Spectral stability of periodic waves in dispersive models

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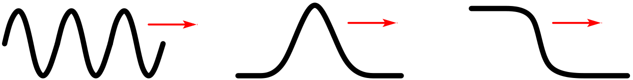
One-dimensional nonlinear waves

- **Standing and travelling waves**

$$u(x - ct) \quad \text{with} \quad c = 0 \quad \text{or} \quad c \neq 0$$



periodic wave — pulse/solitary wave — front/kink

- *found as solutions of an ODE with "time" $x - ct$*

- **Modulated waves**

, . . .

Questions

- **Existence:** *no time dependence*
  - solve a steady PDE
  - 1d waves: solve an ODE
- **Stability:** *add time*
  - solve an initial value problem

- initial data $u_*(x) + \varepsilon v(x)$

? what happens as $t \to \infty$?

Questions

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  - 1d waves: solve an ODE
- **Stability:** *add time*
  - solve an initial value problem

  - initial data $u_*(x) + \varepsilon v(x)$

  ? what happens as $t \to \infty$?
- Interactions: *initial value problem*

  - initial data: superposition of two/several nonlinear waves

  ? what happens ?
- *Role in the dynamics of the PDE*

Stability problems

- **spectral stability**
- **linear stability**
- **nonlinear stability**
  - *orbital stability*
  - *asymptotic stability*

Stability problems

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Answers depend upon

- *type of the wave: localized, periodic, front,...*
- *type of the PDE*

PDEs

- **Dissipative models:** e.g. reaction diffusion systems

$$U_t = D\,\Delta U + F(U)$$

$U(x,t) \in \mathbb{R}^N$; $t \geq 0$ time; $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ space

D diffusion matrix: $D = \mathrm{diag}\,(d_1, \ldots, d_N) > 0$

$F(U)$ kinetics (smooth map)

- **Dispersive models:** e.g. the Korteweg-de Vries equation

$$u_t = u_{xxx} + uu_x, \quad u(x,t) \in \mathbb{R}, \; x \in \mathbb{R}, \; t \in \mathbb{R}$$

- **Mixed models:** dissipation and dispersion

Dispersive models

- **Generalized KdV equation**

$$\boxed{u_t + (u_{xx} + u^{p+1})_x = 0} \quad p \geq 1$$

- *other 1d models: Kawahara, BBM, NLS,...*

- **Kadomtsev-Petviashvili equations**

$$\boxed{(u_t - u_{xxx} - uu_x)_x + \sigma u_{yy} = 0} \qquad x, y \in \mathbb{R}$$

  - **KP-I equation:** $\sigma = 1$ *(positive dispersion)*
  - **KP-II equation:** $\sigma = -1$ *(negative dispersion)*
  - *2d generalization of the KdV-equation*

$$u_t - u_{xxx} - uu_x = 0$$

Answers

- **localized waves and fronts**
  - well established methods
  - nonlinear stability is well understood for many different models
- **periodic waves**
  - nonlinear stability is quite well understood for dissipative models

    [Schneider, Gallay & Scheel,. . .]
  - very recent results for mixed models

    [Noble, Johnson, Rodrigues & Zumbrun, . . .]
  - partial results for dispersive models

Classes of perturbations for periodic waves

- **periodic perturbations (same period as the wave): orbital stability**

  [Angulo, Bona, & Scialom; Gallay & H.; Hakkaev, Iliev, & Kirchev; . . .]

- **localized/bounded perturbations: spectral stability**

  [H., Lombardi, & Scheel; Serre; Gallay & H.; H. & Kapitula; Bottman & Deconinck; Bronski, Johnson & Zumbrun; Ivey & Lafortune; Noble; . . .]

- **intermediate class:** periodic perturbations with period a multiple of the period of the wave
  - orbital stability for the KdV equation [Deconinck & Kapitula]
  - relies on integrability of KdV

Spectral stability of periodic waves

– Operator theory –

alternative tool: Evans function

- Bloch-wave decomposition (Floquet theory)
- Perturbation methods for linear operators
- Use of the Hamiltonian structure

