DIFFRACTION OF ELECTROMAGNETIC BLOCH WAVES

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Outline

1. Some ideas from geometrical optics.

2. Some Maxwell background.

3. The method of Floquet-Bloch in the exactly periodic case.

4. Not quite periodic media.

5. A Theorem.

6. Ingredients of the proof.

Short Wavelength Wave Packets

Talk is about *wave packets of short wavelength*,

$$u^h \ = \ e^{i(\tau(\xi)t + x.\xi)/h}\, a(h,t,x)\,, \qquad\qquad \tau, \xi \text{ real}$$
$$a(h,t,x) \ \sim \ a_0(t,x) + h\, a_1(t,x) + \cdots$$

Ex. For d'Alembert's wave equation or Maxwell in vacuum $\tau = \pm|\xi|$.

Question. Why short wavelength?

Answer 1. Wavelength of light is comparable to size of bacteria.

Answer 2. Oscillation determines a velocity, For $\tau = \pm|\xi|$,

$$\text{group vel.} = -\nabla_\xi \tau = \mp \xi/|\xi| := \mathcal{V}, \qquad \big(\partial_t + \mathcal{V}.\partial_x\big) a_0 \ = \ 0\,.$$

Answer 3. Rectilinear propagation. Solutions above propagate on rays for $t = O(1)$. Lines in example above. Creates shadows. Wave/particle duality.

Maxwell I. Div, Curl, and Energy

$\epsilon(x), \mu(x) \in C^\infty(\mathbb{R}^3)$ positive definite symmetric matrices with bounded derivatives. Dynamic Maxwell:

$$\partial_t \, \epsilon(x) E \;=\; \operatorname{curl} B, \qquad \partial_t \, \mu(x) B \;=\; -\operatorname{curl} E$$

$\operatorname{div}(\operatorname{curl}) = 0$ so

$$\partial_t \operatorname{div} \epsilon(x) E \;=\; \partial_t \operatorname{div} \mu(x) B \;=\; 0$$

$$\operatorname{div} \epsilon(x) E \;=\; \operatorname{div} \mu(x) B \;=\; 0 \tag{1}$$

$$\partial_t \int_{\mathbb{R}^3} \big\langle \epsilon(x) E \,,\, E \big\rangle \;+\; \big\langle \mu(x) B \,,\, B \big\rangle \; dx \;=\; 0 \tag{2}$$

$\exists$ infinite dimensional space of stationary solutions

$$\operatorname{curl} E \;=\; \operatorname{curl} B \;=\; 0, \quad E, B \text{ are gradients}$$

Denote by $L^2(\mathbb{R}^3_{\epsilon,\mu})$ the space normed by (2)

The orthogonal complement of the steady solutions is invariant and consists exactly of the fields satisfying (1).

Maxwell II. Stability.

We study ϵ, μ that oscillate on small scale $0 < h << 1$. If you estimate $\partial_x E, \partial_x B$ by differentiating the equations wrt x, you find $e^{t/h}$ growth from the commutator $[\partial_x, \epsilon]$.

L^2 does not grow.

E_t, B_t are also solutions (when coefficients independent of t). So $\|E_t, B_t\|_{L^2}$ does not grow

So $\|\mathrm{curl}\, E(t)\,,\, \mathrm{curl}\, B(t)\|_{L^2(\mathbb{R}^3)}$ does not grow.

Also $\mathrm{div}\, \epsilon E = \mathrm{div}\, \mu B = 0$. Use coercivity in semiclassical Sobolev space to estimate $h\partial_x$ derivatives without growth.

Thm. $u := (E, B)$. $\forall s \in \mathbb{R},\ \exists c = c(s)$

$$\|u\|_{s,h} \leq c\big(\|h\, \mathrm{curl}\, u\,,\ h\, \mathrm{div}\{\epsilon(x)E, \mu(x)B\}\|_{s-1,h} + \|u\|_{s-1,h}\big)$$

Pf. A scaling and induction reduces to $h = 1$, $s = 0$. For $s = 0$, rhs is a quadratic form in derivatives.

In contrast with scalar wave equation, it is not postive definite.

It is postive definite if one freezes x and takes Fourier Transform.

