Stability of periodic wave trains: part I, Whitham's modulation equations and spectral stability

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Outline of the talk

- Introduction
  - Hydrodynamics instabilities and periodic waves
  - Canonical equations: Swift-Hohenberg, Kuramoto-Sivashinsky,...
- Model equation: Ginzburg Landau equation
  - Spectral stability
  - Phase dynamics
  - Generalizations
- Conservation laws: KdV/KS as a model study
  - Spectral problem/Necessary conditions
  - Whitham's modulation eqs/ Spectral stability
  - Numerical analysis of the spectral problem
  - KdV limit
- Conclusions and perspectives.

Introduction: Hydrodynamic instabilities.

rolls in Taylor Couette flows

rolls in Rayleigh-Bénard convection

Model: Navier-Stokes eqs
Instability: angular velocities (TC) or temperature (RB) gradient

Introduction: Free surface instabilities

Left: regular roll-waves in an open channel
Right: "roll-waves" on a sea of ice

Introduction: Free surface instabilities.

Irregular roll-waves in a spillway

- **Models:** free surface Navier-Stokes equations or shallow water equations with bottom friction
- **Instability:** competition between inertial and viscous effects

- **Aim:** stability of periodic waves
- Unstable wavetrains= transition to turbulence?

Canonical equations: Rayleigh Bénard convection

2d Navier-Stokes equations (Boussinesq approximation)

$$\begin{aligned}
\partial_t u + (u.\nabla)u + \nabla p &= \Delta u + \mathcal{R}\theta e_2, \\
\mathcal{P}\left(\partial_t \theta + (u.\nabla)\theta\right) &= \Delta\theta + \mathcal{R} u_2, \quad \forall x \in \mathbf{R} \times (0,\, \pi) \\
\operatorname{div} u &= 0. \\
\partial_{x_2} u_1 = u_2 = \theta &= 0, \quad x_2 = 0, \pi.
\end{aligned}$$

- Non dimensional numbers: $Ra = \mathcal{R}^2$ Rayleigh, $\mathcal{P}$ Prandlt number.
- Dispersion relation

$$\mathcal{P}\lambda^2 + (\mathcal{P}+1)(k^2 + m^2\pi^2)\lambda + (k^2 + m^2\pi^2)^2 - \frac{\mathcal{R}^2 k^2}{(k^2 + m^2\pi^2)} = 0.$$

- **Instability:** $\mathcal{R}^2 \geqslant \mathcal{R}_c^2(k^2) = (k^2 + m^2\pi^2)^3/k^2$

Canonical equations: Rayleigh Bénard convection.

- **Transition to instability:** $0 < \mathcal{R} - \mathcal{R}_c(k_0^2) \ll 1$, $k_0 = \pi/\sqrt{2}$ $(m = 1)$.
- Dispersion relation at the transition to instability

$$\lambda(k) = \frac{2\mathcal{P}(k_0^2 + \pi^2)}{\mathcal{R}_c(k_0)(\mathcal{P}+1)^2}\left(\mathcal{R}^2 - \mathcal{R}_c^2(k_0^2) - \frac{(\mathcal{R}_c^2)''(k_0)}{2}(k^2 - k_0^2)^2\right) + (h.o.t).$$

Swift Hohenberg equation (heuristic cubic nonlinearity)

$$\partial_\tau u = (Ra - Ra_c)u - (k_0^2 + \partial_x^2)^2 u + f(u, \partial_x u).$$

- Small amplitude/wavenumber limit:

Ginzburg Landau equation (formal derivation from 2d RB or SH)

$$\partial_T A = \partial_{XX} A + A - A|A|^2$$

Canonical equations: thin film flows

Free surface Navier Stokes equations down a ramp

$$\partial_t u + (u.\nabla)u + \frac{\nabla p}{F^2} = \frac{1}{F^2}\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} + \frac{1}{Re}\Delta u,$$
$$\mathrm{div u} = 0 \quad \text{in } \Omega(t) = \{(x_1, x_2)/0 \leqslant x_2 \leqslant h(x_1, t)\}$$

Shallow water regime: $\varepsilon = H/X_1 \ll 1$

Consistent shallow water equations (J.P. Vila)

$$\partial_t h + \partial_x(h\bar{u}) = 0,$$
$$\partial_t(h\bar{u}) + \partial_x(h\bar{u}^2 + \frac{3\mathrm{cotan}(\theta)\, h^2}{2Re} + 2h^5/25) = \frac{1}{\varepsilon Re}(h - \frac{\bar{u}}{h}) + \frac{\varepsilon}{Re}\partial_x(h\partial_x\bar{u})$$

