Stability of periodic wavetrains: part II, nonlinear stability and asymptotic behavior

Pascal Noble & L.Miguel Rodrigues

Institut Camille Jordan
Université Lyon 1
France

Basque meeting on PDE's,
January 2012

General aim.

Goal: Under assumptions of non degeneracy and spectral stability, prove stability and examine asymptotic behavior.

Expectations: Asymptotic behavior is of slow modulation type, obeying at first-order a Whitham system. The nature of this system is determinant.

Outline:

Model case: reaction-diffusion systems.

$t > 0$ time, $x \in \mathbf{R}$ space, $u(x,t) \in \mathbf{R}^d$ unkown.

Reaction diffusion

$$u_t = f(u) + D\, u_{xx}.$$

D positive definite. $D = I_d$ for simplicity.

f regular.

Periodic traveling wave.

Wave

$$u(x,t) \;=\; \bar{u}(k(x-ct)),$$

with $\bar{u}$ of period 1.

k wavenumber,
c phase velocity,
$\omega = -kc$ time frequency.

Wave parametrization.

Profile equation:

$$-kc\bar{u}' = f(\bar{u}) + k^2\bar{u}''$$

$\bar{u}$ of period 1.

Assumption (H)

Poincaré return map is full-rank and a parametrization by shift α and wavenumber k

$$\bar{u}^k(\cdot - \alpha), \qquad c(k)$$

is available.

Non-localized perturbation.

Fix a wave with $k_\star$ and $c_\star$.

Co-moving frame

$$(k_\star(x - c_\star t), k_\star t).$$

Goal: allow $\bar{u} \circ \Psi$ with $\|\partial_x(\Psi - \mathrm{Id})\|_{L^1(\mathbf{R})} << 1$.

Initial datum

$\tilde{u}_0$ such that there is an h_0 such that

$$\|\tilde{u}_0(\cdot - h_0(\cdot)) - \bar{u}\|_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} + \|\partial_x h_0\|_{L^1(\mathbf{R})\cap H^K(\mathbf{R})} << 1.$$

Some insights through formal (incorrect) expectations.

Expected behavior

Slow motion along the set of periodic traveling waves,

$$\bar{u}^{k_\star \partial_x \Psi(x,t)}(\Psi(x,t)), \qquad \Psi \ \text{low-frequency},$$

obeying averaged equations.

Expected decay

Main perturbation term,

$$(\Psi(t) - \mathrm{Id})\ \bar{u}' \sim_{L^p(\mathbf{R})} t^{\frac{1}{2p}}.$$

Averaged modulation equations.

Approximate modulation equation for the local wavenumber κ,

$$k_\star \kappa_t - k_\star(\omega(\kappa) + c_\star \kappa)_x = k_\star^2 (d(\kappa)\kappa_x)_x.$$

$\kappa = k_\star \Psi_x$.

Approximate modulation equation for the local phase Ψ,

$$k_\star \Psi_t - (\omega(k_\star \Psi_x) + c_\star k_\star \Psi) = k_\star^2 (d(k_\star \Psi_x)\Psi_{xx}).$$

Averaged modulation equations: quadratic approximants.

Approximate equations for perturbations.

$(W)_k$

$$k_\star k_t - k_\star\Big((\omega'_\star + c_\star)k + \frac{1}{2}\omega''_\star k^2\Big)_x = k_\star^2 d_\star k_{xx}.$$

Linear group velocity: $\omega'_\star = \omega'(k_\star)$. Linear group diffusion: $d_\star = d(k_\star)$.

$(W)_h$

$$k_\star h_t - \Big((\omega'_\star + c_\star)k_\star h_x + \frac{1}{2}\omega''_\star k_\star^2 h_x^2\Big) = k_\star^2 d_\star h_{xx}.$$

Some assumptions and a normalization.

(H) Parametrization by wavenumber.

(D1) Critical spectrum is reduced to $\{0\}$.

(D2) Diffusive spectral stability.

(D3) Minimal dimension of the generalized co-periodic kernel.

(N) Suitable normalization of parametrization (shift).

Johnson-Noble-Rodrigues-Zumbrun (preprint 2011).

Theorem (Stability)

Let $\eta > 0$ *and* $K \geqslant 3$. *Let*

$$E_0 := \|\tilde{u}_0(\cdot - h_0(\cdot)) - \bar{u}(\cdot)\|_{L^1(\mathbf{R}) \cap H^K(\mathbf{R})} + \|\partial_x h_0\|_{L^1(\mathbf{R}) \cap H^K(\mathbf{R})} << 1$$

for some h_0. *Then, there exists* $\tilde{u}$ *with initial data* $\tilde{u}_0$ *and a phase function* ψ *such that, for* $t > 0$ *and* $2 \leqslant p \leqslant \infty$,

$$\begin{aligned} \|\tilde{u}(\cdot - \psi(\cdot, t), t) - \bar{u}(\cdot)\|_{L^p(\mathbf{R})} &\lesssim E_0 (1+t)^{-\frac{1}{2}(1-1/p)} \\ \|\nabla_{x,t}\psi(t)\|_{L^p(\mathbf{R})} &\lesssim E_0 (1+t)^{-\frac{1}{2}(1-1/p)}, \end{aligned}$$

and

$$\|\tilde{u}(t) - \bar{u}\|_{L^\infty(\mathbf{R})}, \quad \|\psi(t)\|_{L^\infty(\mathbf{R})} \lesssim E_0.$$

Johnson-Noble-Rodrigues-Zumbrun (preprint 2011).