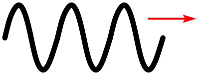
Example: gKdV equation

$$\boxed{u_t + (u_{xx} + u^{p+1})_x = 0} \quad p \geq 1$$

Application: KP equation

Periodic waves of the gKdV equation

- **Travelling periodic waves:** $u(x,t) = q(x-ct)$



  - $q_{yy} = cq - q^{p+1} + b, \quad y = x - ct$
  - three parameter family $(a,\ b,$ speed $c)$ – *up to spatial translations*
  - scaling invariance $\rightsquigarrow$ $c = 1$
  - *KdV equation, $p = 1$: Galilean invariance* $\rightsquigarrow$ $b = 0$
- **Family of periodic waves:** $\boxed{q_{a,b}(y) = P_{a,b}(k_{a,b}y)}$

  with $P_{a,b}$ a 2π-periodic even solution of

$$\boxed{k_{a,b}^2 v_{zz} - v + v^{p+1} = b}$$

Small periodic waves

$$q_{a,b}(y) = P_{a,b}(k_{a,b}y)$$

with $P_{a,b}$ a 2π-periodic even solution of

$$k_{a,b}^2 v_{zz} - v + v^{p+1} = b$$

- **Small amplitude**

$$P_{a,b}(z) = Q_b + \cos(z)\, a - \frac{p+1}{4}\, a^2 + \frac{p+1}{12}\, \cos(2z)\, a^2 + O(|a|(a^2+b^2))$$

$$Q_b = 1 + \frac{1}{p}\, b - \frac{p+1}{2p^2}\, b^2 + O(|b|^3)$$

$$k_{a,b}^2 = p + (p+1)b - \frac{p(p+1)(p+4)}{12}\, a^2 - \frac{p+1}{p}\, b^2 + O(|a|^3 + |b|^3)$$

- **Question: spectral stability ?**

Spectral stability

- **Linearized operator**

$$\mathcal{A}_{a,b}\mathbf{v} = -k_{a,b}^2\partial_{zzz}\mathbf{v} + \partial_z\mathbf{v} - (p+1)\partial_z(P_{a,b}^p\mathbf{v})$$

- **Spectrum in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$**

$$\sigma(\mathcal{A}_{a,b}) = \{\lambda \in \mathbb{C}\ ;\ \lambda - \mathcal{A}_{a,b} \text{ is not invertible }\}$$

- *The periodic wave is* **spectrally stable** *if*

$$\sigma(\mathcal{A}_{a,b}) = \{\lambda \in \mathbb{C}\ ;\ \mathrm{Re}\,\lambda \leq 0\}$$

Spectral stability

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- **Question: locate the spectrum ?**
  - *first difficulty: continuous spectrum*

Bloch-wave decomposition

- *reduces the spectral problem for localized/bounded perturbations to the study of the spectra of an (infinite) family of operators with* **point spectra** [Reed & Simon; Scarpelini; Mielke; . . .]
  - *one spatial dimension:* **Floquet theory**

Bloch-wave decomposition

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  - *one spatial dimension:* **Floquet theory**

Theorem

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}_{a,b}) = \sigma_{C_b^0(\mathbb{R})}(\mathcal{A}_{a,b}) = \bigcup_{\gamma\in(-\frac{1}{2},\frac{1}{2}]} \sigma_{L^2(0,2\pi)}(\mathcal{A}_{a,b,\gamma})$$

where

$$\mathcal{A}_{a,b,\gamma} = -k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^3 + (\partial_z + \mathrm{i}\gamma) - (p+1)(\partial_z + \mathrm{i}\gamma)(P_{a,b}^p\cdot)$$

Notice that the operator $\mathcal{A}_{a,b,\gamma}$ has compact resolvent $\Longrightarrow$ its spectrum consists of eigenvalues with finite algebraic multiplicity.

Floquet theory

$$\lambda v = \mathcal{A}_{a,b} v = -k_{a,b}^2 \partial_{zzz} \mathbf{v} + \partial_z \mathbf{v} - (p+1)\partial_z (P_{a,b}^p \mathbf{v})$$

- **First order system**

$$\frac{\mathrm{d}}{\mathrm{d}z} W = A(z,\lambda) W, \qquad W = \begin{pmatrix} v \\ w_1 = v_z \\ w_2 = v_{zz} \end{pmatrix}$$

$A(z,\lambda)$ matrix with 2π-periodic coefficients

- **Floquet theory:** *any solution is of the form*

$$\boxed{W(z) = Q_\lambda(z) \mathrm{e}^{C(\lambda) z} W(0)}$$

  - $Q_\lambda(\cdot)$ *is a* 2π*-periodic matrix function*
  - $C(\lambda)$ *matrix with constant coefficients* [1]

