercore of an abstract system is the intersection of all subsolutions, the result follows. □

Clearly, this general observation is valid in particular for the case where X is any subset of (finite) set of strategy profiles in a normal-form game, and the dominance relation R is the one of the contingent threat situation (restricted to X).

3.2. *The supercore for the n-person prisoner's dilemma*

We follow Nishihara's [4] formulation of the n-person prisoner's dilemma. Let N be the set of players in which every player has two actions: C (cooperation) and D (defection). Let a be player i's action and let r be the number of other players who select action C. The payoff of player i is given by

$$f_i(a\,|r), a = C, D, \text{ and } r = 0, \ldots, n-1,$$

which satisfies the following conditions: (i) For all $i \in N$: $f_i(C\,|r) < f_i(D\,|r)$ for all $r = 0, \ldots, n-1$; (ii) For all $i \in N$: $f_i(C\,|n-1) > f_i(D\,|0)$, and (iii) Functions $f_i(C\,|r)$ and $f_i(D\,|r)$ are increasing in r.

The application of the procedure described earlier allows us to determine the supercore for the n-person prisoner's dilemma. As an illustration, consider the following 3-person prisoner's dilemma[3]:

	C	D
C	3,3,3	1,5,1
D	5,1,1	4,4,0

C

	C	D
C	1,1,5	0,4,4
D	4,0,4	2,2,2

D

The set of NE profiles is $S_0^* = \{(D, D, D)\}$. The dominion of S_0^* is $\mathcal{D}(S_0^*) = \{(C, D, D), (D, D, C), (D, C, D)\}$. Since $S_1^* = S_0^* \cup \{(D, C, C), (C, C, D), (C, D, C)\}$ and (C, C, C) belongs to the dominion of S_1^*, we may conclude that the supercore is $\{(D, D, D), (D, C, C), (C, C, D), (C, D, C)\}$ and that it is a *vN&M* stable set.

More generally, the following result can be established:

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[3] See Arce [1] for the vN&M stable set of a 3-person prisoner's dilemma.

Proposition 2. *The supercore for the n-person prisoner's dilemma is the unique vN&M stable set of its associated system. It is formed by the unique NE strategy profile* $(D, \ldots, D)$ *and by those strategy profiles such that the number of players who choose C is even.*

Proof. It is straightforward from the application of the procedure to compute the supercore. $\square$

4. The supercore for $(\Delta(S), \succ)$

As the previous example shows, if we restrict attention to pure strategies then the supercore can contain profiles that are not equilibria of the original game. However, if we move to the mixed extension of normal-form games then, provided that the set of NE is finite, the supercore coincides with the set of NE.

The role of the mixed extension can be illustrated using the standard 2-person prisoner's dilemma game. In both the pure strategy game and its mixed extension the unique NE is (D, D). However, in the finite case this NE is not the supercore, while in the mixed extension this NE is the supercore. To see why this is the case, note that in the finite game $\mathcal{U}(S^*) = \{(D, D), (C, C)\}$ and therefore $(C, C) \in \mathcal{U}^2(S^*)$. On the other hand, in the mixed extension game $\mathcal{U}(\{(D, D)\}) = \{(1, 0, 1, 0)\} \cup \{(p, 1-p, q, 1-q) : 0 \leqslant p < 1, 0 \leqslant q < 1\}$, where p and q are the probabilities of choosing D for players 1 and 2, respectively. The set $\mathcal{P}(\{(D, D)\}) = \{(p, 1-p, q, 1-q) : 0 \leqslant p < 1, 0 \leqslant q < 1\}$ and $\mathcal{P}(\{(D, D)\}) \subseteq \mathcal{D}(\mathcal{P}(\{(D, D)\}))$. Therefore, $\mathcal{U}^2(\{(D, D)\}) = \{(D, D)\}$.

We turn now to show the general result formally. To do so, we first associate an abstract system (X, R) to the mixed extension of the game Γ^N. Let the set of elements equal the set of strategy profiles, $X = \Delta(S)$. An *individual contingent threat situation of the mixed extension of game* Γ^N is a 4-tuple $\langle N, \Delta(S), (U_i)_{i \in N}, (\gamma_i)_{i \in N} \rangle$ where γ_i is the correspondence from $\Delta(S)$ into itself defined by

$$\gamma_i(\sigma) = \{(\sigma'_i, \sigma_{-i}) \text{ for some } \sigma'_i \in \Delta(S_i)\}.$$

Definition 2. The system associated to an individual contingent threat situation of the mixed extension of a game Γ^N is the pair $(\Delta(S), \succ)$, where $\succ$ is the binary relation defined on $\Delta(S)$ such that

$$\sigma' \succ \sigma \text{ if there exists } i \in N \text{ such that } \sigma' \in \gamma_i(\sigma) \text{ and } U_i(\sigma') > U_i(\sigma).$$

Let Σ^* be the set of NE strategy profiles of the mixed extension of the game Γ^N and let $\eta \in \Sigma^*$. The dominion of η is $\mathcal{D}(\eta) = \{\sigma \in \Delta(S) : \eta \in \gamma_i(\sigma) \text{ and } U_i(\eta) > U_i(\sigma) \text{ for some } i \in N\}$. Then, the dominion of Σ^* will be $\mathcal{D}(\Sigma^*) = \bigcup_{\eta \in \Sigma^*} \mathcal{D}(\eta)$.