Theorem (Asymptotic behavior)

Moreover, let k and h satisfy $(W)_k$ and $(W)_h$ with initial data $k|_{t=0} = k_\star \partial_x h_0$, $h|_{t=0} = h_0$. Then, for $t > 0$, $2 \leqslant p \leqslant \infty$,

$$\|\tilde{u}(\cdot - \psi(\cdot,t),t) - \bar{u}^{k_\star(1+\psi_x(\cdot,t))}(\,\cdot\,)\|_{L^p(\mathbf{R})} \lesssim E_0 \ln(2+t)\ (1+t)^{-\frac{3}{4}},$$

$$\|k_\star \psi_x(t) - k(t)\|_{L^p(\mathbf{R})} \lesssim E_0 (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}+\eta},$$

$$\|\psi(t) - h(t)\|_{L^p(\mathbf{R})} \lesssim E_0 (1+t)^{-\frac{1}{2}(1-1/p)+\eta}.$$

Transform.

Fourier transform:

$$\hat{g}(\xi) := \frac{1}{2\pi} \int_{\mathbf{R}} e^{-i\xi x} g(x) dx.$$

From Floquet theory: piece together modes corresponding to the same periodic shift.

Bloch transform

$$\check{g}(\xi, x) \;=\; \sum_{j \in \mathbf{Z}} e^{\mathrm{i}\, 2j\pi x}\, \widehat{g}(\xi + 2j\pi)$$

$\check{g}(\xi)$ of period 1 with Floquet parameter $\xi \in [-\pi, \pi]$.

Inverse Bloch transform

$$g(x) \;=\; \int_{-\pi}^{\pi} e^{\mathrm{i}\xi x}\, \check{g}(\xi, x)\, \mathrm{d}\xi\ .$$

Bloch transform: simple observations.

Observations

- If g is periodic of period 1

$$(g\,h)\check{}\,(\xi,x) \;=\; g(x)\,\check{h}(\xi,x).$$

-

$$(\partial_x g)\check{}\,(\xi,\cdot) \;=\; (\partial_x + \mathrm{i}\xi)\check{g}(\xi,\cdot).$$

First consequence: if g is periodic of period 1 and h is **low-frequency**

$$(g\,h)\check{}\,(\xi,x) \;=\; g(x)\,\widehat{h}(\xi).$$

Provides two-scale analysis. Slow modulation equations from averaging.

Bloch symbol.

If L is a differential operator with 1-periodic coefficients then so are

$$(L_\xi g)(x) \;=\; e^{-\mathrm{i}\xi x}\, L(e^{\mathrm{i}\xi\cdot} g(\cdot))(x), \qquad \xi \in [-\pi,\pi],$$

and

$$(Lg)(x) \;=\; \int_{-\pi}^{\pi} e^{\mathrm{i}\xi x}\, (L_\xi \check{g}(\xi,\cdot))(x)\, \mathrm{d}\xi\ .$$

Here we use it with

$$Lg \;=\; k_\star g_{xx} + k_\star^{-1} \mathrm{d}f(\bar{u}) g + c_\star g_x .$$

Spectral splitting:

$$\sigma(L) \;=\; \overline{\bigcup_{\xi\in[-\pi,\pi]} \sigma_{per}(L_\xi)}.$$

Union of discrete spectra.

Spectral stability assumptions.

(D1) $\sigma(L) \subset \{\lambda \mid \mathrm{Re}\lambda < 0\} \cup \{0\}$.

(D2) For some $\theta > 0$, any $\xi \in [-\pi, \pi]$,

$$\sigma_{per}(L_\xi) \;\subset\; \{\lambda \mid \mathrm{Re}\lambda \leqslant -\theta|\xi|^2\}.$$

(D3) $\lambda = 0$ is a simple eigenvalue of L_0.

Outline:

1. Modulations of scalar type
    - Nonlocalized data: statement
    - Nonlocalized data: into the proof
    - Comments and further references
2. Modulations of system type
    - Some comments
    - A spectral lemma
3. Open questions

Structure of the proof.

1. Separation of the critical mode: introduction of the phase.
2. Nonlinear iteration for stability's proof.

3. Refined separation.
4. Estimate of the remainder.
5. Comparison with (W).

Spectral preparation.

Operator:

$$Lg = k_\star g_{xx} + k_\star^{-1} \mathrm{d}f(\bar{u})g + c_\star g_x.$$

Semi-group: $S(t) = e^{tL}$.

