# A refinement of the Strichartz inequality for the wave equation with applications

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- The Strichartz refinement
- The linear profile decomposition
- Maximizers of the Strichartz inequality

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# The wave equation

The wave equation $\partial_{tt} u = \Delta u$, in $\mathbb{R}^{d+1}$, with initial data $u(\cdot, 0) = u_0$, $\partial_t u(\cdot, 0) = u_1$, has solution which can be written as

$$\begin{aligned} u(\cdot,t) &= S(u_0,u_1)(\cdot,t) \\ &= \frac{1}{2}\big(e^{it\sqrt{-\triangle}}u_0 + \frac{1}{i}\frac{e^{it\sqrt{-\triangle}}u_1}{\sqrt{-\triangle}}\big) + \frac{1}{2}\big(e^{-it\sqrt{-\triangle}}u_0 - \frac{1}{i}\frac{e^{-it\sqrt{-\triangle}}u_1}{\sqrt{-\triangle}}\big), \end{aligned}$$

where

$$e^{\pm it\sqrt{-\triangle}}u_0(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi \pm t|\xi|)}\widehat{u_0}(\xi)d\xi$$

$$\frac{e^{\pm it\sqrt{-\triangle}}u_1}{\sqrt{-\triangle}}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi \pm t|\xi|)}\frac{\widehat{u_1}(\xi)}{|\xi|}d\xi.$$

Setting $\widehat{g}(\xi) = f(\xi, |\xi|)$ and $d\sigma(\xi, \tau) = \delta(|\xi| - \tau)d\xi$, we have that

$$\begin{aligned} e^{it\sqrt{-\triangle}}g(x) &= \frac{1}{(2\pi)^d}\int e^{i(x\cdot\xi + t|\xi|)}\widehat{g}(\xi)d\xi \\ &= \frac{1}{(2\pi)^d}\int e^{i(x\cdot\xi + t|\xi|)}f(\xi, |\xi|)d\xi \\ &= \frac{1}{(2\pi)^d}\int_{\mathbf{C}} e^{i(x\cdot\xi + t\tau)}f(\xi, \tau)d\xi d\tau \\ &= \widehat{fd\sigma}(x, t), \end{aligned}$$

where $\mathbf{C} := \left\{(\xi, \tau) \in \mathbb{R}^{d+1} : \ |\xi| = \tau\right\}$.

Therefore, if $\widehat{g}$ is supported in a set $A$, we can interpret $e^{it\sqrt{-\triangle}}g$ as the Fourier transform of a measure supported in

$$\left\{(\xi, |\xi|) \in \mathbb{R}^{d+1} : \ \xi \in A\right\}.$$

In 1977, Strichartz proved his fundamental inequality

$$\|S(u_0,u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \le C(\|u_0\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} + \|u_1\|^2_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)})^{\frac{1}{2}},$$

where

$$\|f\|_{\dot{H}^s} = (\sum_k 2^{2ks}\|P_k f\|_2^2)^{\frac{1}{2}},$$

with $\widehat{P_k f} = \chi_{\mathcal{A}_k}\widehat{f}$ and $\mathcal{A}_k = \{\xi \in \mathbb{R}^d;\ \ 2^k \le |\xi| \le 2^{k+1}\}$.

We improve this inequality to

$$\|S(u_0,u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \le C(\|u_0\|^2_{\dot{B}^{\frac{1}{2}}_{2,q}(\mathbb{R}^d)} + \|u_1\|^2_{\dot{B}^{-\frac{1}{2}}_{2,q}(\mathbb{R}^d)})^{\frac{1}{2}},$$

where $q = 2\frac{d+1}{d-1}$ for $d \ge 3$, and $q = 3$ for $d = 2$.

Here $\dot{B}^s_{2,q}$ is defined by

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# Better refinement

Let $S = \{w_m\}_m \subset \mathbb{S}^{d-1}$ be maximally $2^{-j}$-separated, and define $\tau_m^{j,k}$ by

$$\tau_m^{j,k} := \left\{ \xi \in \mathcal{A}_k : \left|\frac{\xi}{|\xi|} - w_m\right| \leq \left|\frac{\xi}{|\xi|} - w_{m'}\right| \text{ for every } w_{m'} \in S,\ m' \neq m \right\}.$$

We also set $\widehat{P_k g_m^j} = \chi_{\tau_m^{j,k}} \widehat{g}$.
For our applications the following refinement will be of more use.

There exist $p < 2$ and $\theta > 0$ such that

$$\|S(u_0,u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C\Big( \sup_{j,k,m} 2^{k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_0)_m^j}\|_p^{\theta} \|u_0\|^{1-\theta}_{B^{\frac{1}{2}}_{2,q(1-\theta)}}$$
$$+ \sup_{j,k,m} 2^{-k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_1)_m^j}\|_p^{\theta} \|u_1\|^{1-\theta}_{B^{-\frac{1}{2}}_{2,q(1-\theta)}} \Big).$$

Note: Quilodrán posted in the arxiv recently a similar result for the case of dimension $d = 2$.