Notation: $c = \mathrm{cotan}(\theta)$

Canonical equations: thin film flows

- Dispersion relation (stability of steady states)

$$\lambda^2 + (1 + 2ik + \frac{k^2}{Re^2})\lambda + 3 - 3ik(\frac{c}{Re} - \frac{1}{5}) + \frac{k^2}{Re^2} = 0.$$

- Bifurcation from $(\lambda, k) = (0,0)$, $6/5 - c/Re = \varepsilon^2\Gamma$

$$\lambda(\varepsilon k) = -3i\varepsilon k - \frac{2}{Re^2}(i\varepsilon k)^3 + \varepsilon^4(3\Gamma k^2 - \frac{8k^4}{Re^2}) + (h.o.t).$$

- Small amplitude/wavenumber and $0 < \varepsilon^2\Gamma \ll 1$

Korteweg de Vries/Kuramoto-Sivashinsky equation

$$\partial_T v + \beta\partial_X \frac{v^2}{2} + \frac{2}{Re^2}\partial_X^3 v + \varepsilon\left(3\Gamma\partial_{XX} v + \frac{8}{Re^2}\partial_{XXXX} v\right) = 0.$$

A model problem: Ginzburg Landau equation

Ginzburg Landau equation

$$\partial_T A = \partial_{XX} A + A - A|A|^2.$$

- Periodic stationary waves: $A_{per}[q,\phi_0](X) = \sqrt{1-q^2}^{iqX+i\phi_0}$.
- Polar coordinates: $A(X,T) = r(X,T)e^{i\phi(X,T)}$:

$$\partial_T r = \partial_{XX} r + r - (\partial_X \phi)^2 r - r^3, \quad \partial_T \phi = \partial_{xx}\phi + \frac{2(\partial_X r)(\partial_X \phi)}{r}.$$

- A closed system on $(r,q) = (r, \partial_X \phi)$:

$$\partial_T r = \partial_{XX} r + r - q^2 r - r^3, \quad \partial_T q = \partial_{XX} q + 2\partial_X \left(\frac{(\partial_X r)q}{r} \right). \qquad (1)$$

Ginzburg Landau: spectral stability of periodic waves

- Periodic solutions $A_{per}(q_0, \phi_0)$= steady states $(r, q) = (\sqrt{1 - q_0^2}, q_0)$ of (1).
- Linearized equations ($r_0 = \sqrt{1 - q_0^2}$):

$$\partial_T \dot{r} = \partial_{XX} \dot{r} - 2r_0^2 \dot{r} - 2q_0 r_0 \dot{q}, \quad \partial_T q = \partial_{XX} \dot{q} + \frac{2q_0}{r_0} \partial_{XX} \dot{r}.$$

- Dispersion relation:

$$(\lambda + k^2 + 2(1 - q_0^2))(\lambda + k^2) - 4q_0^2 k^2 = 0.$$

- **Remark:** the spectrum is continuous (no point spectrum).
- Spectral stability: $q_0 < 1/\sqrt{3}$ (sufficient condition).
- Expansion of the eigenvalue bifurcating from $(\lambda, k) = (0, 0)$

$$\lambda(k) = -\frac{(1 - 3q_0^2)}{1 - q_0^2} k^2 + O(k^4).$$

Ginzburg Landau: phase dynamics/spectral stability

- Ansatz: $r(X,T) = \tilde{r}(\delta^2 T, \delta X), q(X,T) = \tilde{q}(\delta^2 T, \delta X)$

$$\tilde{r}(1-\tilde{q}^2-\tilde{r}^2) = O(\delta^2), \quad \partial_\tau \tilde{q} = \partial_{\xi\xi}\tilde{q} + \partial_\xi\left(\tilde{q}\frac{\partial_\xi \tilde{r}}{\tilde{r}}\right).$$

- Modulation equation ($\tilde{r} = \sqrt{1-\tilde{q}^2}$):

$$\partial_\tau \tilde{q} = \partial_\xi\left(\frac{1-3\tilde{q}^2}{1-\tilde{q}^2}\partial_\xi \tilde{q}\right) + O(\delta^2).$$

- Dispersion relation:

$$\tilde{\lambda}_{mod}(k) = -\frac{1-3q_0^2}{1-q_0^2}k^2.$$

Phase Dynamics/Spectral Stability

$$\lambda(k) = \tilde{\lambda}_{mod}(k) + O(k^4).$$

Phase Dynamics/Spectral stability: applications

- Spectral stability results: approximation by GL equations
  1. Stability of small amplitude periodic waves in SH (Mielke, 97)
  2. Stability of small rolls in Rayleigh Bnard convection (Mielke, 97)
  3. Stability of small rolls in Taylor Couette flows (Schneider, 98)
- Spectral stability results: parabolic equations (reaction diffusion)
  1. Structure of the spectrum: Gardner ('93)
  2. Large period results: expansion of the Evans function near $(0, 0)$, computation of a stability index (Gardner/Sandstede).