Critical eigen elements of L_ξ

For ξ sufficiently small, $\lambda(\xi)$; $\phi(\xi)$, $\tilde{\phi}(\xi)$ dual right-left.

Normalizations: $\phi(0) = \bar{u}'$;

$$\langle \tilde{\phi}(0), \phi(\xi) \rangle_{L^2([0,1])} = 1.$$

Separation: semi-group level.

Expand

$$\phi(\xi) \;=\; \bar{u}' + \mathcal{O}(\xi).$$

Then introduce

$$(s^{\mathrm{p}}(t)g)(x) \;=\; \int_{-\pi}^{\pi} e^{\mathrm{i}\xi x}\;\alpha(\xi)\;e^{\lambda(\xi)t}\;\langle\tilde{\phi}(\xi),\check{g}(\xi)\rangle_{L^2([0,1])}\mathrm{d}\xi$$

with α a high-Floquet cut-off, and split

$$(S(t)g)(x) \;=\; \bar{u}'(x)\;(s^{\mathrm{p}}(t)g)(x) + (\tilde{S}(t)g)(x).$$

Some of the estimates.

For all $t > 0$, $2 \leqslant p \leqslant \infty$,

$0 \leqslant n \leqslant K+1$

$$\left\|\partial_x^l \partial_t^m s^{\mathrm{p}}(t)\partial_x^n g\right\|_{L^p(\mathbf{R})} \lesssim \min\begin{cases}(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}}\|g\|_{L^1(\mathbf{R})},\\ (1+t)^{-\frac{1}{2}(1/2-1/p)-\frac{l+m}{2}}\|g\|_{L^2(\mathbf{R})};\end{cases} \tag{1}$$

and for some $\theta' > 0$

$0 \leqslant l+2m, n \leqslant K+1$

$$\left\|\partial_x^l \partial_t^m \tilde{S}(t)\partial_x^n g\right\|_{L^p(\mathbf{R})} \lesssim \min\begin{cases}(1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}-\frac{1}{2}}\|g\|_{L^1(\mathbf{R})\cap H^{l+2m+1}(\mathbf{R})},\\ e^{-\theta' t}\|\partial_x^n g\|_{H^{l+2m+1}(\mathbf{R})} + (1+t)^{-\frac{1}{2}(1/2-1/p)-\frac{l+m}{2}-\frac{1}{2}}\|g\|_{L^2(\mathbf{R})}.\end{cases} \tag{2}$$

Some of the estimates: nonlocalized data.

For all $t > 0$, $2 \leqslant p \leqslant \infty$,

$l + m \geqslant 1$ or ($l = m = 0$ and $p = \infty$)

$$\|\partial_x^l \partial_t^m s^{\mathrm{p}}(t)(h_0 \bar{u}')\|_{L^p(\mathbf{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)+\frac{1}{2}-\frac{l+m}{2}} \|\partial_x h_0\|_{L^1(\mathbf{R})}; \qquad (3)$$

$0 \leqslant l + 2m \leqslant K + 1$

$$\|\partial_x^l \partial_t^m \tilde{S}(t)(h_0 \bar{u}')\|_{L^p(\mathbf{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}} \|\partial_x h_0\|_{L^1(\mathbf{R}) \cap H^{l+2m+1}(\mathbf{R})}. \qquad (4)$$

Bloch transform: estimates.

Parseval's identity

$$\|g\|_{L^2(\mathbf{R})} = \sqrt{2\pi}\|\check{g}\|_{L^2([-\pi,\pi],L^2([0,1]))}.$$

Haussdorf-Young's inequalities

$$\|g\|_{L^p(\mathbf{R})} \leqslant \|\check{g}\|_{L^{p'}([-\pi,\pi],L^p([0,1]))}, \quad 2 \leqslant p \leqslant \infty.$$

Some of the estimates: one proof.

$$\partial_x(s^{\mathrm{p}}(t)(\bar{u}'h_0))(x)$$

$$= \int_{-\pi}^{\pi} e^{\mathrm{i}\xi x+\lambda(\xi)t}\ \mathrm{i}\xi\alpha(\xi)\,\langle\tilde{\phi}(\xi)\bar{u}', \check{h}_0(\xi)\rangle_{L^2([0,1])}\mathrm{d}\xi$$

$$= \sum_{j\in\mathbf{Z}} \int_{-\pi}^{\pi} e^{\mathrm{i}\xi x+\lambda(\xi)t}\ \alpha(\xi)\frac{\xi}{\xi+2\pi j}(\bar{u}'\tilde{\phi}(\xi))\hat{}_j^{*}\ \widehat{\partial_x h_0}(\xi+2j\pi)\mathrm{d}\xi,$$

with $(\widehat{g}_j)_{j\in\mathbf{Z}}$ denotes Fourier's series of g.

$$\|\partial_x(s^{\mathrm{p}}(t)(\bar{u}'h_0))\|_{L^p(\mathbf{R})}$$

$$\lesssim (1+t)^{-\frac{1}{2}(1-1/p)}\|\partial_x h_0\|_{L^1(\mathbf{R})}\,\sup_\xi \|\tilde{\phi}(\xi)\bar{u}'\|_{L^2([0,1])}.$$

