Prescribing the gaussian curvature on a surface with conical points.

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joint work with Andrea Malchiodi (SISSA).

Outline

The Kazdan Warner problem

A old problem in geometric analysis consists in prescribing the curvature of a surface. In other words, given Σ a compact surface and $\tilde{K} : \Sigma \to \mathbb{R}$, does there exist $\tilde{g}$ so that $\tilde{K}(x)$ is the curvature of $(\Sigma, \tilde{g})$?

If we fix a metric g, we can look for the new metric $\tilde{g}$ as a conformal deformation of g, that is, $\tilde{g} = e^{2u} g$. In such case, we get:

$$-\Delta_g u + K(x) = \tilde{K}(x) e^{2u}. \tag{1}$$

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By integrating, we obtain:

$$\int_\Sigma K(x)\, dV_g = \int_\Sigma \tilde{K}(x) e^{2u}\, dV_g = \rho = 2\pi\chi(\Sigma).$$

Here we restrict ourselves to the positive case, $\rho > 0$. Hence, a necessary condition is that $\tilde{K}$ is positive elsewhere.

The case $\rho \leq 0$ has been completely described:

J. Kazdan and F. Warner, Ann. of Math. 1974.

We can rewrite the equation:

$$-\Delta_g u + K(x) = \rho \frac{\tilde{K}(x)e^{2u}}{\int_\Sigma \tilde{K}(x)e^{2u}\,dV_g}.$$

This is the Euler-Lagrange equation of the energy functional $I_\rho : H^1(\Sigma) \to \mathbb{R}$,

$$I_\rho(u) = \int \left(|\nabla u|^2 + 2K(x)u\right) dV_g - \rho \log\left(\int_\Sigma \tilde{K}(x)e^{2u}\,dV_g\right).$$

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The Moser-Trudinger inequality implies that I_ρ is bounded from below for $\rho \in (0, 4\pi]$. Moreover, I_ρ is coercive for $\rho \in (0, 4\pi)$, and then a solution can be found as a minimizer. For instance, that solves the problem for $\Sigma = \mathbb{RP}^2$, since $\chi(\mathbb{RP}^2) = 1$.

J. Moser, Dynamical Systems (M. Peixoto ed.), Academic Press, New York, 1973.

The case of the sphere is intrinsically more complicated, and there are counterexamples, already found by Kazdan and Warner. Here, $\rho = 4\pi$, which is critical for the Moser-Trudinger inequality.

S.Y.A. Chang and P. C. Yang, Acta Math. 1987.

C. C. Chen and C. S. Lin, CPAM 2003.

Moreover, if $\rho > 4\pi$, $I_\rho(\varphi_\lambda) \to -\infty$ as $\lambda \to +\infty$, where:

$$\varphi_\lambda(x) = \log\left(\frac{\lambda}{1+\lambda^2 dist(x,x_0)^2}\right), \quad x_0 \in \Sigma, \ \tilde{K}(x_0) > 0.$$

The previous choice is not casual: indeed, the functions defined in $\mathbb{R}^2$:

$$U(x) = \log\left(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\right), \quad \lambda > 0,\ x_0 \in \mathbb{R}^2,$$

are the unique entire solutions of the problem:

$$-\Delta U(x) = 4e^{2U(x)}, \quad x \in \mathbb{R}^2$$

with $e^{2u} \in L^1(\mathbb{R}^2)$.

W. Chen and C. Li, Duke Math. J. 1991.

Compact surfaces with conical singularities

We say that p is a conical point of (Σ, g) of degree $\alpha > -1$ if in a neighborhood of p,

$$g = h(x) g_0,$$

where g_0 is a smooth metric on Σ, and $h(x) \sim |x - p|^{2\alpha}$.

For instance, the vertex of a cone of total angle θ is a conical point of order α, with $\theta = 2\pi(1 + \alpha)$. The case $\alpha > 0$ gives rise to non-embedded conical points.

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Now we are interested in prescribing both the curvature on Σ and the conical points p_i (with degree α_i). We can build a metric g with those conical points by using partitions of unity. The equation becomes, as previously:

$$-\Delta_g u + K(x) = \tilde{K}(x)e^{2u}.$$

As a first difference with respect to the smooth case, the Gauss-Bonnet formula implies now that $\rho = 2\pi(\chi(\Sigma) + \sum_i \alpha_i)$.

Recall the expression of I_ρ:

$$I_\rho(u) = \int_\Sigma \left(|\nabla u|^2 + K(x)u\right) dV_g - \rho \log\left(\int_\Sigma \tilde{K}(x) e^{2u}\, dV_g\right).$$

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If we write this expression in terms of g_0, we obtain:

$$I_\rho(u) = \int_\Sigma \Big(|\nabla u|^2 + K(x)h(x)u \Big)\, dx - \rho \log \Big(\int_\Sigma h(x)\tilde{K}(x) e^{2u}\, dx \Big),$$

where $h(x)$ is positive outside p_i and $h(x) \sim |x - p_i|^{2\alpha_i}$.

A second difference: now I_ρ is bounded from below if $\rho \leq 4\pi min\{1, 1 + \alpha_i : \ i = 1 \dots k\}$.