Large period results

As $L \to \infty$, assume periodic waves converge to a **stable** solitary wave. If the tail is "monotone", necessary conditions of stability are satisfied. If the tail is "oscillatory", there are ranges of unstable periodic wave trains

- Modulation equations (phase dynamics):
  1. Formal phase dynamics: SH/RB (Cross/Hohenberg '84)
  2. Rigorous connection: GL (Schneider '98), RD (DSSS '09)

General case: KdV/KS as a model study

KSKdV

$$\partial_t u + \partial_x(f(u)) + \varepsilon\partial_x^3 u + \delta\left(\partial_x^2 u + \partial_x^4 u\right) = 0, \quad x \in \mathbf{R}, t > 0.$$

Parameters: $\varepsilon > 0$, $\delta > 0$.

Particular case: Si $f(u) = u^2/2$

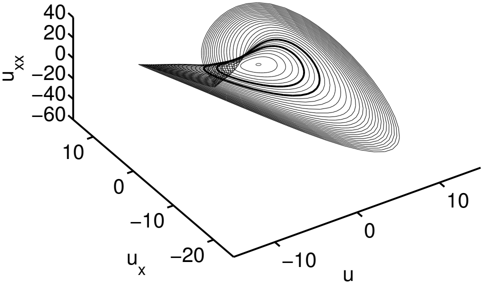
- $\varepsilon = 0$, $\delta = 1$: classical Kuramoto-Sivashinsky equation
- $\varepsilon = 1$, $\delta = 0$: Korteweg-de Vries equation (water-waves).
- $\varepsilon = 1$, $0 < \delta \ll 1$: thin film equation .

Remark: ALL steady states UNSTABLE.

Periodic Wavetrains I

Travelling waves: $u(x,t) = U(\xi), \xi = x - ct$.

$$f(U) - cU + \varepsilon U'' + \delta\left(U' + U'''\right) = q, \quad q \in \mathbf{R}.$$



Periodic wavetrains bifurcating from a Hopf point. Periodic waves near Hopf point and solitary waves are unstable (constant states unstable)

Periodic Wavetrains II.

Geometric assumption:
$H : \mathbf{R}^6 \to \mathbf{R}^3, (X, b, c, q) \mapsto (U, U', U'')(X, b, c, q) - b$ where $U(., b, c, q)$ is the unique solution of the profile equation such that $U(0, b, c, q) = b$ is a submersion at $(\bar{X}, \bar{b}, \bar{c}, \bar{q}) \in H^{-1}(0)$.

Structure pf periodic wavetrains:

$$U(x - \alpha - c(\beta)t, \beta), \quad \alpha \in \mathbf{R}, \beta \in \mathbf{R}^2$$

Parameterization of periodic waves:

- Natural parameters: X (period), q (constant of integration)
- Modulation parameters: $k = X^{-1}$ (wavenumber), $M = \langle U \rangle$ (spatial mean).

Spectral problem: Bloch Transform

- Linearized (KdV/KS) about $\bar{U} = U(.,\bar{X},\bar{q})$:

$$\partial_t v = Lv := -\partial_x \left((f'(\bar{U}) - \bar{c})v \right) - \varepsilon \partial_x^3 v - \delta \left(\partial_x^2 v + \partial_x^4 v \right),$$

- Spectrum of a differential operator with **periodic coefficients**?
- Bloch Transform ($X = 1$):

$$\check{u}(\xi, x) = \sum_k e^{2i\pi kx} \hat{u}(\xi + 2\pi k).$$

Properties

$$\check{u}(\xi, x+1) = \check{u}(\xi, x), \quad \check{u}(\xi + 2\pi, x) = e^{-2i\pi x} \check{u}(x, \xi),$$

$$\check{\partial_x u}(\xi, x) = (\partial_x + i\xi)\check{u}(\xi, x),$$

Suppose that A is a 1-periodic function

$$(\check{Au})(x, \xi) = A(x)\check{u}(\xi, x).$$

Formulation of the spectral problem

- Bloch in space/Laplace in time transform of linearized (KdV/KS)