Some of the estimates: another one.

$$\tilde{S}(t)(h_0\bar{u}')(x)$$

$$= \sum_{j\in\mathbf{Z}} \int_{-\pi}^{\pi} e^{i\xi x} \frac{1-\alpha(\xi)}{i(\xi+2\pi j)} \left(e^{tL_\xi}(\bar{u}' e^{2i\pi j \cdot})\right)(x)\ \widehat{\partial_x h_0}(\xi+2j\pi)\mathrm{d}\xi$$
$$+ \sum_{j\in\mathbf{Z}} \int_{-\pi}^{\pi} e^{i\xi x}\alpha(\xi) \frac{\left(e^{tL_\xi}\tilde{\Pi}(\xi)(\bar{u}' e^{2ij\pi \cdot})\right)(x)}{i(\xi+2\pi j)}\ \widehat{\partial_x h_0}(\xi+2j\pi)\mathrm{d}\xi$$
$$+ \sum_{j\in\mathbf{Z}} \int_{-\pi}^{\pi} e^{i\xi x+\lambda(\xi)t}\alpha(\xi)\frac{\phi(\xi,x)-\phi(0,x)}{i(\xi+2\pi j)}(\tilde{\phi}(\xi)\bar{u}')\hat{\ }^{*}_{j}\widehat{\partial_x h_0}(\xi+2j\pi)\mathrm{d}\xi$$

with $\tilde{\Pi}(\xi)$ projector on non critical modes. $(1-\alpha(\xi)) \lesssim |\xi|$, $\tilde{\Pi}(0)(\bar{u}')=0$,

$$|\tilde{\Pi}(\xi)-\tilde{\Pi}(0)|_{H^1([0,1])\to H^1([0,1])} \lesssim |\xi|, \quad \|\phi(\xi)-\phi(0)\|_{H^1([0,1])} \lesssim |\xi|.$$

Some of the estimates: another one.

- $$\begin{aligned} \left|(1-\alpha(\xi))e^{tL_\xi}\right|_{H^1([0,1])\to H^1([0,1])} &\lesssim e^{-\theta' t},\\ \left|e^{tL_\xi}\tilde{\Pi}(\xi)\right|_{H^1([0,1])\to H^1([0,1])} &\lesssim e^{-\theta' t}. \end{aligned}$$

- $$\|\bar{u}' e^{2i\pi j\cdot}\|_{H^1([0,1])} \lesssim (1+j)\|\bar{u}'\|_{H^1([0,1])}.$$

- $$\|\xi \mapsto \sum_j |\widehat{\partial_x h_0}(\xi+2j\pi)|\|_{L^{p'}([-\pi,\pi])} \lesssim \|\partial_x h_0\|_{H^1(\mathbf{R})}.$$

Recall.

For all $t > 0$, $2 \leqslant p \leqslant \infty$,

$l + m \geqslant 1$ or ($l = m = 0$ and $p = \infty$)

$$\|\partial_x^l \partial_t^m s^{\mathrm{p}}(t)(h_0 \bar{u}')\|_{L^p(\mathbf{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)+\frac{1}{2}-\frac{l+m}{2}} \|\partial_x h_0\|_{L^1(\mathbf{R})};$$

$0 \leqslant l + 2m \leqslant K + 1$

$$\|\partial_x^l \partial_t^m \tilde{S}(t)(h_0 \bar{u}')\|_{L^p(\mathbf{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{l+m}{2}} \|\partial_x h_0\|_{L^1(\mathbf{R}) \cap H^{l+2m+1}(\mathbf{R})}.$$

Introduction of the phase: wrong way.

The natural way it does not work:

$$v(x,t) \;=\; \tilde{u}(x,t) - \bar{u}(x+\psi(x,t)).$$

Issue: it introduces ψ in nonlinear terms, yet ψ does not decay.

Introduction of the phase: correct way.

New unkowns: ψ and

$$v(x,t) \;=\; \tilde{u}(x-\psi(x,t),t)-\bar{u}(x).$$

Implicit change of variables.
Valid as long as $\|\psi\|_{L^\infty(\mathbf{R})}$ is bounded and $\|\psi_x\|_{L^\infty(\mathbf{R})}$ is **small**.

New equation.

$$\begin{aligned} k_\star(1-\psi_x)v_t \quad &+ \quad k_\star(-c_\star+\psi_t)(\bar{u}+v)_x \\ &= \quad k_\star^2\left(\tfrac{1}{1-\psi_x}(\bar{u}+v)_x\right)_x \;+\; (1-\psi_x)f(\bar{u}+v). \end{aligned}$$

Quasilinear.

Linear contribution of ψ: not in form for separation.

A priori energy estimate.