Theorem follows from Gårding inequality. ∎

Floquet-Bloch (Fourier) Decomposition

$$\text{“every”} \quad u(x) \;=\; \int e^{ix\xi}\; a(\xi)\; d\xi \qquad\qquad \text{(Fourier)}$$

Identify ξ modulo $\mathbb{Z}^N$,

$$\xi = \theta + n, \qquad n \in \mathbb{Z}^N, \quad \theta \in [0,1[^N$$

$$\begin{aligned} u(x) \;&=\; \int_{[0,1[^N} \sum_n a(\theta+n)\; e^{ix(\theta+n)}\; d\theta \\ &=\; \int_{[0,1[^N} e^{i\theta.x} g(x,\theta)\; d\theta, \quad g \text{ period } 2\pi \text{ in } x, \end{aligned}$$

Definition. *$f(x)$ is θ-periodic iff $e^{-i\theta.x} f$ is periodic.*

Every u is a sum over $\theta \in [0,1[^N$ of θ-periodic functions.

$$g \text{ is } \theta\text{−periodic} \quad \Longrightarrow \quad \partial_x g \;\; \theta\text{−periodic.}$$

g is $\theta-$periodic and $\epsilon(x)$ is periodic $\Rightarrow$ $\epsilon(x)g$ $\theta-$periodic

θ-periodic data yields θ-periodic solution.

Every solution of periodic Maxwell is a superposition of θ-periodic solutions.

Bloch Plane Waves

For each $\theta \neq 0$, there is an infinite dimensional space of stationary θ-periodic solutions.
The $\perp$ compl. with respect to the ϵ, μ scalar prod. is invariant and consists of sols. satisfying the diverg. constraints.

They are linear combinations of eigensolutions

$$e^{i\omega(\theta)t}\, u(x), \qquad \psi \ \text{ is } \ \theta\text{−periodic}$$

The *nonzero* eigenvalues $i\omega(\theta)$ can be ordered

$$\cdots < \omega_{-2}(\theta) < \omega_{-1}(\theta) < 0 < \omega_1(\theta) < \omega_2(\theta) < \cdots$$

tending to infinity in both directions and have finite multiplicity (coercivity). $\psi := e^{-i\theta x} u$ is vector valued periodic. Eigenvalue iff $\psi \in \mathbb{K} := \ker \mathbb{L}(\omega, \theta, y, \partial_y)$

$$\mathbb{L} := i\omega \begin{pmatrix} \epsilon_0(y) & 0 \\ 0 & \mu_0(y) \end{pmatrix} - \begin{pmatrix} 0 & (i\theta + \partial_y)\wedge \\ -(i\theta + \partial_y)\wedge & 0 \end{pmatrix}$$

Spectrum is a union $\cup_j i\omega_j\big([0,1]^N\big)$. Closed intervals called **bands**. Separated by open **gaps** (that may be absent).

Maxwell solutions are superpositions in θ, n of the **Bloch plane waves**

$$e^{i(\omega_j(\theta)t + \theta.x)}\, \psi_n(x, \theta), \qquad \textbf{Bloch disp rel}$$

Three Regimes in Periodic Media

Consider a periodic medium with period $h \ll 1$. Waves with wavelength ℓ.

For **long waves** $\ell \gg h$, the medium is approximated by a homogenized medium that does not vary on the small scale. The effective coefficients are computed as in the static case.

The dispersion relation and group velocities are those of the wave equation with homogenized elliptic part.

Very short waves $\ell \ll h$, see a medium slowly varying on their length scale, and the approximations of standard geometric optics are appropriate. The group velocities vary on the scale $h \gg \ell$.

For **resonant waves**, $\ell \sim h$. The dispersion relation can be very different from the preceding regimes.

Bloch Wave Packets in Periodic Media

Fix $\underline{\theta}$*, eigenvalue* $\omega_j(\underline{\theta}) \neq 0$ *that has constant multiplicity on a neighborhood of* $\underline{\theta}$*.* (a.e.)
Then choose ψ_n analytic in θ on a neighborhood of $\underline{\theta}$.