# Previous Strichartz's refinements in the literature

- For the Schrödinger equation:
  - Bourgain in 1989 in dimension $d = 2$.
  - Moyua–Vargas–Vega first in 1996 and then in 1999 improved that refinement.
  - Begout–Vargas in 2007 extended the result to dimensions $d \geq 2$.
- For other equations:
  - Rogers–Vargas in 2006 for the nonelliptic Schrödinger equation.
  - Chae–Hong-Lee in 2009 for higher order Schrödinger equations.
  - Shao in 2009 for the airy equation.
  - Killip-Stovall-Visan in 2011 for the Klein–Gordon equation.

2 key points:

- how to get the $L^p$ norm with $p < 2$ on the right hand side?

$$\|S(u_0,u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C\Big( \sup_{j,k,m} 2^{k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_0)}_m^j\|_{p}^{\theta} \|u_0\|_{B^{\frac{1}{2}}_{2,q(1-\theta)}}^{1-\theta} + \sup_{j,k,m} 2^{-k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_1)}_m^j\|_{p}^{\theta} \|u_1\|_{B^{-\frac{1}{2}}_{2,q(1-\theta)}}^{1-\theta} \Big).$$

- how to improve from $\ell^2$ to $\ell^q$ with $q > 2$?

$$\|S(u_0,u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C\Big( \sup_{j,k,m} 2^{k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_0)}_m^j\|_{p}^{\theta} \|u_0\|_{B^{\frac{1}{2}}_{2,q(1-\theta)}}^{1-\theta} + \sup_{j,k,m} 2^{-k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_1)}_m^j\|_{p}^{\theta} \|u_1\|_{B^{-\frac{1}{2}}_{2,q(1-\theta)}}^{1-\theta} \Big).$$

We use the bilinear method to get the $L^p$ norm. Questions:

- What is a bilinear estimate?
- How do we pass from bilinear to linear?
- Why do we want to use a bilinear estimate?

## What is a bilinear estimate for the wave equation?

Asuming that $|w_m - w_{m'}| \sim 1$, it is an estimate of the form

$$\|e^{it\sqrt{-\triangle}} P_0 g_m^1 \, e^{it\sqrt{-\triangle}} P_0 g_{m'}^1\|_{L^q(\mathbb{R}^{d+1})} \lesssim \|\widehat{P_0 g_m^1}\|_{L^2(\mathbb{R}^d)} \|\widehat{P_0 g_{m'}^1}\|_{L^2(\mathbb{R}^d)}.$$

Of course, by Cauchy–Schwarz a linear estimate

$$\|e^{it\sqrt{-\triangle}} P_0 g_m^1\|_{L^{2q}(\mathbb{R}^{d+1})} \lesssim \|\widehat{P_0 g_m^1}\|_{L^2(\mathbb{R}^d)}.$$

implies the bilinear one.
But while $q = \frac{d+1}{d-1}$ is the lowest value of $q$ for the linear estimate (Strichartz estimate), on the other hand, we can push down this value in the bilinear case thanks to the angular separation.

## How do we pass from linear to bilinear?

Tao–Vargas–Vega in 1998 gave a way to deduce linear estimates from bilinear ones. The key ingredient is the Whitney decomposition. Let $\widehat{g}$ supported in $\mathcal{A}_1$, we have

$$\mathcal{A}_1 \times \mathcal{A}_1 = \bigcup_{j} \bigcup_{m,m' : \tau_m^j \sim \tau_{m'}^j} \tau_m^j \times \tau_{m'}^j$$

where we write $\tau_m^j \sim \tau_{m'}^j$ if $|w_m - w_{m'}| \sim 2^{-j}$. Therefore

$$\begin{aligned}
\|e^{it\sqrt{-\triangle}} P_0 g\|^2_{L^{2q}(\mathbb{R}^{d+1})} &= \|e^{it\sqrt{-\triangle}} P_0 g \; e^{it\sqrt{-\triangle}} P_0 g\|_{L^q(\mathbb{R}^{d+1})} \\
&\lesssim \|\sum_j \sum_{m,m' : \tau_m^{j,k} \sim \tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\triangle}} P_0 g_m^j \; e^{it\sqrt{-\triangle}} P_0 g_{m'}^j \|_{L^q(\mathbb{R}^d)} \\
&\lesssim \sum_j \sum_{m,m' : \tau_m^j \sim \tau_{m'}^j} \|e^{it\sqrt{-\triangle}} P_0 g_m^j \; e^{it\sqrt{-\triangle}} P_0 g_{m'}^j \|_{L^q(\mathbb{R}^d)}
\end{aligned}$$

A rescaling argument permits us to use the bilinear estimate. The step when we use the triangle inequality can be improved, as we will see.