Provided

$$\sup_{[0,t]}(\|v\|^2_{H^K(\mathbf{R})} + \|\psi_t\|^2_{H^K(\mathbf{R})} + \|\psi_x\|^2_{H^{K+1}(\mathbf{R})}) << 1$$

for some $\theta'' > 0$

$$\begin{aligned}\|v(t)\|^2_{H^K(\mathbf{R})} &\leqslant C\, e^{-\theta'' t}\|v(0)\|^2_{H^K(\mathbf{R})} \\ &\quad + C\int_0^t e^{-\theta''(t-s)}\left(\|v(s)\|^2_{L^2(\mathbf{R})} + \|(\psi_t,\psi_x)(s)\|^2_{H^K(\mathbf{R})}\right)\mathrm{d}s.\end{aligned}$$

Separation: equation.

Equation

$$k_\star(\partial_t - L)\ (v(t) + \bar{u}'\psi(t)) = k_\star\mathcal{N}(t).$$

with $k_\star\mathcal{N} = \mathcal{Q} + \mathcal{R}_x + (k_\star\partial_t + k_\star^2\partial_x^2)\mathcal{S} + \mathcal{T}$, where

$$\mathcal{Q} = f(v+\bar{u}) - f(\bar{u}) - df(\bar{u})v,$$

$$\mathcal{R} = -k_\star v\psi_t - k_\star^2 v\psi_{xx} + k_\star^2(\bar{u}_x + v_x)\tfrac{\psi_x^2}{1-\psi_x},$$

$$\mathcal{S} := v\psi_x,$$

$$\mathcal{T} := -\left(f(v+\bar{u}) - f(\bar{u})\right)\psi_x.$$

Integral form

$$v(t) + \bar{u}'\psi(t) = S(t)\ (v(0) + \bar{u}'h_0) + \int_0^t S(t-s)\mathcal{N}(s)\mathrm{d}s$$

Separation: nonlinear level.

Integral equations

$$\begin{aligned}\psi(t) &= s^{\mathrm{p}}(t)(v(0)+\bar{u}'h_0)+\int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)\mathrm{d}s\\ &- \chi(t)\left[s^{\mathrm{p}}(t)(v(0)+\bar{u}'h_0)-h_0+\int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)\mathrm{d}s\right]\end{aligned}$$

$$\begin{aligned}v(t) &= \tilde{S}(t)(v(0)+\bar{u}'h_0)+\int_0^t \tilde{S}(t-s)\mathcal{N}(s)\mathrm{d}s\\ &+ \chi(t)\bar{u}'\left[s^{\mathrm{p}}(t)(v(0)+\bar{u}'h_0)-h_0+\int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)\mathrm{d}s\right]\end{aligned}$$

with χ a large-time cut-off.

Stability.

Close nonlinear iteration scheme with

$$\sup_{0\leqslant s\leqslant t}(1+s)^{1/4}\|(v,\psi_t,\psi_x)(s)\|_{H^K(\mathbf{R})}.$$

Improve bounds.

Structure of the proof.

1. Separation of the critical mode: introduction of the phase.
2. Nonlinear iteration for stability's proof.

3. Refined separation.
4. Estimate of the remainder.
5. Comparison with (W).

Refined spectral preparation.

Key relation

$$\partial_\xi \phi(0) \;=\; \mathrm{i}k_\star \partial_k \bar{u}.$$

Thanks to $L_0 \partial_\xi \phi(0) \;=\; L_0(\mathrm{i}k_\star \partial_k \bar{u})$ and

Assumption (N): normalization of shift

$$\langle \tilde{\phi}(0), \partial_k \bar{u} \rangle_{L^2([0,1])} = 0.$$

Refined linear separation.

Expand

$$\phi(\xi) = \bar{u}' + k_\star \partial_k \bar{u}\ (\mathrm{i}\xi) + \mathcal{O}(\xi^2)$$

and split

$$S(t) \;=\; (\bar{u}' + k_\star \partial_k \bar{u}\ \partial_x)\ s^{\mathrm{p}}(t) \;+\; \tilde{R}(t).$$

Remainder.

$$z(t) \;=\; v(t) \;-\; k_\star \partial_k \bar{u}\; \psi_x(t).$$

Equation for remainder

$$\begin{aligned} z(t) \;&=\; \tilde{R}(t)(v(0) + \bar{u}' h_0) + \int_0^t \tilde{R}(t-s)\mathcal{N}(s)\mathrm{d}s \\ &+\; \chi(t)\; (\bar{u}' + k_\star \partial_k \bar{u}\; \partial_x) \\ &\quad \cdot \left[s^{\mathrm{p}}(t)(v(0) + \bar{u}' h_0) - h_0 + \int_0^t s^{\mathrm{p}}(t-s)\mathcal{N}(s)\mathrm{d}s \right]. \end{aligned}$$

Comparison with (W): linear estimates.