Scale to period h and consider Bloch plane waves with wave-length $\sim h$,

$$e^{i(\omega(\theta)t+\theta.x)/h}\ \psi_n(x/h,\theta).$$

Superposition yields exact solutions, ($\alpha \in C_0^\infty$)

$$\int_{[0,2\pi[^N} e^{i(\omega(\underline{\theta}+h\zeta)t+(\underline{\theta}+h\zeta).x)/h}\ \psi_n(x/h,\underline{\theta}+h\zeta)\ \alpha\big(\zeta\big)\, d\zeta.$$

Taylor expansion of $\omega(\underline{\theta}+h\zeta)$ yields rectilinear propagation,

$$e^{i(\omega(\underline{\theta})t+\underline{\theta}.x)/h}\ \psi_n(x/h,\underline{\theta})\ \int_{[0,2\pi[^N} e^{\zeta.(x-\mathcal{V}t)}\ \alpha(\zeta)\ d\zeta\ +\ O(h),$$

$$\mathcal{V} := -\nabla_\theta \omega(\underline{\theta})\,. \qquad \textbf{n.b.}$$

The leading term is

$$e^{i(\omega(\underline{\theta})t+\underline{\theta}.x)/h}\ \psi_n(x/h,\underline{\theta})\ a(x-\mathcal{V}t)\,.$$

A Bloch plane wave times a slowly varying envelope that is transported at the group velocity $\mathcal{V}$. This is a **Bloch wave packet**. *N.B. resonant scaling.*

Three Important Consequeces

1. Designer materials. The dispersion relation $\tau = \omega_j(\theta)$ does not resemble the dispersion relation of the original problem. *Not* polynomial in τ, θ.

Can engineer periodic materials with properties **radically** different than those of the constituent materials.

2. Slow light For the original equations, group velocities are bounded below by a strictly positive quantity.

For Bloch wave packets, one can have $\mathcal{V} = 0$. For example if $\omega_j(\underline{\theta})$ is a value at the edge of a band. Then ω_j has a max or min so $\nabla_\theta \omega_j$ vanishes. Light has been slowed to 17m/sec.

3. Forbidden temporal spectrum. If I is in a *gap* in the spectrum, and, $u(t,x)$ solves the $2\pi h$-periodic wave equation, then

$$\int_{-\infty}^{\infty} e^{-i\tau t/h}\ u(t,x)\ dt$$

vanishes for $\tau \in I$.

When a wave packet with temporal spectrum contained in the forbidden region, arrives at a large piece of $2\pi h$-periodic medium it is totally reflected.

Two More Important Consequences

4. Photonic crystal fibers. Using periodic materials, large and therefore large capacity monomode optical fibers have been constructed. This is important for high energy laser projects.

5. Enhanced dispersion. Second order Taylor $\Rightarrow$ for times $t \sim 1/h$ wave packets are approximated as

$$e^{i(\omega(\underline{\theta})t+\underline{\theta}.x)/h} \; \psi_n(x/h, \underline{\theta}) \; a(ht, x - \mathcal{V}t)$$

with $a(\mathcal{T}, x)$ determined from its initial data by

$$i\,\partial_{\mathcal{T}} a \;=\; \nabla_\theta^2 \omega(\underline{\theta})(\partial_x, \partial_x) a\,.$$

The rank of $\nabla_\theta^2 \omega(\underline{\theta})$ is generically equal to 3. Wave packets decay as $\sim t^{-3/2}$.

For Maxwell waves decay at the slower rate t^{-1}.

Some Harder Problems

It is **very** hard to make (nearly) perfectly periodic materials.

It is important to understand slightly nonperiodic materials. In particular the impact of imperfections. To ascertain precision requirements.

For fibers and other applications, it is important to understand propagations over long time intervals. Where diffractive effects are important. The remainder of the talk addresses these issues.

Open problem. It is important to understand diffraction in weakly random perturbations of periodic media.

Maxwell in Perturbed Periodic Media

$\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$. Matrix valued $\epsilon_0(y)$ and $\mu_0(y)$ are $C^\infty(\mathbb{T}^3)$, symmetric positive definite. Perturbations satisfy

$$\forall\alpha,\ \partial^\alpha_{t,x,y}\{\epsilon_1, \mu_1, M\}(t,x,y) \in L^\infty(\mathbb{R}^{1+3}\times\mathbb{T}^3)$$