[1] eigenvalues of $C(\lambda)$: *Floquet exponents*

Spectral problem

$$\frac{\mathrm{d}}{\mathrm{d}z}W = A(z,\lambda)W, \qquad W(z) = Q_\lambda(z)\mathrm{e}^{C(\lambda)z}W(0)$$

The ODE has a nontrivial bounded solution for $\lambda \in \mathbb{C}$

$\Longleftrightarrow$ $\boxed{\ker_{C_b^0(\mathbb{R})}(\mathcal{A}_{a,b} - \lambda) \neq \{0\}}$

$\Longleftrightarrow$ *ex. solution of the form* $W(z) = Q(z)\mathrm{e}^{\mathrm{i}\gamma z}$

$$\gamma \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right), \quad Q(\cdot) \ \ 2\pi - \textit{periodic}$$

$\Longleftrightarrow$ *the eigenvalue problem has a nontrivial solution*

$$v(z) = q(z)\mathrm{e}^{\mathrm{i}\gamma z}, \quad \gamma \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right), \quad q(\cdot) \ \ 2\pi - \textit{periodic}$$

$\Longleftrightarrow$ *ex. nontrivial* 2π-**periodic solution** *to*

$$\lambda q = \mathcal{A}_{a,b,\gamma}q = -k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^3 q + (\partial_z + \mathrm{i}\gamma)q - (p+1)(\partial_z + \mathrm{i}\gamma)(P_{a,b}^p q)$$

for $\gamma \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right]$

Spectral problem

$$\frac{\mathrm{d}}{\mathrm{d}z}W = A(z,\lambda)W, \qquad W(z) = Q_\lambda(z)\mathrm{e}^{C(\lambda)z}W(0)$$

The ODE has a nontrivial bounded solution for $\lambda \in \mathbb{C}$

$$\iff \boxed{\ker_{C_b^0(\mathbb{R})}(\mathcal{A}_{a,b} - \lambda) \neq \{0\}}$$

$\iff$ *ex. nontrivial* 2π-**periodic solution** *to*

$$\lambda q = \mathcal{A}_{a,b,\gamma} q = -k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^3 q + (\partial_z + \mathrm{i}\gamma)q - (p+1)(\partial_z + \mathrm{i}\gamma)(P_{a,b}^p q)$$

for $\gamma \in (-\frac{1}{2}, \frac{1}{2}]$

$\iff$ *the* **linear operator** $\lambda - \mathcal{A}_{a,b,\gamma}$ **has a nontrivial kernel in** $L^2(0,2\pi)$

$$\iff \boxed{\lambda \in \sigma_{L^2(0,2\pi)}(\mathcal{A}_{a,b,\gamma}), \quad \gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right)}$$

Bloch-wave decomposition

Lemma

1. $$\ker_{C_b^0(\mathbb{R})}(\mathcal{A}_{a,b}-\lambda)\neq 0 \iff \lambda\in\bigcup_{\gamma\in[-\frac12,\frac12)}\sigma_{L^2(0,2\pi)}(\mathcal{L}_{a,c,\gamma})$$
2. $$\ker_{C_b^0(\mathbb{R})}(\mathcal{A}_{a,b}-\lambda)\neq 0 \iff \lambda\in\sigma_{C_b^0(\mathbb{R})}(\mathcal{L}_{a,c})=\sigma_{L^2(\mathbb{R})}(\mathcal{L}_{a,c})$$

Proof of (2). Solve $\lambda v-\mathcal{A}_{a,b}v=f$ using

- *Floquet theory*
- *variation of constant formula . . .*

Bloch-wave decomposition

Theorem

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}_{a,b}) = \sigma_{C_b^0(\mathbb{R})}(\mathcal{A}_{a,b}) = \bigcup_{\gamma \in (-\frac{1}{2}, \frac{1}{2}]} \sigma_{L^2(0,2\pi)}(\mathcal{A}_{a,b,\gamma})$$

where

$$\mathcal{A}_{a,b,\gamma} = -k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^3 + (\partial_z + \mathrm{i}\gamma) - (p+1)(\partial_z + \mathrm{i}\gamma)(P_{a,b}^p \cdot)$$

Next question: locate the point spectra of $\mathcal{A}_{a,b,\gamma}$?