Linear group velocity (in moving frame): $a_\star \;=\; \omega'_\star + c_\star$.
Let $\sigma(t)$ be the solution operator of

$$u_t \;=\; a_\star u_x + k_\star d_\star u_{xx}.$$

For all $t > 0$, $2 \leqslant p \leqslant \infty$,

f periodic of period 1, $l \in \mathbf{N}$

$$\begin{aligned}\|\partial_x^l s^{\mathrm{p}}(t)(h_0 f) \;&-\; \langle \tilde{\phi}(0), f\rangle\, \sigma(t)(\partial_x^l h_0)\|_{L^p(\mathbf{R})} \\ &\lesssim\; (1+t)^{-\frac{1}{2}(1-1/p)}\, t^{-\frac{l}{2}} \|\partial_x h_0\|_{L^1(\mathbf{R})\cap L^2(\mathbf{R})}.\end{aligned}$$

Comparison with (W): linear group velocity.

For all $t > 0$, $2 \leqslant p \leqslant \infty$,

$$\|(\partial_t - a_\star\partial_x)(s^{\mathrm{p}}(t)g)\|_{L^p(\mathbf{R})} \lesssim \min \begin{cases} (1+t)^{-\frac{1}{2}(1-1/p)-1}\|g\|_{L^1(\mathbf{R})}, \\ (1+t)^{-\frac{1}{2}(1/2-1/p)-1}\|g\|_{L^2(\mathbf{R})}; \end{cases}$$

and

$$\|(\partial_t - a_\star\partial_x)(s^{\mathrm{p}}(t)(h_0\bar{u}'))\|_{L^p(\mathbf{R})} \lesssim (1+t)^{-\frac{1}{2}(1-1/p)-\frac{1}{2}}\|\partial_x h_0\|_{L^1(\mathbf{R})}.$$

Comparison with (W): quadratic terms.

$$\|\psi_t(t) - a_\star \psi_x(t)\|_{L^2(\mathbf{R})} \lesssim E_0 (1+t)^{-3/4}.$$

$$k_\star \mathcal{N}(t) = f^{\mathrm{p}}\ k_\star^2 \psi_x(t)^2 + r(t)$$

where $\|r(t)\|_{L^1(\mathbf{R})} \lesssim E_0^2 (1+t)^{-1}$ and

$$f^{\mathrm{p}} = \frac{1}{2}\mathrm{d}^2 f(\bar{u})(\partial_k \bar{u}, \partial_k \bar{u}) + k_\star \partial_k \bar{u}'' - \frac{1}{k_\star}\mathrm{d}f(\bar{u})\partial_k \bar{u} + \bar{u}'' - a_\star \partial_k \bar{u}'.$$

$$\langle \tilde{\phi}(0), f^{\mathrm{p}} \rangle_{L^2([0,1])} \ = \ \frac{1}{2}\omega_\star''.$$

Comparison with (W): nonlinear level.

Continuity argument on

$$\sup_{p\in[2,\infty]}\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{1}{2}(1-1/p)+\frac{1}{4}-\eta}\|k_\star\psi_x(s)-k(s)\|_{L^p(\mathbf{R})}.$$

Then bound $\psi - h$.

Outline:

Some comments.

- Not optimized: $K \geqslant 3$ may be replaced with $K \geqslant 1$.

- Very robust. Needed (besides explicit assumptions): an a priori estimate, resolvent estimates to generate C^0-semi-groups.

Quasilinear higher-order parabolic equations. Swift-Hohenberg.

References.

$h_0 \equiv 0$

Schneider (CMP 1996)
Schneider (Tohuku Math.Pub. 1997)
Schneider (ARMA 1998)
Johnson-Zumbrun (Annales IHP 2011).

h_0 non trivial

Sandstede-Scheel-Schneider-Uecker (JDE 2011)
Johnson-Noble-Rodrigues-Zumbrun (Preprint 2011).

Weighted spaces.

Same assumptions (H), (D1)-(D3).

$\rho(x) = (1 + x^2)^{1/2}$.

$$\|g\|_{H^m(m')} \;=\; \|g\,\rho^{m'}\|_{H^m(\mathbf{R})}.$$

Group velocity: $a_\star$.

Sandstede-Scheel-Schneider-Uecker (JDE 2011).

Stability and behavior of the phase ($\omega''_\star \neq 0$)

Let $0 < \eta < \frac{1}{2}$. Let

$$\widetilde{E}_0 := \|\tilde{u}_0(\cdot - h_0(\cdot)) - \bar{u}(\cdot)\|_{H^2(2)} + \|\partial_x h_0\|_{H^2(2)} << 1$$

for some h_0. Then, there exists $\tilde{u}$ with initial data $\tilde{u}_0$ such that

$$\|\tilde{u}(\cdot, t) - \bar{u}(\cdot + \psi^{\mathrm{p}}(\cdot - a_\star t, t))\|_{L^\infty(\mathbf{R})} \lesssim \widetilde{E}_0 t^{-1/2+b}$$

where

$$\psi^{\mathrm{p}}(x,t) \;=\; \frac{k_\star d_\star}{-\frac{1}{2}\omega''_\star} \ln\left(1 + z\,\mathrm{errf}\left(\frac{x}{\sqrt{k_\star d_\star t}}\right)\right)$$

with $z = \exp(\int_{\mathbf{R}} \partial_x h_0) - 1$, and $\mathrm{errf}(x) = (4\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/4} \mathrm{d}y$.