Diffractive time scale hypothesis. Maxwell with lower order term $M^h u$ and

$$\begin{aligned} M^h(t,x) &= h\, M(t,x,x/h)\,, \\ \epsilon^h(t,x) &= \epsilon_0(x/h) + h^2\epsilon_1(t,x,x/h) \\ \mu^h(t,x) &= \mu_0(x/h) + h^2\mu_1(t,x,x/h) \\ A_0 &= A_0^0 + h^2 A_0^1 \qquad\qquad (\text{periodic} + \text{perturbation}) \end{aligned}$$

Definition. $\Pi :=$ *the projection operator onto*

$$\mathbb{K} := \ker \mathbb{L}(\omega(\underline{\theta}), \underline{\theta}, y, \partial_y)$$

along the image of $\mathbb{L}$*.*

Definition *For each* t,x *define the linear map* $\gamma(t,x) \in \mathrm{Hom}(\mathbb{K})$ *by*

$$\gamma \ := \ \big(\Pi\, A_0^0(y)\, \Pi\big)^{-1}\ \Pi\,(i\omega(\underline{\theta})A_0^1 + M)\,\Pi\,.$$

The Ray Average Hypothesis

Assume that the ray averages of γ exist

$$\widetilde{\gamma}(x) \ := \ \lim_{T\to+\infty} \ \frac{1}{T} \int_0^T \gamma(t, x + \mathcal{V}t) \ dt$$

The solution of the transport equation

$$\big(\partial_t \ + \ \mathcal{V}.\partial_x)g \ = \ \gamma(t,x) - \widetilde{\gamma}(x - \mathcal{V}t), \quad g\big|_{t=0} = 0$$

is then *sublinear* in time.

Ray Average Hypothesis. *There is a* $\beta \in [0, 1[$ *so that for all* α*, the soln.* $g_\alpha(t, x)$ *of*

$$\Big(\partial_t \ + \ \mathcal{V}.\partial_x\Big) g_\alpha \ = \ \partial^\alpha_{t,x}\big(\gamma(t,x) - \widetilde{\gamma}(x - \mathcal{V}t)\big), \quad g_\alpha\big|_{t=0} \ = \ 0,$$

satisfies $t^{-\beta}\, g_\alpha \in L^\infty([0, \infty[\times\mathbb{R}^N)$.

Examples. i. If γ is periodic with any period.
ii. For quasiperiodic γ satisfying a $\mathcal{V}$-dependent small divisor hypothesis. Rarely violated.
iii. For random perturbations with $\beta = 1/2$.
iv. Linear combinations.

Theorem Statement

Assume that γ satisfies the ray average hypothesis with parameter $0 \le \beta < 1$.

For $f \in \mathcal{S}(\mathbb{R}^3;\mathbb{K})$ define $\widetilde{w}_0 \in C^\infty(\mathbb{R}\,;\,\mathcal{S}(\mathbb{R}^3;\mathbb{K}))$, solution of the Schrödinger initial value problem

$$\Big(\partial_{\mathcal{T}} + \frac{1}{2} i\ \partial_\theta^2 \omega(\partial_x,\partial_x) + \widetilde{\gamma}(x)\Big)\widetilde{w}_0 \ = \ 0, \qquad \widetilde{w}_0(0,x) = f(x).$$

Define the $\mathbb{K}$ valued (independent variable y) profiles

$$w_0(\mathcal{T},t,x) \ := \ \widetilde{w}_0(\mathcal{T}, x - \mathcal{V}t)\,.$$

$$P^h u = P^h(E,B) := \Big(\partial_t\, \epsilon E - \operatorname{curl} B\,,\ \partial_t\, \mu B + \operatorname{curl} B\Big)$$

Theorem. *Define a family of approximate solutions*

$$\underline{v}^h(t,x) \ := \ e^{i(\omega(\underline{\theta})t + \underline{\theta}x)/h}\ w_0(ht,t,x,x/h)$$

and let $\underline{u}^h$ *denote the exact solution of the Maxwell equations* $P^h \underline{u}^h = 0$ *with* $\underline{u}^h|_{t=0} = \underline{v}^h|_{t=0}$. *Then* $\forall\, \alpha \in \mathbb{N}^4$, $\exists\, C(\alpha)$, $\forall\, 0 < h < 1$,

$$\sup_{t \in [0,T/h]} \ \big\|(x\,,\, h\,\partial_{t,x})^\alpha(\underline{u}^h - \underline{v}^h)\big\|_{L^2(\mathbb{R}^3)} \ \le \ C(\alpha)\, h^{1-\beta}\,.$$

Rmks. Each term in the difference is $O(1)$. The divergence constraint is satisfied because $\widetilde{w}$ takes values in $\mathbb{K}$.