## Why do we want to use a bilinear estimate?

As we have seen, in the linear restriction if we have the $L^2$ norm in the right hand we can not push down the $L^{2\frac{d+1}{d-1}}$ norm of the left hand side to some $L^q$ with $q < 2\frac{d+1}{d-1}$, while in the bilinear setting we can push down the $L^{\frac{d+1}{d-1}}$ norm. Moreover, as a consequence of this and interpolating we can get estimates

$$\|e^{it\sqrt{-\triangle}}P_0g_m^1\, e^{it\sqrt{-\triangle}}P_0g_{m'}^1\|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \lesssim \|\widehat{P_0g_m^1}\|_{L^p(\mathbb{R}^d)}\|\widehat{P_0g_{m'}^1}\|_{L^p(\mathbb{R}^d)}.$$

with $p < 2$.

# The Cone Bilinear Restriction Theorem

More precisely, we have the trivial bilinear estimate

$$\| e^{it\sqrt{-\triangle}} P_0 g^1_m \; e^{it\sqrt{-\triangle}} P_0 g^1_{m'} \|_{L^\infty(\mathbb{R}^{d+1})} \lesssim \| \widehat{P_0 g^1_m} \|_{L^1(\mathbb{R}^d)} \| \widehat{P_0 g^1_{m'}} \|_{L^1(\mathbb{R}^d)}. \quad (1)$$

And the (sharp except for the end-point) following estimate

**Theorem (Wolff 2000)**

*Let $\frac{d+3}{d+1} < r_1$, and suppose that $\angle(w_m, w_{m'}) \sim 1$. Then,*

$$\| e^{it\sqrt{-\triangle}} P_0 g^1_m \; e^{it\sqrt{-\triangle}} P_0 g^1_{m'} \|_{L^{r_1}(\mathbb{R}^{d+1})} \lesssim \| \widehat{P_0 g^1_m} \|_{L^2(\mathbb{R}^d)} \| \widehat{P_0 g^1_{m'}} \|_{L^2(\mathbb{R}^d)}. \quad (2)$$

interpolating (1) and (2):

$$\| e^{it\sqrt{-\triangle}} P_0 g^1_m \; e^{it\sqrt{-\triangle}} P_0 g^1_{m'} \|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \lesssim \| \widehat{P_0 g^1_m} \|_{L^p(\mathbb{R}^d)} \| \widehat{P_0 g^1_{m'}} \|_{L^p(\mathbb{R}^d)}.$$

with $p = \frac{2(d+1)}{2(d+1)-(d-1)r_1} < 2$.

A naive attempt: Wolff's estimate needs (up to rescaling) that the input functions are Fourier supported in the same annulus. We could use Littlewood-Paley theory and get

$$\|e^{it\sqrt{-\triangle}}g\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = \sum_k \|e^{it\sqrt{-\triangle}}P_k g\|^2_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}$$
$$= \sum_k \|e^{it\sqrt{-\triangle}}P_k g \; e^{it\sqrt{-\triangle}}P_k g\|_{L^{\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}.$$

which would yield to (with $p < 2$ and $\theta > 0$)

$$\|e^{it\sqrt{-\triangle}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \Big(\sum_k 2^k \big(\sum_j \sum_m |\tau_m^{j,k}|^{q\frac{p-2}{2p}} \|\widehat{P_k g_m^j}\|_p^q\big)^{\frac{2}{q}}\Big)^{\frac{1}{2}}.$$

$$\|e^{it\sqrt{-\triangle}}g\|_{L^q(\mathbb{R}^{d+1})} \lesssim \Big(\sum_k 2^k \big(\sup_{j,k,m} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k g_m^j}\|_p^\theta \|\widehat{P_k g}\|_2^{1-\theta}\big)^2\Big)^{\frac{1}{2}}.$$

and it is not possible to take a supremum in $k$ without losing some regularity !!!