Theorem

Assume that $0 < \rho < 4\pi min\{1, 1 + \alpha_i : \ i = 1 \dots k\}$, *and take* $\tilde{K}(x)$ *positive elsewhere. Then there exists a minimizer for* I_ρ.

M. Troyanov, TAMS 1991.

There are also some non-existence results. It is known that there does not exist a metric on the sphere with one conical point and constant positive curvature.

Moreover, a metric on the sphere with constant positive curvature and two conical points exists if and only if $\alpha_1 = \alpha_2$.

M. Troyanov, Lect. Notes Math., 1410, Springer, NY, 1989.

Motivation from physics

This equation arises also from physical models such as the abelian Chern-Simons-Higgs theory and the Electroweak theory. In this framework, the points p_i represent vortexes.

- J. Hong, Y. Kim, P. Y. Pac, Phys. Rev. Lett. 1990.
- R. Jackiw and E. J. Weinberg, Phys. Rev. Lett. 1990.
- C. H. Lai (ed.), Selected Papers on Gauge Theory of Weak and Electromagnetic Interactions, World Scientific, Singapore, 1981.
- G. Dunne, *Self-dual Chern-Simons Theories,* Lecture Notes in Physics, Springer-Verlag 1995.
- G. Tarantello, *Self-Dual Gauge Field Vortices: An Analytical Approach,* PNLDE 72, Birkhäuser Boston, 2007.
- Y. Yang, *Solitons in Field Theory and Nonlinear Analysis,* Springer Monographs in Mathematics, 2001.

The result

Theorem

Assume that $\tilde{K}(x)$ is strictly positive. We rearrange the conical points so that $0 = \alpha_0 < \alpha_1 \leq \alpha_2 \leq \ldots \alpha_k \leq 1$. Assume $\rho \in (4\pi, 8\pi)$, $\rho \neq 4\pi(1 + \alpha_i)$ for any i.
Assume also that either Σ is not homeomorphic to a sphere or $\rho \notin (4\pi(1 + \alpha_{k-1}), 4\pi(1 + \alpha_k))$.
Then problem (1) has a solution.

A. Malchiodi and D. Ruiz, Geom. and Funct. Anal. 2011.

The second condition may look technical. However, there holds:

Theorem

Assume that $k = 1$, $\Sigma = S^2$. If $\tilde{K}(x) = \tilde{K} > 0$, then (1) does not admit any solution for any $\rho \in (4\pi, 4\pi(1 + \alpha))$.

D. Bartolucci, C.S. Lin and G. Tarantello, DCDS 2010.

The variational argument

The argument of the proof is reminiscent of Morse theory. For any $L \in \mathbb{R}$, we define:

$$I_\rho^L = \{u \in H^1(\Sigma) : \ I_\rho(u) < L\}.$$

By Morse theory, if I_ρ^a is not homotopically equivalent to I_ρ^b, then there exists a critical point $u \in H^1(\Sigma)$ with $I_\rho(u) \in [a, b]$.

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It is easy to show that I_ρ^L is contractible for $L >> 1$. Therefore, we will be done if we prove that low sub-levels of I_ρ are not contractible.

W. Ding, J. Jost, J. Li and G. Wang, AIHP 1999.

Z. Djadli and A. Malchiodi, Ann. of Math. 2008.

Case 1: $\rho \in (4\pi(1+\alpha_k), 8\pi)$.

Proposition

Assume $h : \Sigma \to \mathbb{R}$ with $0 \le h(x) \le C_0$. Let Ω_1, Ω_2 be two subsets of Σ with $dist(\Omega_1, \Omega_2) \ge \delta_0 > 0$, and fix $\gamma_0 > 0$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(C_0, \varepsilon, \delta_0, \gamma_0)$ such that

$$\log \int_\Sigma h(x)e^{2u} \le C + \frac{1}{8\pi - \varepsilon} \int_\Sigma |\nabla u|^2 + 2 \int_\Sigma u.$$

for all functions $u \in H^1(\Sigma)$ satisfying

$$\frac{\int_{\Omega_j} h(x)e^{2u}}{\int_\Sigma h(x)e^{2u}} \ge \gamma_0, \qquad j = 1, 2. \tag{2}$$

W. Chen and C. Li, J. Geom. Anal. 1991.

As a consequence we have that for any $u_n \in H^1$ such that $I_\rho(u_n) \to -\infty$,

$$\frac{he^{u_n}}{\int he^{u_n}} \rightharpoonup \delta_x$$

for some $x \in \Sigma$. This provides us with a continuous map:

$$\Psi : I_\rho^{-M} \to \Sigma,$$

for M large enough.

Moreover, for $\lambda > 0$ large we define $\Phi_\lambda : \Sigma \to I_\rho^{-M}$ as:

$$\Phi_\lambda[x_0](x) = \log\left[\frac{\lambda}{1+\lambda^2|x-x_0|^{2(1+\alpha_k)}}\right]$$

for x in a neighborhood of x_0.
Again, those functions correspond to the unique entire solutions of the problem

$$-\Delta u = 4(1+\alpha_k)^2|x-x_0|^{2\alpha_k}e^{2u}.$$

J. Prajapat and G. Tarantello, PRSE 2001.

In order to compute that indeed $I_\rho(\Phi_\lambda[x_0]) < -M$ for any $x_0 \in \Sigma$ we use strongly that $\rho > 4\pi(1+\alpha_k)$.