Remarks on the Proof

1. (WKB) A three term three scale approximate solution is constructed,

$$v^h(t,x) \;:=\; e^{2\pi i S/h}\, W^\epsilon(ht, x - \mathcal{V}t, x/h)\,,$$

$$W^h(\mathcal{T},x,y) \;:=\; w_0(\mathcal{T},x,y) + h\,w_1(\mathcal{T},x,y) + h^2\,w_2(\mathcal{T},x,y)$$

Apply P^h and set coefficients of powers of h equal to zero.

2. The construction of w_1 fails when the ray average hypothesis is not satisfied.

Open problem. This may signal an interesting instability. Or it may mean that there is a better *ansatz*.

3. The derivation of the profile equations is in the style of Joly-Métiver-R. We give a few elements.

Derivation of Profile Equations I

Ansatz for times $t = O(1/h)$

$$v^h(t,x) \ := \ e^{iS/h} W\Big(h, ht, t, x, \frac{x}{h}\Big), \quad S(t,x) = \omega\, t + \theta\,.x\,,$$

$$W = w_0(\mathcal{T}, t, x, y) + h w_1(\mathcal{T}, t, x, y) + h^2 w_2(\mathcal{T}, t, x, y)$$

periodic in y.

To preserve the ordering for $t = O(1/h)$ need $w_j(\mathcal{T}, t, x, y)$ *sublinear in* t.

Compute

$$P^h(t, x, \partial_t, \partial_x) v^h \ = \ e^{iS/h}\, Z^h(h\mathcal{T}, t, x, x/h)\,,$$

$$Z^h = h^{-1}\, r_{-1} + r_0 + h\, r_1 \, + \, h^2\, r_2 + h^3\, r_3, \quad r_j \ = \ r_j(t,x,y)$$

$$r_{-1} \ = \ \mathbb{L} w_0, \quad r_0 \ = \ \mathbb{L} w_1 + \mathbb{M} w_0, \quad r_1 \ = \ \mathbb{L} w_2 + \mathbb{M} w_1 + \mathbb{N} w_0$$

$$\mathbb{M}(y, \partial_t, \partial_x) \ := \ A_0^0(y)\partial_t \, - \begin{pmatrix} 0 & \partial_x \wedge \\ -\partial_x \wedge & 0 \end{pmatrix}$$

$$\mathbb{N} \ := \ \partial_{\mathcal{T}} A_0^0 + i\omega A_0^1 + M$$

$$\mathbb{L}\, w_0 \ = \ 0 \qquad\qquad \text{(yields Bloch dispersion relation)}$$

$\Pi := L^2 \perp$ proj on $\mathbb{K}$., $\quad \Pi w_0 = w_0, \quad \mathbb{Q} :=$ partial inverse

Profile Equations II

$r_j = 0$ iff $\Pi r_j = 0$ and $(I - \Pi) r_j = 0$.

Find, (nontrivial!)

$$\Pi\, \mathbb{M}\, \Pi\, w_0 \;=\; 0, \qquad \big(\Pi\, \mathbb{N}\, \Pi \;-\; \Pi\, \mathbb{M}\, Q\, \mathbb{M}\, \Pi\big) w_0 \;=\; 0.$$

Miraculous Algebraic Lemmas (nontrivial!)

$$\Pi\, \mathbb{M}\, \Pi\, w \;=\; \Pi\, A_0^0\, \Pi \left(\partial_t + \mathcal{V}\, \partial_x\right) w$$

whence $w_0(\mathcal{T}, t, x) = \widetilde{w}_0(\mathcal{T}, x - \mathcal{V} t)$.

$$\big(\Pi\, \mathbb{N}\, \Pi - \Pi\, \mathbb{M}\, Q\, \mathbb{M}\, \Pi\big)\, w \;= \\ \Pi A_0^0 \Pi \Big[\partial_{\mathcal{T}} + \frac{1}{2} i\ \partial_\theta^2 \omega(\partial_x, \partial_x) + \gamma(t, x)\Big] w$$

whence the Schrodinger equation for $\widetilde{w}_0$.

Milesker entzuteagatik

Thank you for your attention