Point spectra of $\mathcal{A}_{a,b,\gamma}$

- **Perturbation arguments for linear operators**
  - *small perturbations of operators with constant coefficients* *(restrict to small waves)*
  - *symmetries: spectra are symmetric with respect to the imaginary axis*
- **Hamiltonian structure**
  - *operator* $\mathcal{A}_{a,b,\gamma} = \mathcal{J}_\gamma \mathcal{L}_{a,b,\gamma}$
  - $\mathcal{J}_\gamma$ is **skew-adjoint**
  - $\mathcal{L}_{a,b,\gamma}$ is **self-adjoint**
- **Other ways:**
  - use integrability and compute spectra explicitly
  - numerical calculations

Small periodic waves

Spectrum of

$$\mathcal{A}_{a,b,\gamma} = -(\partial_z + \mathrm{i}\gamma)^3 + \frac{1}{k_{a,b}^2}(\partial_z + \mathrm{i}\gamma) - \frac{p+1}{k_{a,b}^2}(\partial_z + \mathrm{i}\gamma)(P_{a,b}^p \cdot) \text{ ?}$$

- **Waves with small amplitude**

$$\boxed{P_{a,b}(z)} = Q_b + \cos(z)\,a - \frac{p+1}{4}\,a^2 + \frac{p+1}{12}\,\cos(2z)\,a^2 + O(|a|(a^2+b^2))$$

$$Q_b = 1 + \frac{1}{p}\,b - \frac{p+1}{2p^2}\,b^2 + O(|b|^3)$$

$$k_{a,b}^2 = p + (p+1)b - \frac{p(p+1)(p+4)}{12}\,a^2 - \frac{p+1}{p}\,b^2 + O(|a|^3 + |b|^3)$$

- a, b **small**

Step 0: perturbation argument

Spectrum of

$$\mathcal{A}_{a,b,\gamma} = -(\partial_z + \mathrm{i}\gamma)^3 + \frac{1}{k_{a,b}^2}(\partial_z + \mathrm{i}\gamma) - \frac{p+1}{k_{a,b}^2}(\partial_z + \mathrm{i}\gamma)(P_{a,b}^p \cdot) \ ?$$

- *a, b small $\longrightarrow$ $\mathcal{A}_{a,b,\gamma}$ is a "small perturbation" of $\mathcal{A}_{0,0,\gamma}$*

$$\mathcal{A}_{a,b,\gamma} = \mathcal{A}_{0,0,\gamma} + \mathcal{A}^1_{a,b,\gamma}, \qquad \mathcal{A}_{0,0,\gamma} = -(\partial_z + \mathrm{i}\gamma)^3 + (\partial_z + \mathrm{i}\gamma)$$

  - $\mathcal{A}_{0,0,\gamma}$ *operator with constant coefficients*
  - $\mathcal{A}^1_{a,b,\gamma}$ *operator with 2π-periodic coefficients*

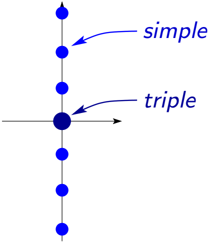
$$\|\mathcal{A}^1_{a,b,\gamma}\|_{H^1 \to L^2} = \mathrm{O}(|a| + |b|)$$

– $\mathcal{A}_{a,b,\gamma}$ is a *small relatively bounded* perturbation of $\mathcal{A}_{0,0,\gamma}$ –

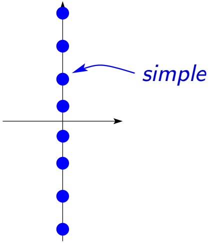
Step 1: spectrum of $\mathcal{A}_{0,0,\gamma}$

- *Fourier analysis*

$$\sigma(\mathcal{A}_{0,0,\gamma}) = \left\{ \mathrm{i}\omega_{n,\gamma} = \mathrm{i}\left((n+\gamma)^3 - (n+\gamma)\right) \; ; \; n \in \mathbb{Z} \right\}$$

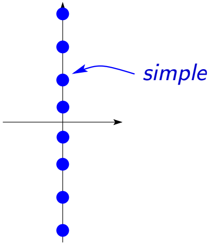


$\gamma = 0$



$\gamma \neq 0$

Step 2: $\gamma_* \leq |\gamma| \leq \frac{1}{2}$



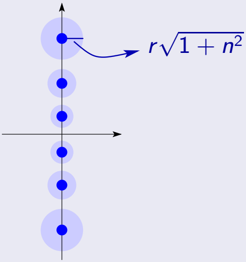
- all eigenvalues are simple
- *picture persists for small a, b* **?**