Topological properties of sublevels

We can compute the composition:

$$\Sigma \xrightarrow{\Phi_\lambda} I_\rho^{-M} \xrightarrow{\Psi} \Sigma.$$

It is possible to show that $\Psi \circ \Phi_\lambda \to Id$ as $\lambda \to +\infty$. So, the composition is $\Psi \circ \Phi_\lambda$ is homotopycally equivalent to the identity. Since Σ is not contractible, neither is I_ρ^{-M}.

Case 2: $\rho \in (4\pi(1+\alpha_i), 4\pi(1+\alpha_{i+1}))$

In this case it seemed impossible to us to define a map $\Phi_\lambda : \Sigma \to I_\rho^{-M}$ as before. The problem appears when defining $\Phi_\lambda[p_j]$, $j > i$.
A result that may explain why is the following:

Theorem

Let B be the unit ball of $\mathbb{R}^2$. Then, for any $u \in H^1(B)$ radial, there holds:

$$\log \int_B |x|^{2\alpha} e^{2u} \leq \frac{1}{4(1+\alpha)\pi} \int_B |\nabla u|^2 + 2 \fint_B u + C.$$

J. Dolbeault, M. J. Esteban and G. Tarantello, Ann. Sc. Norm. Sup. Pisa Cl. Sci 2008.

That makes one think that now sublevels will not contain functions concentrated around p_j, $j > i$.

Theorem

Fix $\rho \in (4\pi, 8\pi)$. There exists $M > 0$ and a map $\beta : I_\rho^{-M} \to \Sigma$ such that for any $\varepsilon > 0$

$$\log \int_\Sigma |x-p|^{2\alpha} e^{2u} \leq \frac{1}{4\pi(1+\alpha)-\varepsilon} \int_\Sigma |\nabla u|^2 + 2\int_\Sigma u + C_\varepsilon,$$

for any $u \in I_\rho^{-M}$ with $\beta(u) = p$.

Actually the map β applies to $f = \frac{he^u}{\int he^u}$. Let us define:

$$\mathcal{A} = \{f \in L^1(\Sigma),\ f(x) > 0 \ a.e.,\ \int_\Sigma f = 1\},$$

$$\sigma : \Sigma \times \mathcal{A} \to (0, +\infty),$$

where $\sigma = \sigma(x, f)$ is chosen such that (for some $C > 0$ large):

$$\int_{B_x(\sigma)} f = \int_{B_x(C\sigma)^c} f.$$

A barycenter map

Let us define $T : \Sigma \times \mathcal{A} \to (0, +\infty)$,

$$T(x, f) = \int_{B_x(\sigma(x,f))} f.$$

Both σ and T are continuous.

There exists $\tau = \tau(C) > 0$ such that

$$\max_{x \in \Sigma} T(x, f) > 2\tau \ \forall \ f \in \mathcal{A}.$$

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By Nash embedding Theorem, we can assume that $\Sigma \subset \mathbb{R}^N$ isometrically. Let us define:

$$\eta : I_\rho^{-M} \to \mathbb{R}^N, \ \ \eta(u) = \frac{\int_\Sigma [T(x, f) - \tau]^+ x}{\int_\Sigma [T(x, f) - \tau]^+}.$$

Moreover, the integrand is nonzero in a very small region for M large. So, we can simply define $\beta(u)$ as the projection of $\eta(u)$ onto Σ.

That map β defined in that way is somehow a barycenter of u. In particular, if $u \in H^1(B)$ is radial, $\beta(u) = 0$.

Now we need to show that if $\beta(u) = p$, then

$$\log \int_\Sigma |x - p|^{2\alpha} e^{2u} \leq \frac{1}{4\pi(1+\alpha) - \varepsilon} \int_\Sigma |\nabla u|^2 + 2 \int_\Sigma u + C_\varepsilon.$$

The proof of that is quite technical and will be skipped in this talk.

Therefore, we can use the barycenter map:

$$\beta : I_\rho^{-M} \to \Sigma \setminus \{p_{i+1}, \dots p_k\}$$

for M large enough. By using a deformation retract, we can define:

$$\Psi : I_\rho^{-M} \to \Sigma \setminus \left(\cup_{j=i+1}^{k} B_{p_j}(\delta)\right),$$

for small δ fixed. Moreover, we can also define:

$$\Phi_\lambda : \Sigma \setminus \left(\cup_{j=i+1}^{k} B_{p_j}(\delta)\right) \to I_\rho^{-M},$$

$$\Phi_\lambda[x_0](x) = \log\left[\frac{\lambda}{1 + \lambda^2 |x - x_0|^{2(1+\alpha_{i+1})}}\right]$$

So, we conclude again whenever $\Sigma \setminus \left(\cup_{j=i+1}^{k} B_{p_j}(\delta)\right)$ is not contractible.

Thank you for your attention!