Spectral problem

$$(L_\xi - \lambda)\check{v} = 0, \quad \text{with} \quad L_\xi := e^{-i\xi.x} L e^{i\xi.x}.$$

$$\mathrm{spec}_{L^2(\mathbf{R})} L = \bigcup_{\xi \in]-\pi,\pi]} \mathrm{spec}(L_\xi) = \bigcup_{\xi \in]-\pi,\pi]} \{\lambda_j(\xi),\ j \in \mathbf{N}\}\,.$$

- **Remark 1:** 0 is an eigenvalue of L_0 with algebraic multiplicity $\geqslant 2$.

$$L_0[\bar{U}'] = 0 \quad \text{translation invariance,}$$
$$L_0[1] = \bar{U}' \quad \text{galilean invariance/mass conservation.}$$

- **Remark 2:** In reaction diffusion eqs, RB convection, TC flows, 0 is a simple eigenvalue of L_0 (analysis is simpler).

Stability of periodic waves of conservation laws

- The structure of periodic waves is richer than in RD setting
- Phase Dynamics replaced by Modulation theory
  1. Formal derivation: KdV (Whitham, '65), Euler-Korteweg (Gavrilyuk, Serre '96), KS (Frisch, She, Thual '98).
  2. Viscous Conservation laws (Serre: '05): connection modulation equations/expansion of the Evans function near $(0, 0)$
  3. Other applications: NLS (Schneider/Dull), GKdV (Zumbrun/Johnson), St Venant, KdV/KS (N/Rodrigues)
- Spectral Stability results KdV/KS
  1. KdV: spectral stability (pure imaginary) [Bottman/Deconinck '09]
  2. KdV/KS (limit $\delta \to 0$): partial spectral stability [Bar/Nepomnyashchy '95, general perturbations], [Ercolani, McLaughlin, Roitner '98, periodic perturbations]
  3. Direct numerical simulations: (Chang/Demekhin '02)
- **Aim: use modulation equations to study spectral stability**

Some "Necessary" Spectral Stability Assumptions

- **Assumption 2 (Geometric Assumption) :** 0 is non semi simple eigenvalue of multiplicity 2.

Expansions of 2 eigenvalues bifurcating from $(\lambda, \xi) = (0, 0)$

Near $\xi = 0$, $\mathrm{spec}(L_\xi) = \left\{ \lambda_j(\xi) = i\,\lambda_j^0\,\xi + o(\xi),\ j = 1, 2 \right\}$.

- **Assumption 3 (Hyperbolicity):** $\lambda_j^0 \in \mathbf{R}, j = 1, 2$

- **Assumption 4 (Dissipation):** $\mathrm{Re}\,(\mathrm{spec}(L_\xi)) \leqslant -\theta|\xi|^2$.

Main Purpose: Interpretation of Assumption 3,4 with the help of (Whitham's) modulation equations.

Modulations: derivation Whitham's equations I

Aim. Determine the evolution of large scale perturbations.

"Hyperbolic" scaling (Serre '05): $(T, X) = (\eta t, \eta x), \quad \eta \ll 1.$

$$\partial_T u + \partial_X(\frac{u^2}{2}) + \varepsilon\eta^2\partial_X^3 u + \delta\left(\eta\partial_X^2 u + \eta^3\partial_X^4 u\right) = 0.$$

Ansatz (Non linear WKB computation):

$$u(T, X) \; = \; \sum_j \eta^j u^j\left(\frac{\phi(T, X)}{\eta}; T, X\right)$$

with $u^j(y; T, X)$ 1-periodic in y and

$$\phi(T, X) \; = \; \sum_j \eta^j \phi^j(T, X) \; .$$

Modulations: derivation of Whitham's equations II.

- Collect $O(\eta^{-1})$ terms

$$\omega_0 \partial_y u^0 + k_0 u^0 \partial_y u^0 + \varepsilon k_0^3 \partial_y^3 u^0 + \delta \left(k_0^2 \partial_y u^0 + k_0^4 \partial_y^4 u^0 \right) = 0,$$

with $k_0 = \partial_X \phi^0$, $\omega_0 = \partial_T \phi 0 = -k_0 c(k_0, M_0)$ and $M_0 = \langle u^0 \rangle$.