Difficulties

- infinitely many simple eigenvalues
- *relatively* bounded perturbation

Step 2: $\gamma_* \leq |\gamma| \leq \frac{1}{2}$

Lemma



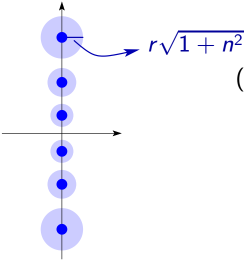
For all $\gamma_* > 0$, $r > 0$, ex. $\varepsilon_* > 0$ such that

$$\sigma(\mathcal{A}_{a,b,\gamma}) \subset \bigcup_{n\in\mathbb{Z}} B(\mathrm{i}\omega_{n,\gamma}, r\sqrt{1+n^2}),$$

for $|a| + |b| \leq \varepsilon_*$ and $\gamma_* \leq |\gamma| \leq \frac{1}{2}$.

Step 2: $\gamma_* \leq |\gamma| \leq \frac{1}{2}$

Proof.



Resolvent formula

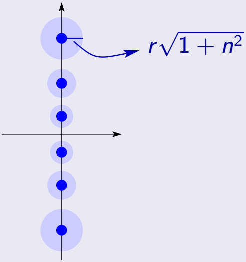
$$(\lambda - \mathcal{A}_{a,b,\gamma})^{-1} = (\lambda - \mathcal{A}_{0,0,\gamma})^{-1}\left(\mathrm{id} - \mathcal{A}^1_{a,b,\gamma}(\lambda - \mathcal{A}_{0,0,\gamma})^{-1}\right)^{-1}$$

- $\|(\lambda - \mathcal{A}_{0,0,\gamma})^{-1}\|_{L^2 \to H^1} \leq \frac{1}{r}$
- $\|\mathcal{A}^1_{a,b,\gamma}\|_{H^1 \to L^2} = \mathrm{O}(|a| + |b|)$

$\longrightarrow$ $\lambda - \mathcal{A}_{a,b,\gamma}$ *is invertible for* λ *outside these balls.*

Step 2: $\gamma_* \leq |\gamma| \leq \frac{1}{2}$

Lemma



Fix $\gamma_* > 0$ and choose $r > 0$ small. Then

- *the balls are mutually disjoints;*
- *$\mathcal{A}_{a,b,\gamma}$ has precisely* **one simple eigenvalue inside each ball** *$B(\mathrm{i}\omega_{n,\gamma}, r\sqrt{1+n^2})$, for a, b sufficiently small, and $\gamma_* \leq |\gamma| \leq \frac{1}{2}$. This eigenvalue is* **purely imaginary**.

Step 2: $\gamma_* \leq |\gamma| \leq \frac{1}{2}$

Proof. Choose a ball $B(\mathrm{i}\omega_{n,\gamma}, r\sqrt{1+n^2})$, r sufficiently small.

- *$\mathcal{A}_{a,b,\gamma}$ has precisely one simple eigenvalue inside this ball.*
  - Construct spectral projectors
    $$\Pi^n_{0,0,\gamma} \text{ for } \mathcal{A}_{0,0,\gamma} \text{ and } \Pi^n_{a,b,\gamma} \text{ for } \mathcal{A}_{a,b,\gamma}$$
  - Show that
    $$\|\Pi^n_{a,b,\gamma} - \Pi^n_{0,0,\gamma}\| < \min\left(\frac{1}{\|\Pi^n_{0,0,\gamma}\|}, \frac{1}{\|\Pi^n_{a,b,\gamma}\|}\right)$$
  - Conclude that $\Pi^n_{a,b,\gamma}$ **and** $\Pi^n_{0,0,\gamma}$ **have the same finite rank**, equal to 1.

Step 2: $\gamma_* \leq |\gamma| \leq \frac{1}{2}$

Proof. Choose a ball $B(\mathrm{i}\omega_{n,\gamma}, r\sqrt{1+n^2})$, r sufficiently small.