Instead of using Wolff's estimate, we need to use the following bilinear estimate

**Theorem (Tao 2001)**

*Let* $\frac{d+3}{d+1} \leq r_1 \leq 2$, *and suppose that* $\angle(w_m, w_{m'}) \sim 1$. *Then for all* $\epsilon > 0$,

$$\|e^{it\sqrt{-\triangle}}P_0 g_m^1 e^{it\sqrt{-\triangle}}P_\ell g_{m'}^1\|_{L^{r_1}(\mathbb{R}^{d+1})} \lesssim 2^{\ell(\frac{1}{r_1}-\frac{1}{2}+\epsilon)}\|\widehat{P_0 g_m^1}\|_{L^2(\mathbb{R}^d)}\|\widehat{P_\ell g_{m'}^1}\|_{L^2(\mathbb{R}^d)}$$

There is some gain when working at different Fourier scales!
We explote this to improve the $\ell^2$ summation. Therefore, instead of Littlewood–Paley we have

$$\begin{aligned}
\|e^{it\sqrt{-\triangle}}g\|_{L^q(\mathbb{R}^{d+1})} &= \|e^{it\sqrt{-\triangle}}g\ e^{it\sqrt{-\triangle}}g\|_{L^2(\mathbb{R}^{d+1})}^{\frac{1}{2}} \\
&= \|\sum_{k>\ell} e^{it\sqrt{-\triangle}}P_k g\ e^{it\sqrt{-\triangle}}P_\ell g + \sum_{k\leq\ell} e^{it\sqrt{-\triangle}}P_k g\ e^{it\sqrt{-\triangle}}P_\ell g\|_{L^2(\mathbb{R}^{d+1})}^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{\ell>0}\|\sum_k e^{it\sqrt{-\triangle}}P_k g\ e^{it\sqrt{-\triangle}}P_{k+\ell} g\|_{L^r(\mathbb{R}^{d+1})}\Big)^{\frac{1}{2}},
\end{aligned}$$

and we have to deal with the summation in $\ell$.

When we decompose with Whitney we have to deal with

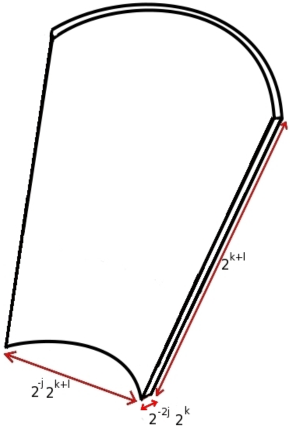
$$\|e^{it\sqrt{-\triangle}}P_k g \; e^{it\sqrt{-\triangle}}P_{k+\ell} g\|_{L^r(\mathbb{R}^{d+1})}$$
$$= \|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}.$$

Where are the functions $\{e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\}_{j,m,m'}$ Fourier supported?
Answer: surprisingly they are disjoint Fourier supported!
Wolff observed it for a fixed index $j$ (in the case of $\ell = 0$).

A calculation shows that they are supported in

$$H^{k,\ell}_{j,m,m'} = \Big\{(\xi,\tau) \in \tilde{\mathcal{A}}_{k+\ell} \times \mathbb{R} : \quad d\big((\xi,\tau), \mathbf{C}\big) \sim 2^{-2j}2^{k}, \quad \angle(w_m, \xi) \lesssim 2^{-j}\Big\}$$

That is, the index $j$ gives the distance to the cone.



$2^{k+l}$
$2^{-j}2^{k+l}$
$2^{-2j}\ 2^{k}$

How can we take advantage of this Fourier orthogonality?

In $L^2$ if a collection of functions $(f_k)_k$ has disjoint Fourier support, then

$$\|\sum_k f_k\|_{L^2}^2 \leq \sum_k \|f_k\|_2^2$$

This orthogonality is only valid in dimension $d = 3$:

$$\begin{aligned}
&\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^2(\mathbb{R}^{3+1})}^2 \\
&\qquad \lesssim \sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^2(\mathbb{R}^{3+1})}^2.
\end{aligned}$$

For different norm than $L^2$, Tao–Vargas–Vega used a substitute of the $L^2$ orthogonality:

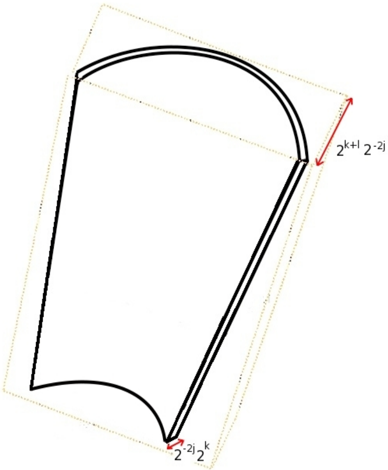
**Lemma (Vargas-Vega-Tao 1998)**

*Let $R_k$ be a collection of rectangles in frequency space such that the dilates $(1+c)R_k$ with $c>0$ are almost disjoint, and suppose that $f_k$ are collection of functions whose Fourier transforms are supported on $R_k$. Then for all $1 \leq p \leq \infty$*

$$\|\sum_k f_k\|_p \lesssim \Big(\sum_k \|f_k\|_p^{p^*}\Big)^{\frac{1}{p^*}}$$

*where $p^* = \min(p, p')$.*

In our case we can not find a collection of rectangles $R_{j,m,m'}$ almost disjoints such that $H^{k,\ell}_{j,m,m'} \subset R_{j,m,m'}$.