- One has $u^0(y, X, T) = U(y, k_0(X, T), M_0(X, T))$.

- Compatibility condition $\partial_T k_0 = \partial_X \omega_0$ yields

Whitham's modulation equation I

$$\partial_T k_0 + \partial_X \left(k_0 \, c(k_0, M_0) \right) = 0.$$

Modulations: derivation of Whitham's equations III.

- Collect $O(1)$ terms:

$$L_0[u^1 - k^1 \partial_k u^0] + (\omega_1 + k_1 \partial_k (kc)^0)\partial_y u^0 + \partial_T u^0 + u^0 \partial_X u^0 + \partial_y g^0(y, X, T) = 0.$$

- The solvability conditions yields

Whitham's modulation equation II

$$\partial_T \langle u^0 \rangle + \partial_X \langle \frac{(u^0)^2}{2} \rangle = 0.$$

Modulation Equations/Spectral Stability: Hyperbolicity

Theorem

The eigenvalues $\lambda_j(\xi), j = 1, 2$ bifurcating from $(0, 0)$ expand as:

$$\lambda_j(\xi) = i\bar{k}\xi\,\lambda_j^0 + o(\xi), \quad j = 1, 2,$$

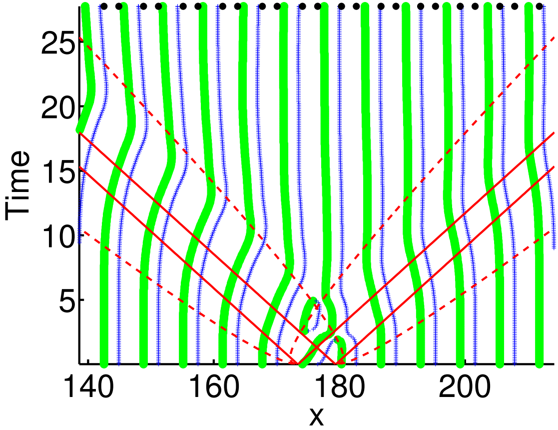
where $\lambda_j^0$are the characteristic wave speeds at $(\bar{k}, \bar{M})$ of

Whitham's modulation system

$$\partial_T k + \partial_X \left(k\, c(k, M)\right) = 0, \quad \partial_T M + \partial_X \langle \frac{U(., k, M)^2}{2} \rangle = 0.$$

- Assumption 4 is equivalent to hyperbolicity of Whitham's equations.
- Assuming **strict hyperbolicity**: eigenvalues are **analytic** in ξ

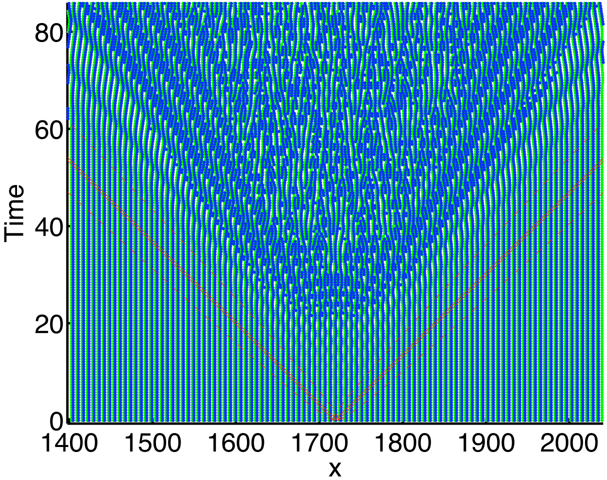
Numerical validation: direct simulations



Time evolution: Stable case

Red line: characteristics of the modulation equations

Numerical validation: direct simulations



Time evolution: one unstable case (loss of dissipativity)
Red line: characteristics of the modulation equations

Modulations: higher order corrections I

Equation on u^1 reads (with $L_0[f_0] + \partial_T u^0 + u^0 \partial_X u^0 + \partial_y g^0 = 0$)

$$L_0[u^1 - k_1 \partial_k u^0 + f_0 - \langle f_0 \rangle + M_1] + (\omega_1 + d(kc)_0[k_1, M_1] + k_0 \langle f_0 \rangle) \partial_y u^0 = 0.$$

- One choses M_1 so that $\omega_1 + d(kc)_0[k_1, M_1] + k_0 \langle f_0 \rangle = 0$.
- Then $u^1 = du^0[k_1, M_1] + f_0 - \langle f_0 \rangle + A_1 \partial_y u^0$.
- The compatibility condition $\partial_T k_1 = \partial_X \omega_1$ reads