Spectral separation.

Same introduction of the phase. Exact projection.

Diagonalization.

$$\begin{aligned}\partial_t v^c &= \lambda^c v^c + \partial_x(\pi^c[\tilde{\mathcal{N}}(v^c, v^s))])\\ \partial_t v^s &= \Lambda^s v^s + \Pi^s[\tilde{\mathcal{N}}(v^c, v^s)]\end{aligned}$$

Not easily read in physical variables.

Scaling.

$(W)_k$

$$k_\star k_t - k_\star \Big(a_\star k + \frac{1}{2} \omega''_\star k^2 \Big)_x = k_\star^2 d_\star k_{xx}.$$

In a frame **moving with linear group velocity**, viscous Burgers' equation

$$k_t - \Big(\frac{1}{2} \omega''_\star k^2 \Big)_x = k_\star d_\star k_{xx}.$$

Scaling: if k is a solution, so is k_L,

$$k_L(x, t) \ = \ L\, k(L\, x, L^2\, t).$$

Renormalization process.

Moving with linear group velocity:

$$(\check{u}^c, \check{u}^s)(\xi, x, t) \;=\; e^{\mathrm{i}a_\star \xi t}(\check{v}^c, \check{v}^s)(\xi, x, t).$$

$L > 1$ sufficiently large, γ small.

$$\begin{aligned} \check{u}_n^c(\xi, x, t) &= \check{u}^c(L^{-n}\xi, x, L^{2n}t - 1), \\ \check{u}_n^s(\xi, x, t) &= L^{n(1-\gamma)}\ \check{u}^s(L^{-n}\xi, x, L^{2n}t - 1). \end{aligned}$$

Sequence of systems on $[L^{-2}, 1]$.

$$u_n \quad \text{on} \quad [L^{-2}, 1] \qquad \text{gives} \quad u \quad \text{on} \quad [L^{2(n-1)} - 1, L^{2n} - 1].$$

Discrete iteration.

Main part and remainder:

$$(\check{u}_n^c, \check{u}_n^s)(\xi, x, 1) \ = \ ((u_n^{\mathrm{p}})\check{}(\xi, x, 1), 0) + (\check{r}_n^c, \check{r}_n^s)(\xi, x).$$

Close estimates on

$$\|\check{r}_n^c\|_{X_{L^n}}, \qquad \|\check{r}_n^s\|_{X_{L^n}}.$$

Critical terms are scaled without change, others are damped.

Outline:

1. Modulations of scalar type
   - Nonlocalized data: statement
   - Nonlocalized data: into the proof
   - Comments and further references
2. Modulations of system type
   - Some comments
   - A spectral lemma
3. Open questions

Some examples.

Kuramoto-Sivashinsky Korteweg-de Vries equation.

Saint-Venant system on an inclined plane.

Model case: conservation laws.

$t > 0$ time, $x \in \mathbf{R}$ space, $u(x,t) \in \mathbf{R}^d$ unknown.

Conservation laws

$$u_t = (f(u))_x + D\, u_{xx}.$$

D positive definite. $D = I_d$ for simplicity.

f regular.

Wave parametrization.

Profile equation:

$$-kc\bar{u}' = k(f(\bar{u}))' + k^2\bar{u}''$$

$\bar{u}$ of period 1.

Mean $M \in \mathbf{R}^d$

$$M \; = \; \int_0^1 u.$$

Assumption (H1)

Poincaré return map is full-rank and a parametrization by shift α, wavenumber k and mean M

$$\bar{u}^{k,M}(\cdot - \alpha), \qquad c(k,M)$$

is available.

Some assumptions and a normalization.

(H1) Parametrization by wavenumber and mean.

(D1) Critical spectrum is reduced to $\{0\}$.

(D2) Diffusive spectral stability.

(D3) Minimal dimension of the generalized co-periodic kernel.

(N) Suitable normalization of parametrization (shift).

(H2) Strict hyperbolicity of (W) or linear group velocities are distinct.

Bloch symbol.

Operator

$$Lg \;=\; k_\star g_{xx} + (\mathrm{d}f(\bar{u})g)_x + c_\star g_x.$$

Bloch symbol

$$(L_\xi g)(x) \;=\; e^{-\mathrm{i}\xi x}\, L(e^{\mathrm{i}\xi\cdot} g(\cdot))(x), \qquad \xi \in [-\pi, \pi].$$

Assumption (D3)

$\lambda = 0$ is an eigenvalue of L_0 of algebraic multiplicity $d+1$.

Critical eigenspace.

Right eigenfunctions:

$$L_0 \bar{u}' = 0, \qquad L_0 \partial_M \bar{u} = -k_\star \bar{u}' \partial_M c_\star.$$

Left eigenfunctions:

$$g \text{ constant}, \qquad L_0^{adj} g = 0.$$

Linearly decoupled case

$\dim \ker L_0 = d + 1$ (no Jordan block)

Similar to scalar modulation. $h_0 \equiv 0$ increases decay.