$2^{k+l}\,2^{-2j}$
$2^{-2j}2^{k}$

We need to measure the loss if we use different sets than rectangles.

## Lemma

*Let $(E_k)_{k\in\mathbb{Z}}$ be a collection of sets such that there exist almost disjoint $(F_k)_{k\in\mathbb{Z}}$, with $E_k \subset F_k$ for every $k$, such that there exist bump functions $\phi_{E_k}$ equal to 1 on $E_k$ and 0 outside $F_k$, and such that*

$$\int |\widehat{\phi_{E_k}}(\xi)| d\xi \leq C \tag{3}$$

*uniformly in $k$. Suppose that $(f_k)_{k\in\mathbb{Z}}$ are a collection of functions whose Fourier transforms are supported on $(E_k)_{k\in\mathbb{Z}}$. Then for all $1 \leq p \leq \infty$, we have*

$$\|\sum_k f_k\|_p \lesssim C^{1-\frac{2}{p^*}} \left(\sum_k \|f_k\|_p^{p_*}\right)^{\frac{1}{p_*}}$$

*where $p_* = \min(p, p')$ and $p^* = \max(p, p')$.*

Therefore by last Lemma

$$\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}$$

$$\lesssim 2^{\ell\frac{d-1}{2}(r_*-2\frac{r_*}{r^*})}\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*},$$

and also by orthogonality at single scale:

$$\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}$$

$$\lesssim (\sum_j ( \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*})^{\frac{1}{r_*}})^{r_*}.$$

The first one is not sufficient as the constant $2^{\ell\frac{d-1}{2}(r_*-2\frac{r_*}{r^*})}$ does not permit to sum in $\ell$

$$\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell}g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}$$
$$\lesssim 2^{\ell\frac{d-1}{2}(r_*-2\frac{r_*}{r^*})}\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell}g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}.$$

The second one is not sufficient due to the power of $\frac{1}{r_*}$ that appears

$$\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\triangle}}P_k g_m^j \, e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}$$

$$\lesssim \Big(\sum_j \Big( \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\triangle}}P_k g_m^j \, e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}\Big)^{\frac{1}{r_*}}\Big)^{r_*},$$

The combination of both instead will permit to obtain the result:

$$\|\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}$$

$$\lesssim \Big(\sum_j \Big(\sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}\Big)^{\frac{1}{r_*}}\Big)^{\alpha r_*}$$

$$\Big(2^{\ell\frac{d-1}{2}(r_*-2\frac{r_*}{r^*})}\sum_j \sum_{m,m':\tau_m^{j,k}\sim\tau_{m'}^{j,k+\ell}} \|e^{it\sqrt{-\triangle}}P_k g_m^j \; e^{it\sqrt{-\triangle}}P_{k+\ell} g_{m'}^j\|_{L^r(\mathbb{R}^{d+1})}^{r_*}\Big)^{1-\alpha}$$

for some $0 < \alpha < 1$.

The first one is bad because of the $\frac{1}{r_*}$ exponent, but even though it does not permit to get the $L^p$ norm on the right hand side with $p < 2$, we can still get the $L^2$ norm.

The key Lemma is

**Lemma**

*Let $q > 2$, and $1 < p < 2$. Then*

$$\sum_j \Big(\sum_m |\tau_m^{j,k}|^{q\frac{p-2}{2p}} \|\widehat{P_k g_m^j}\|_p^q\Big)^{\frac{2}{q}} \lesssim \|P_k g\|_2^2.$$

The last lemma is proved using the atomic decomposition of Keel and Tao.

## Lema (Keel-Tao 1999)

*Let $f \in L^p(\mathbb{R}^d)$ for some $1 < p < \infty$. Then, we can decompose*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n \chi_n(x),$$

*where $\chi_n$ are functions bounded in magnitude by 1 and supported in disjoint sets of measure at most $2^n$, and $c_n$ are non-negative real numbers such that*

$$\sum_{n\in\mathbb{Z}} 2^n |c_n|^p \sim \|f\|_p^p.$$

Using some polarization arguments we get finally the result.

**Theorem**

*There exist $p < 2$ and $\theta > 0$ such that*

$$\|S(u_0,u_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \leq C\Big( \sup_{j,k,m} 2^{k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_0)}_m^j\|_p^\theta \|u_0\|^{1-\theta}_{B^{\frac{1}{2}}_{2,q(1-\theta)}} + \sup_{j,k,m} 2^{-k\frac{\theta}{2}} |\tau_m^{j,k}|^{\frac{\theta}{2}\frac{p-2}{p}} \|\widehat{P_k(u_1)}_m^j\|_p^\theta \|u_1\|^{1-\theta}_{B^{-\frac{1}{2}}_{2,q(1-\theta)}} \Big).$$

# Introduction of the Profile decomposition

Let $F : X \to Y$ be a linear transformation between two Banach spaces.