Linearized Whitham eqs with r.h.s I

$$\partial_T k_1 + \partial_X \left(d(kc)_0[k_1, M_1] \right) = -\partial_X \langle k_0 f_0 \rangle,$$

with $f_0 := F_0(k_0, M_0)[\partial_X k_0, \partial_X M_0, \partial_T k_0, \partial_T M_0]$

Modulation: higher order corrections II

- Collect $O(\eta)$ terms:

$$L_0[u^2] + \partial_T u^1 + \partial_X(u^0 u^1) + \delta\partial_{XX} u^0 + \partial_y g^1(y, X, T) = 0.$$

- The solvability condition yields

Linearized Whitham's modulation quation with r.h.s. II

$$\partial_T M_1 + \partial_X \left(d\langle \frac{(u^0)^2}{2} \rangle [k_1, M_1] \right) = \partial_X \langle u^0 (f_0 - \langle f_0 \rangle) \rangle - \delta\partial_{XX} M_0.$$

- The linearized modulation equations are well posed.

- **Continuing: approximate solutions of KdV/KS to any order**

Viscous modulation equations/Spectral stability

- Let $k = k_0 + \eta k_1$ and $M = M_0 + \eta M_1$, and drop $O(\eta^2)$ terms:

Viscous Whitham's modulation equations

$$\partial_T k + \partial_X \left(kc(k,M)\right) = \eta \partial_X \left(L_1(k,M)[\partial_X k, \partial_X M]\right),$$
$$\partial_T M + \partial_X \langle \frac{U(k,M)^2}{2} \rangle = \eta \partial_X \left(L_2(k,M)[\partial_X k, \partial_X M]\right),$$

- Relation with the spectral problem:

Theorem

Let $\bar{\Lambda}_j(\xi), j = 1,2$ (ξ classical Fourier coeff) the eigenvalues of linearized viscous Whitham's equations about $(\bar{k}, \bar{M})$. Then:

$$\lambda_j(\xi) = \bar{\Lambda}_j(\xi) + O(\xi^3), \quad j = 1,2, \quad |\xi| \ll 1.$$

- Assumption 5: dissipativity of the viscous modulation system.

The complete set of spectral assumptions

- **Spectral Assumptions**
  1. (G0) Poincaré return map (defining periodic wave trains) is full rank.
  2. (D1) $\sigma(L_\xi) \subset \{\mathrm{Re}(\lambda) < 0\}$ for $\xi \neq 0$
  3. (D2) $\mathrm{Re}\sigma(L_\xi) \leqslant -\theta\xi^2$, $\theta > 0$, for $|\xi|$ sufficiently small
  4. (D3) 0 is an eigenvalue of algebraic multiplicity 2
  5. (D4) $\lambda_j(\xi) = ia_j\xi + o(\xi), j = 1, 2$ with $a_1 \neq a_2$

- **Remark:** in the reaction diffusion setting, (D3),(D4) substituted by 0 simple eigenvalue (or semi simple if more symmetries)

- **Part II (L.-M. Rodrigues):** one can prove "nonlinear stability" under these assumptions

- **Aim:** verify (numerically/analytically) these assumptions.

Numerical analysis of the spectral problem: Evans function

If $Lv = \lambda v$ is written $Y' = H(.,\lambda)Y$ and $\Phi(.,\lambda)$ is the resolvente matrix:

Evans Function definition

$$\lambda \in \mathrm{spec}(L_\xi) \quad \text{iff} \quad D(\lambda,\xi) = \det\left(\Phi(X,\lambda) - e^{i\xi.X} Id\right) = 0,$$

- If $\Gamma = \partial\Omega$ is a closed contour, the number of zeros of D insideΩ is

$$n(\xi,\Gamma) := \frac{1}{2\pi i}\int_\Gamma \frac{\partial_\lambda D(\lambda,\xi)}{D(\lambda,\xi)} d\lambda.$$

- Expansion of eigenvalues: one expands D near $(\lambda,\xi) = (0,0)$:

$$\partial_\lambda^r \partial_\xi^s D(0,0) = \frac{r!s!}{4\pi^2}\int_{\partial B(0,h)}\int_{\partial B(0,h)} D(\mu,k)\mu^{-r-1}k^{-s-1} d\mu dk.$$

Numerical proof of spectral stability.