Linearly coupled case

$\dim \ker L_0 = d$ and there is a **Jordan block** of height 1.

Even if $h_0 \equiv 0$, decay is slow.

Stability, $h_0 \equiv 0$.

Uncoupled case

Johnson-Zumbrun (SIAM J. Appl. Dyn. Sys. 2011)

Coupled case

Johnson-Zumbrun (JDE 2010)
Johnson-Zumbrun-Noble (SIAM J. Math. Anal. 2011)

Outline:

Johnson-Zumbrun (JDE 2010)

Regularity of critical spectrum

Assume (H1) and (D3). Then critical eigenvalues $\lambda(\xi)$ expand about 0 as C^1 functions of the Floquet parameter ξ.

Assume moreover (H2). Then, for ξ small non zero, there exist dual right and left eigenfunctions $w_j(\xi,\cdot)$ and $\tilde{w}_j(\xi,\cdot)$ of L_ξ associated with $\lambda_j(\xi)$, for $j = 1, \cdots, d+1$, of form

$$\begin{aligned} w_j(\xi,\cdot) &= \sum_{k=1}^{d} \beta_{j,k}(\xi)\phi_k(\xi,\cdot) + \xi^{-1}\beta_{j,d+1}(\xi)\,\phi_{d+1}(\xi,\cdot)\ , \\ \tilde{w}_j(\xi,\cdot) &= \sum_{k=1}^{d} \tilde{\beta}_{j,k}(\xi)\tilde{\phi}_k(\xi,\cdot) + \xi\ \beta_{j,d+1}(\xi)\,\tilde{\phi}_{d+1}(\xi,\cdot) \end{aligned}$$

Johnson-Zumbrun (JDE 2010)

where

- $(\phi_k(\xi,\cdot))_{k=1,\cdots,d+1}$ and $(\tilde{\phi}_k(\xi,\cdot))_{k=1,\cdots,d+1}$ are dual bases of the total critical eigenspace of L_ξ, analytic in ξ, and such that, $j \neq d+1$,

$$\tilde{\phi}_j(0,\cdot)\ \text{constant}\ e_j, \quad \phi_j(0,\cdot) = \partial_M \bar{u}\cdot e_j, \quad \phi_{d+1}(0,\cdot) = \bar{u}'\ ;$$

- $\tilde{\beta}_{j,k}(\xi)$, $\beta_{j,k}(\xi)$ analytic in ξ.

Finite-dimensional reduction.

Perturbation theory: $\phi_j(\xi,\cdot)$, $\tilde{\phi}_j(\xi,\cdot)$.

Matrix perturbation:

$$M_\xi \;=\; \left[\langle\tilde{\phi}_k(\xi,\cdot), L_\xi\phi_j(\xi,\cdot)\rangle_{L^2([0,1])}\right]_{j,k}.$$

Expansion

$$L_\xi \;=\; L_0 + (\mathrm{i}\xi)L^{(1)} + (\mathrm{i}\xi)^2 L^{(2)}.$$

Cancellation.

For $j \neq d+1$

$$\langle \tilde{\phi}_j(\xi,\cdot), L_\xi \phi_{d+1}(\xi,\cdot)\rangle_{L^2([0,1])} \;=\; \mathrm{i}\xi\, \langle e_j, L^{(1)}\overline{u}'\rangle_{L^2([0,1])} + \mathcal{O}(\xi^2).$$

Yet

$$\mathrm{i}k_\star L_0(\partial_k \overline{u}) + \mathrm{i}k_\star \overline{u}'\partial_k c + L^{(1)}\overline{u}' = 0$$

thus for $j \neq d+1$

$$\langle \tilde{\phi}_j(\xi,\cdot), L_\xi \phi_{d+1}(\xi,\cdot)\rangle_{L^2([0,1])} \;=\; \mathcal{O}(\xi^2).$$

Scaling.

$$M_\xi^W = (\mathrm{i}\xi)^{-1} S(\xi) M_\xi S(\xi)^{-1}, \qquad S(\xi) := \begin{pmatrix} I_d & 0 \\ 0 & \mathrm{i}\xi \end{pmatrix}.$$

Related to: phase $\longrightarrow$ wavenumber.

Eigenvalues of M_0^W are linear group velocities.

Continuity of spectrum of M_ξ^W gives C^1 regularity.

Analyticity under (H2) yields the rest.

Outline:

1. Modulations of scalar type
   - Nonlocalized data: statement
   - Nonlocalized data: into the proof
   - Comments and further references
2. Modulations of system type
   - Some comments
   - A spectral lemma
3. Open questions

In progress.

- Verification of assumptions (either half-numerical or perturbative).
- System-type modulations: stability and behavior for non localized perturbation (h_0 non trivial).
- Non localized perturbation of parameters (fronts in parameters).

Fully open.

- Free surface Navier-Stokes equation down an incline plane.
- Patterns genuinely multidimensional.
- Dispersive nonlinear stability.