$F$ is called compact if for every bounded sequence $x_n \in X$, the sequence $F(x_n)$ has a convergent subsequence.

Let consider the wave operator

$$\begin{aligned} S : \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}} &\longrightarrow L^{2\frac{d+1}{d-1}} \\ (u_0, u_1) &\longrightarrow S(u_0, u_1) \end{aligned}$$

# Symmetries of the $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ wave equation

Let $(u_0, u_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$, then the following $u(x,t)$ transformations of $S(u_0, u_0)(x,t)$ are also solutions of the wave equation

- $u(x,t) = S(u_0,u_1)(rx, rt)$ with $r > 0$
- $u(x,t) = r\,S(u_0,u_1)(x,t)$ with $r > 0$
- $u(x,t) = S(u_0,u_1)(x + x_0, t + t_0)$ with $x_0 \in \mathbb{R}^d$ and $t \in \mathbb{R}$
- $u(x,t) = S(u_0,u_1)(x - x_v + \frac{x_v - vt}{\sqrt{1-|v|^2}}, \frac{t - vx}{\sqrt{1-|v|^2}})$ with $|v| < 1$ and $x_v$ is the projection of $x$ onto the line parallel to $v$.
- $u(x,t) = S(u_0,u_1)(\theta x, t)$ with $\theta \in SO(d)$
- $u(x,t) = e^{\theta_+ i} S_+(u_0,u_1)(x,t) + e^{\theta_- i} S_-(u_0,u_1)(x,t)$ with $\theta_+, \theta_- \in [0, 2\pi)$.

These cause a defect of compactness in the wave operator.

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These cause a defect of compactness in the wave operator.

For example: Let $u_0, u_1 \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$, we define $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$: $u_{0,n}(x) = n^{\frac{d-1}{2}} u_0(nx)$, $u_{1,n}(x) = n^{\frac{d-1}{2}+1} u_1(nx)$. We have that

$$\|u_{0,n}\|_{\dot{H}^{\frac{1}{2}}} = \|u_0\|_{\dot{H}^{\frac{1}{2}}}, \quad \|u_{1,n}\|_{\dot{H}^{-\frac{1}{2}}} = \|u_1\|_{\dot{H}^{-\frac{1}{2}}}.$$

but

$$S(u_{0,n}, u_{1,n})(x,t) = n^{\frac{d-1}{2}} S(u_0, u_1)(nx, nt)$$

does not have any convergent subsequence.

And in general this defect of compactness comes always from:
Let $(r^n, \ell^n, w^n, x^n, t^n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$, we define the transformations $\Gamma^n$ by

$$\Gamma^n S(x,t) = \left(\frac{r^n}{\ell^n}\right)^{\frac{d-1}{2}} S\Big((T^{\ell^n}_{w^n})^{-1} r^n(x-x^n, t-t^n)\Big),$$

where letting $w \in \mathbb{S}^{d-1}$, and $\ell \in [1,\infty)$, the transformation $(T^{\ell^n}_{w^n})^{-1}$ is a Lorentz transformation rescaled by $\sim \ell^n$ and with $v = (w^n, 1)$.

We want to express the wave operator acting on any bounded subsequence in terms of this defect of compactness, and that it is the so called profile decomposition.

# Idea of the Profile decomposition

Roughly speaking, the profile decomposition states that for any bounded sequence $u_{0,n}, u_{1,n} \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$, we have that $\{S(u_{0,n}, u_{1,n})\}_n$ can be written, up to taking a subsequence, as a sum of transformations $\{\{\Gamma_j^n v_j\}_n\}_j$ called profiles with a small interaction, where $v_j$ are also solutions, and a remainder term $r_n^N$ which is very small in some sense:

$$S(u_{0,n}, u_{1,n}) = \sum_{j=1}^{N} \Gamma_j^n v_j + r_n^N$$

Some previous linear profile decompositions for dispersive equations:
First profile decomposition for the Schrödinger and wave equation

- Bahouri–Gérard (1999) for the $\dot{H}^1 \times L^2$ wave equation in dimension $d = 3$.
- Merle–Vega (1998) for the $L^2$ Schrödinger equation in dimension $d = 2$.