- *$\mathcal{A}_{a,b,\gamma}$ has precisely one simple eigenvalue inside this ball.*
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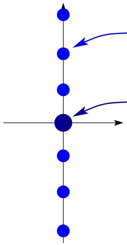
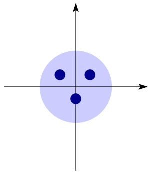
  - Conclude that $\Pi^n_{a,b,\gamma}$ **and** $\Pi^n_{0,0,\gamma}$ **have the same finite rank**, equal to 1.
- *This eigenvalue is purely imaginary.*
  - The spectrum is symmetric with respect to the imaginary axis, so that the simple eigenvalue lies on the imaginary axis.

Step 3: $|\gamma| \leq \gamma_*$ (*γ small*)

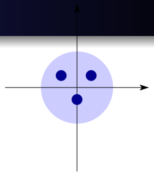


- **Step 3.1:** $|n| \geq 2$ *argue as in Step 2.*
- **Step 3.2:** $n = 0, \pm 1 \longrightarrow \mathcal{A}_{a,b,\gamma}$ *has three eigenvalues inside the ball* $B(0,1)$



Step 3.2: $n = 0, \pm 1$, $|\gamma| \leq \gamma_*$

Locate the three eigenvalues inside $B(0,1)$?



Consider the associated **spectral subspace** *(three-dimensional)*

- compute a **basis** $\left\{\xi^0_{a,b,\gamma},\ \xi^1_{a,b,\gamma},\ \xi^2_{a,b,\gamma}\right\}$;
- compute the 3×3-matrix $\mathcal{M}_{a,b,\gamma}$ representing the **action of** $\mathcal{A}_{a,b,\gamma}$ on this subspace;
- locate the three **eigenvalues** of this matrix [2]

$\Longrightarrow$ **purely imaginary if p < 2**

[2] **Difficulty:** *three small parameters; use the results for $a = b = 0$ and $\gamma = 0$.*

Spectral stability

Theorem

Consider the generalized KdV equation

$$u_t + (u_{xx} + u^{p+1})_x = 0, \quad p \geq 1$$

and the periodic travelling wave $q_{a,b}$ for a and b sufficiently small.

- $p < 2$
  - $\sigma(\mathcal{A}_{a,b,\gamma}) \subset \mathrm{i}\mathbb{R}$
  - **the periodic wave is spectrally stable**
- $p > 2$
  - $\sigma(\mathcal{A}_{a,b,\gamma}) \cap \{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda > 0\} \neq \emptyset$
  - **the periodic wave is spectrally unstable**

Hamiltonian structure

- Bloch operators

$$\mathcal{A}_{a,b,\gamma} = \mathcal{J}_\gamma \mathcal{L}_{a,b,\gamma}$$

where

$$\mathcal{J}_\gamma = \partial_z + \mathrm{i}\gamma, \quad \mathcal{L}_{a,b,\gamma} = -k_{a,b}^2(\partial_z + \mathrm{i}\gamma)^2 + 1 - (p+1)P_{a,b}^p$$

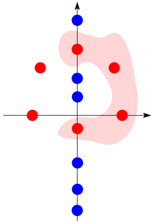
  - $\mathcal{J}_\gamma$ is **skew-adjoint** with compact resolvent, and invertible for $\gamma \neq 0$
  - $\mathcal{L}_{a,b,\gamma}$ is **self-adjoint** with compact resolvent, and invertible for a.a. γ
  - $\mathcal{L}_{a,b,\gamma}$ has a finite number of negative eigenvalues

- **Connection between the spectra of $\mathcal{A}_{a,b,\gamma}$ and $\mathcal{L}_{a,b,\gamma}$?**

Spectra of $\mathcal{A} := \mathcal{A}_{a,b,\gamma}$ and $\mathcal{L} := \mathcal{L}_{a,b,\gamma}$

Spectrum of $\mathcal{A}$

– symmetric w.r.t. imaginary axis –

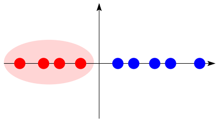


$$k_u := \#\{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\,\lambda > 0\}$$

$$k_i^- := \#\{\lambda \in \sigma(\mathcal{A}) : \mathrm{Re}\,\lambda = 0, \text{ with negative Krein signature}\}$$

Spectrum of $\mathcal{L}$

– real –



$$\mathbf{n}(\mathcal{L}) := \#\{\lambda \in \sigma(\mathcal{L}) : \lambda < 0\}$$

$$\longrightarrow \quad \boxed{\mathbf{n}(\mathcal{L}) = k_u + k_i^-}$$

Krein signature: the sign of $\langle \mathcal{L}v, v\rangle$ for a simple eigenvalue with eigenvector v