- **Step 1:** A priori estimates on possible unstable eigenvalues

Proposition

There exists$R_0 > 0$ such that $\mathrm{spec}(L) \cap \{\lambda \in \mathbf{C} \,|\, \mathrm{Re}(\lambda) \geqslant 0\} \subset B(0, R_0)$.

- **Step 2:** Let $0 < r_0 < R_0$ and $\Omega_0 := B(0, R_0)/B(0, r_0)$

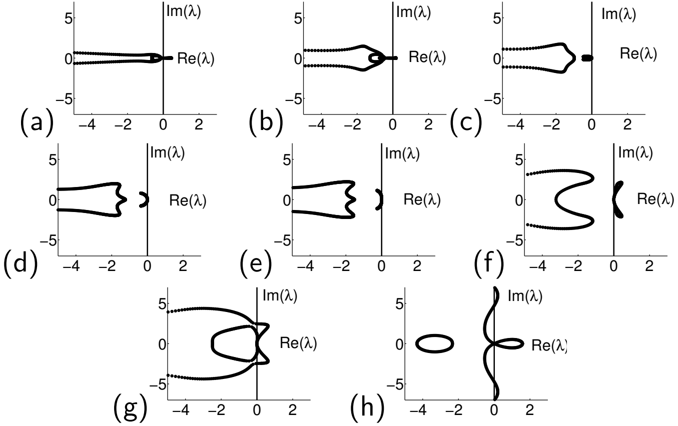
$$\Gamma_0 := \partial \left(\Omega_0 \cap \{\lambda \in \mathbf{C} | \mathrm{Re}(\lambda) \geqslant 0\}\right), \quad \Gamma_1 := \partial \left(B(0, R_0) \cap \{\lambda \in \mathbf{C} | \mathrm{Re}(\lambda) \geqslant 0\}\right).$$

Verify $n(\xi, \Gamma_0) = 0, \forall \xi \in [-\pi, \pi[$ et $n(\xi, \Gamma_1) = 0, k_0 \leqslant |\xi| \leqslant \pi,\ k_0 \ll 1$.

- **Step 3:** for $r_0 < r_1 < R_0$, verify $n(\xi, \partial B(0, r_1)) = 2$ if $|\xi| < k_0$
- **Step 4:** Expansion of D to expand eigenvalues near $(0, 0)$

$$\lambda_j(\xi) = i\lambda_j^0 \xi + \beta_j \xi^2 + O(\xi^3).$$

Numerical Results (SpectrUW: Hill's method)



There is a range of stable periodic wave trains of KdV/KS

Numerical results: stable bands

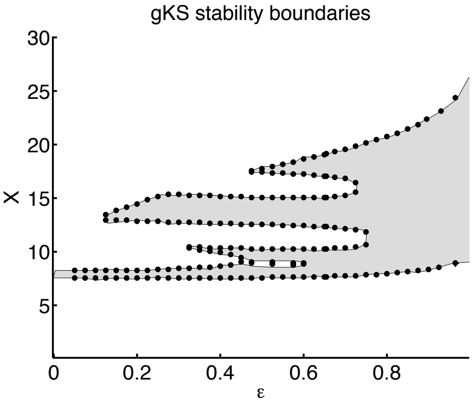


Figure: Stable bands of periodic wave trains for $\varepsilon^2 + \delta^2 = 1$

Remark: In the KdV limit, it seems there is a limit stability band

The KdV limit: modulation revisited I

In (KdV/KS): $\delta = \eta\bar{\delta}$, $(X, T) = (\eta x, \eta t)$ with $\eta \ll 1$

$$\partial_T u + u\partial_X u + \eta^2 \partial_X^3 u + \bar{\delta}\left(\eta^2 \partial_X^2 u + \eta^4 \partial_X^4 u\right) = 0.$$

Ansatz: $u(X, T) = U^{(0)}(\phi(X, T)/\eta, X, T) + \eta U^{(1)}(\phi(X, T)/\eta, X, T)$.