After that, many works on that

- Keraani (2001) for the $\dot{H}^1$ Schrödinger equation in dimension $d \geq 3$.
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- Bégout–Vargas (2007) for the $L^2$ Schrödinger equation in dimension $d \geq 3$.
- Shao (2009) for the Airy equation.
- Bulut (2010) for the $\dot{H}^s \times \dot{H}^{s-1}$ wave equation in dimension $d \geq 3$ and $s \geq 1$.
- Killip–Stovall–Visan (2011) for the $\dot{H}^1 \times L^2$ Klein–Gordon equation
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The interaction of the profiles is small:

$$\limsup_{n\to\infty} \| \sum_{j=1}^{N} \Gamma_j^n S(\phi_0^j, \phi_1^j) \|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} = \sum_{j=1}^{N} \limsup_{n\to\infty} \| \Gamma_j^n S(\phi_0^j, \phi_1^j) \|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}}.$$

That is, by a change of variables

$$\limsup_{n\to\infty} \| \sum_{j=1}^{N} \Gamma_j^n S(\phi_0^j, \phi_1^j) \|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} = \sum_{j=1}^{N} \| S(\phi_0^j, \phi_1^j) \|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}}.$$

How can we ensure this property?
The sequences $(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{j\in\mathbb{N}}$ in $\mathbb{R}^+ \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$ must be orthogonal.

If $(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{j\in\mathbb{N}}$ is a family of sequences in $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$, then we say that the family is orthogonal if one of the following properties is satisfied for all $j \neq k$:

A. Lorentz property

$$\frac{\ell_j^n}{\ell_k^n} + \frac{\ell_k^n}{\ell_j^n} \underset{n\to\infty}{\longrightarrow} +\infty$$

B. Rescaling property

$$\frac{r_j^n}{r_k^n} + \frac{r_k^n}{r_j^n} \underset{n\to\infty}{\longrightarrow} +\infty$$

C. Angular property

$$r_j^n = r_k^n,\ \ell_j^n = \ell_k^n \quad \text{and} \quad \ell_j^n |w_j^n - w_k^n| \underset{n\to\infty}{\longrightarrow} +\infty$$

D. Space-time translation property

$$r_j^n = r_k^n,\ \ell_j^n = \ell_k^n,\ w_j^n = w_k^n \quad \text{and} \quad \left| (T_{w_j^n}^{\ell_j^n})^{-1} r_j^n (x_j^n - x_k^n, t_j^n - t_k^n) \right| \underset{n\to\infty}{\longrightarrow} +\infty$$

The remainder term is small:

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|r_n^N\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = 0.$$

## Theorem

*Let* $(u_{0,n}, u_{1,n})_n$ *be a bounded sequence in* $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ *with* $d \geq 2$*. Then, there exist a subsequence (still denoted* $(u_{0,n}, u_{1,n})_n$*), a sequence* $(\phi_0^j, \phi_1^j)_{j\in\mathbb{N}} \subset \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ *and a family of orthogonal sequences* $(r_j^n, \ell_j^n, w_j^n, x_j^n, t_j^n)_{j\in\mathbb{N}}$ *in* $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$*, such that for every* $N \geq 1$*,*

$$S(u_{0,n}, u_{1,n})(x,t) = \sum_{j=1}^{N} \Gamma_j^n S(\phi_0^j, \phi_1^j)(x,t) + S(R_{0,n}^N, R_{1,n}^N)(x,t),$$

*with*

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S(R_{0,n}^N, R_{1,n}^N)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = 0.$$

*Furthermore, we also have for every* $N \geq 1$*,*

$$\|(u_{0,n}, u_{1,n})\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} = \sum_{j=1}^{N} \|(\phi_0^j, \phi_1^j)\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} + \|(R_{0,n}^N, R_{1,n}^N)\|^2_{\dot{H}^{\frac{1}{2}}\times\dot{H}^{-\frac{1}{2}}} + o(1).$$

Property of compact operator: let $F : X \to Y$ be a linear transformation between two Banach spaces.

For every sequence $u_n \underset{n\to\infty}{\rightharpoonup} 0$, we have $F(x_n) \to 0$ in norm.

## Theorem

*Let $d \geq 2$, and let $(u_{0,n}, u_{1,n})_n$ be a sequence in $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ such that*

$$\|(u_{0,n}, u_{1,n})\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} \leq M \quad \text{and} \quad \|S(u_{0,n}, u_{1,n})\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \geq A.$$

*Then, there exists a sequence $(r^n, \ell^n, w^n, x^n, t^n)$ in $\mathbb{R}^+ \setminus \{0\} \times [1,\infty) \times \mathbb{S}^{d-1} \times \mathbb{R}^d \times \mathbb{R}$ such that, up to a subsequence,*

$$(\Gamma^n)^{-1} S(u_{0,n}, u_{1,n}) \underset{n\to\infty}{\rightharpoonup} \mathtt{U} \quad \text{with} \quad \|\mathtt{U}\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} \geq C(A, M).$$

Steps of the proof of the profile decomposition.