Proposition 3. *If* Σ^* *is finite then* Σ^* *is the supercore for* $(\Delta(S), \succ)$.

Proof. We first prove that Σ^* is a subsolution for $(\Delta(S), \succ)$, that is that $\Sigma^* \subseteq \mathcal{U}(\Sigma^*)$ and $\Sigma^* = \mathcal{U}^2(\Sigma^*)$.

Given that $\Sigma^* \subseteq \mathcal{U}(\Sigma^*)$, if $\Sigma^* = \mathcal{U}(\Sigma^*)$ then $\Sigma^* = \mathcal{U}^2(\Sigma^*)$, and Σ^* is a subsolution. If $\Sigma^* \neq \mathcal{U}(\Sigma^*)$ we have to show that $\mathcal{P}(\Sigma^*) \subseteq \mathcal{D}(\mathcal{P}(\Sigma^*))$ which, given that $\Sigma^* \subseteq \mathcal{U}^2(\Sigma^*)$, is equivalent to showing $\Sigma^* = \mathcal{U}^2(\Sigma^*)$.

Let $\sigma \in \mathcal{P}(\Sigma^*)$. We will see that $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$. Since $\sigma \notin \Sigma^*$, then σ_i will not be the best response to σ_{-i} for some player i. Therefore, there exists a profile $\sigma' \in \gamma_i(\sigma)$ such that $U_i(\sigma') > U_i(\sigma)$. Now, let $\sigma^\lambda = \lambda\sigma + (1-\lambda)\sigma'$ for all $\lambda \in [0, 1)$. By the linearity of U_i we have $U_i(\sigma^\lambda) > U_i(\sigma)$, and since $\sigma^\lambda \in \gamma_i(\sigma)$, it follows that $\sigma^\lambda \succ \sigma$ for all $\lambda \in [0, 1)$. Thus, σ^λ dominates σ, and $\sigma^\lambda \notin \Sigma^*$ given that $\sigma \in \mathcal{P}(\Sigma^*)$.

It remains to show that $\sigma^\lambda \in \mathcal{P}(\Sigma^*)$ for some λ. Note that if $\eta \succ \sigma^\lambda$, where $\eta \in \Sigma^*$, then $\sigma_i^\lambda = \eta_i$. Otherwise, since $\sigma_j^\lambda = \eta_j$ for all $j \neq i$ implies $\eta \succ \sigma$, we would have that $\sigma \notin \mathcal{P}(\Sigma^*)$. Therefore, if $\eta^1 \succ \sigma^{\lambda_1}$ and $\eta^2 \succ \sigma^{\lambda_2}$, where $\eta^1, \eta^2 \in \Sigma^*$ and $\lambda_1 \neq \lambda_2$, then $\eta_i^1 \neq \eta_i^2$ and hence $\eta^1 \neq \eta^2$. It then follows that:

$$\left|\{\sigma^\lambda : \exists \eta \in \Sigma^* \text{ such that } \eta \succ \sigma^\lambda\}\right| \leqslant \left|\Sigma^*\right| .$$

Given that Σ^* is a finite set, $\sigma^\lambda \in \mathcal{P}(\Sigma^*)$ for at least one (in fact, for a continuum of) λ. Therefore, $\sigma \in \mathcal{D}(\mathcal{P}(\Sigma^*))$.

Lastly, since the supercore is the intersection of all subsolutions and any subsolution contains Σ^*, Proposition 3 follows. $\square$

This result no longer holds for the mixed extension of games with infinite number of NE profiles. The following example illustrates that non-Nash equilibrium strategy profiles may belong to the supercore for $(\Delta(S), \succ)$:

	b_1	b_2
a_1	1,0	1,1
a_2	$-1, 1$	1,0

Let p be the probability that player 1 chooses a_1 and let q be the probability that player 2 chooses b_1. It is easy to see that $\Sigma^* = \{(p, 1-p, 0, 1) : \frac{1}{2} \leqslant p \leqslant 1\}$, $\mathcal{D}(\Sigma^*) = \{(p, 1-p, q, 1-q) : \frac{1}{2} < p \leqslant 1, 0 < q \leqslant 1\}$ and $\mathcal{P}(\Sigma^*) = \{(p, 1-p, q, 1-q) : 0 \leqslant p \leqslant \frac{1}{2}, 0 < q \leqslant 1\} \cup \{(p, 1-p, 0, 1) : 0 \leqslant p < \frac{1}{2}\}$. It is then straightforward to show that the supercore for $(\Delta(S), \succ)$ is $\Sigma^* \cup \{(\frac{1}{2}, \frac{1}{2}, q, 1-q) : 0 < q \leqslant 1\} \cup \{(p, 1-p, 0, 1) : 0 \leqslant p < \frac{1}{2}\}$.

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