Collect $O(\eta^{-1})$ terms: **KdV cnoidal waves**

$$U^{(0)} = u_0(X, T) + 12k(X, T)^2 p(X, T)^2 \mathrm{cn}^2(k(X, T)y, p(X, T)),$$
$$c_0(u_0, k, p) = u_0(X, T) + 8k(X, T)^2 p(X, T)^2 - 4k(X, T)^2.$$

Compatibility condition: $\partial_T \partial_X \phi = \partial_X \partial_X \phi$:

Whitham's equation I

$$\partial_T k + \partial_X \left(k\, c_0(u_0, k, p)\right) = 0.$$

The KdV limit: modulation revisited II

Collect $O(1)$ terms: there are **two** solvability conditions

Whitham's modulation equation II and III

$$\partial_T\langle U_0\rangle + \partial_X\langle \frac{U_0^2}{2}\rangle = 0,$$
$$\partial_T\langle \frac{U_0^2}{2}\rangle + \partial_X\langle (\frac{U_0^3}{3} - \frac{3(U_0')^2}{2})\rangle = \bar{\delta}\left(\langle (U_0')^2\rangle - \langle (U_0'')^2\rangle\right).$$

Remark

1. Limit $\bar{\delta} \to 0$: one recovers modulation equations
2. Limit $\bar{\delta} \to \infty$ (limit of relaxation in the modulation system): we get the modulation KdV/KS system with $\delta \to 0$

Subcharacteristic conditions I

- Parameterize KdV profile by k, $M = \langle U_0 \rangle$ and $E = \langle U_0^2/2 \rangle$
- Write modulation equations as

$$\partial_T k + \partial_X(kc_0(k,M,E)) = 0, \quad \partial_T M + \partial_X E = 0,$$
$$\partial_T E + \partial_X F(k,M,E) = \bar{\delta} S(k,M,E).$$

- **Whitham's modulation equation for KdV is hyperbolic:**

$$\alpha_1(k,M,E) < \alpha_2(k,M,E) < \alpha_3(k,M,E)$$

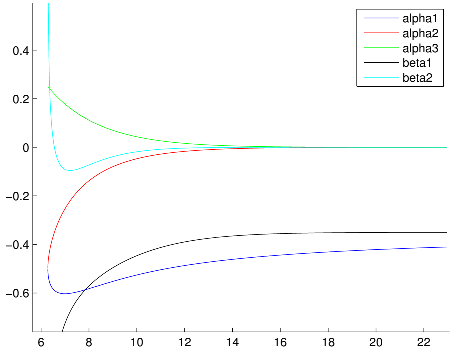
Subcharacteristic condition

1. $\partial_E S(k^*, M^*, E^*) < 0$
2. Subcharacteristic condition 2: **Relaxed system is hyperbolic**

$$\beta_1(k^*, M^*, E^*) < \beta_2(k^*, M^*, E^*) \quad \text{with} \quad S(k^*, M^*, E^*) = 0.$$

3. $\alpha_1^* < \beta_1^* < \alpha_2^* < \beta_2^* < \alpha_3^*$.

Subcharacteristic conditions II



Characteristic wave speeds of KdV Whitham and relaxed modulation equations

Spectral stability in the KdV limit

A priori estimates: unstable eigenvalues are $O(1)$

Case $\delta \ll |\xi|$: expansion of eigenvalues (Bar/Nepomnyaschy '95)

$$\lambda(\xi,\delta) = \lambda_{kdv}(\xi) + \delta\lambda_1(\xi) + O(\delta^2), \quad \lambda_1(\xi) \in \mathbf{R}$$

Theorem (case $\xi^2 + \delta^2 \ll 1$)

Spectral stability assumptions (D2), (D3), (D4) are satisfied if and only if sub characteristic conditions (S1),(S2),(S3) are satisfied

(D1) satisfied if $\lambda_1(\xi) < 0$ (Bar/Nepomnyaschy '95).

Conclusion: There is a range of spectrally stable eigenvalues (X_m, X_M) for δ small enough and $X_m \approx 8$, $X_M \approx 26$.

Conclusions and perspectives

Remark. Analysis adapted for Shallow Water equations:

$$\partial_t h + \partial_x(hu) = 0, \quad \partial_t(hu) + \partial_x(hu^2 + \frac{h^2}{2F}) = h - u^2 + \nu\partial_x(h\partial_x u).$$

Perspectives.

- **Main question: obtain analytical spectral stability results**
  1. First step (done): Spectral stability for KdV/KS in the KdV limit
  2. Shallow Water eqs/Navier-Stokes eqs: transition to instability (like Taylor-Couette/Rayleigh Bénard with Ginzburg Landau)
  3. An other limit: vanishing viscosity limit in St Venant eqs (convergence to explicit profiles)
- Periodic waves in other models.
  1. Viscous Saint Venant with surface tension (comparison with Liu/Schneider/Gollub experiments)
  2. Related (dispersive) models: Euler-Korteweg, water-waves models (Boussinesq)
  3. 2d motions: extended Whitham's modulation equations.