- 1. Obtaining a profile decomposition **but** assuming that the initial data has compact Fourier support away from zero. The transformations on the profiles are space-time translations.
- 2. Using the argumentation of Bourgain (1998) (and Merle–Vega (1998)) to reduce to the case when the initial data has compact Fourier support away from zero, but with an epsilon dependence, that is, for a fixed $\epsilon > 0$ we find some profiles with a remainder term smaller in the Strchartz norm than $\epsilon$.
  Key observation: The transformations $T^{2^j}_{w_m}$ maps a a set $\tau_m^{j,1}$ into a set in $A_0 \cup A_1 \cup A_2$ of measure $\sim 1$.
  Observe that for every cap $\tau_m^{j,k}$ in which the supremun of the refinement is taken, we have that

$$\text{supp}\chi_{\tau_m^{j,k}}((T^{2^j}_{w_m})^{-1}2^k(x)) \subset A_0 \cup A_1 \cup A_2.$$

- 3. Prove a profile decomposition with a weaker condition on the smallness of the remainder term (not using the Strichartz norm).
- 4. Deduce the required smallness condition putting together 2 and 3.

As an application we get that there exists a maximizer for the Strichartz inequality. A lot of related work has been done in the last years, some of them are:

- Kunze (2003) proved the existence of maximizers for the Schrödinger equation in $d = 1$.
- Foschi (2007) for the Schrödinger and wave equation in dimensions $d = 1, 2$ found the maximizers.
- Hundertmark–Zharnitsky (2006), Bennett–Bez–Carbery–Hundertmark (2009) and Carneiro (2009) also found the maximizers in dimension $d = 1, 2$; with different techniques.
- Shao (2009) proved that maximizers exist for the Schrödinger equation in all dimensions, and Bulut for the $\dot{H}^s \times \dot{H}^{s-1}$ wave equation with $s \geq 1$ in dimensions $d \geq 3$; both with the profile decomposition.
- Duyckaerts, Merle and Roudenko (2011) for the nonlinear Schrödinger equation.
- Christ–Shao (2011) for the Fourier extension inequality for the sphere in dimension $d = 2$; and Fanelli–Vega–Visciglia (2011) extended it for more general surfaces and dimensions.
- Bez–Rogers found the maximizers for the $\dot{H}^1 \times L^2$ wave equation in dimension $d = 5$.

# Existence of maximizers for the Stricharz inequality

## Corollary

*Let* $d \geq 2$*, then there exists a maximizing pair* $(\psi_0, \psi_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)$ *such that*

$$\|S(\psi_0, \psi_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} = W(d)\|(\psi_0, \psi_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)},$$

*where*

$$W(d) := \sup\{\|S(\phi_0, \phi_1)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})} : (\phi_0, \phi_1) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}} \text{ with } \|(\phi_0, \phi_1)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} = 1\}.$$

**Proof.** We choose $(u_{0,n}, u_{1,n}) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$ such that $\|(u_{0,n}, u_{1,n})\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)} = 1$ and $\|S(u_{0,n}, u_{1,n})\|_{L^{2\frac{d+1}{d-1}}} \underset{n\to\infty}{\longrightarrow} W(d)$.

$$
\begin{aligned}
W(d)^{2\frac{d+1}{d-1}} &= \limsup_{n\to\infty} \|S(u_{0,n}, u_{1,n})\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} \\
&= \lim_{N\to\infty} \limsup_{n\to\infty} \Big\| \sum_{j=1}^{N} \Gamma_j^n S(\phi_0^j, \phi_1^j) \Big\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} \\
&= \sum_{j=1}^{\infty} \|S(\phi_0^j, \phi_1^j)\|_{L^{2\frac{d+1}{d-1}}(\mathbb{R}^{d+1})}^{2\frac{d+1}{d-1}} \\
&\leq W(d)^{2\frac{d+1}{d-1}} \sum_{j=1}^{\infty} \|(\phi_0^j, \phi_1^j)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^{2\frac{d+1}{d-1}} \\
&\leq W(d)^{2\frac{d+1}{d-1}} \Big(\sum_{j=1}^{\infty} \|(\phi_0^j, \phi_1^j)\|_{\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}}^{2}\Big)^{\frac{d+1}{d-1}} \leq W(d)^{2\frac{d+1}{d-1}}.
\end{aligned}
$$

Therefore, in order to have equalities throughout, there should be exactly one term in the sum, which yields the maximizing pair.

Application of our Strichartz inequality to the nonlinear theory:

- We can prove a concentration pehenomena for solutions of the nonlinear $\partial_{tt} u - \Delta u = \gamma |u|^{\frac{4}{d-1}} u$ wave equation. It is based in works of Bourgain (1998) and Begout–Vargas (2007).
- We can prove a nonlinear profile decomposition (based in the linear decomposition) which permits to prove that there exists a blow-up solution with minimal initial data, based in the work of Keraani (2006).
- We can characterize nonlinear solution with linearizable data based in work of Bahaouri–Gérard (1998) and Carles–Keraani (2007).