Definition of k_i^-

- Take $\lambda \in \sigma(\mathcal{A})$ with $\mathrm{Re}\,\lambda = 0$, and consider the associated spectral subspace E_λ (finite-dimensional)
- Consider the Hermitian matrix $\mathbf{L}(\lambda)$ associated with the quadratic form $\langle \mathcal{L}|_{E_\lambda} \cdot, \cdot \rangle$ on E_λ
- Define $k_i^-(\lambda) = \mathbf{n}(\mathbf{L}(\lambda))$ (the number of negative eigenvalues of the matrix $\mathbf{L}(\lambda)$)
- Set

$$k_i^- := \sum_{\lambda \in \sigma(\mathcal{A}), \mathrm{Re}\,\lambda = 0} k_{\mathrm{i}}^-(\lambda)$$

gKdV equation

- perturbation arguments (a, b small)
  - locate the spectra for $a = b = 0$

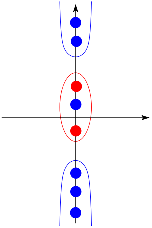
    $\longrightarrow$ operators with constant coefficients

    $\longrightarrow$ use Fourier analysis
- use $\mathbf{n}(\mathcal{L}) = k_u + k_i^-$

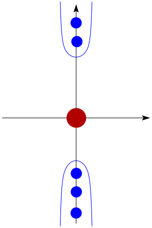
Spectra at $a = b = 0$

- **Spectrum of $\mathcal{A}_{0,0,\gamma}$**

$\gamma \neq 0$



$\gamma = 0$



- **Spectrum of $\mathcal{L}_{0,0,\gamma}$**

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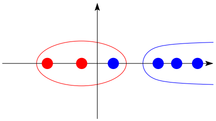

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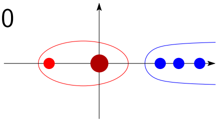
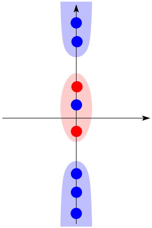
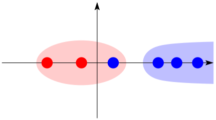


Perturbation arguments: (a, b) small

Spectrum of $\mathcal{A}_{a,b,\gamma}$



Spectrum of $\mathcal{L}_{a,b,\gamma}$

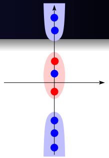


- *Spectral decomposition*

$$\sigma(\mathcal{A}_{a,b,\gamma}) = \sigma_1(\mathcal{A}_{a,b,\gamma}) \cup \sigma_2(\mathcal{A}_{a,b,\gamma})$$

- *The eigenvalues in $\sigma_1(\mathcal{A}_{a,b,\gamma})$ have positive Krein signature*
- $\mathbf{n}(\mathcal{L}) = k_u + k_i^- \implies \boxed{\boldsymbol{\sigma_1(\mathcal{A}_{a,b,\gamma}) \subset i\mathbb{R}}}$

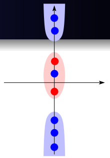
Location of $\sigma_2(\mathcal{A}_{a,b,\gamma})$



- $|\gamma| \geq \gamma_*$

- $\gamma \sim \mathbf{0}$

Location of $\sigma_2(\mathcal{A}_{a,b,\gamma})$



- $|\gamma| \geq \gamma_*$
  - *The three eigenvalues in* $\sigma_2(\mathcal{A}_{a,b,\gamma})$ *are simple*
  - *The spectrum is symmetric w.r.t. the imaginary axis*

  $$\Longrightarrow \quad \boldsymbol{\sigma_2(\mathcal{A}_{a,b,\gamma}) \subset i\mathbb{R}}$$

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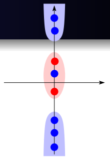

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  $\Longrightarrow$ $\boldsymbol{\sigma_2(\mathcal{A}_{a,b,\gamma}) \subset i\mathbb{R}}$

- $\gamma \sim \mathbf{0}$
  - compute a **basis** $\left\{\xi^0_{a,b,\gamma},\ \xi^1_{a,b,\gamma},\ \xi^2_{a,b,\gamma}\right\}$ for the three dimensional spectral subspace;
  - compute the 3×3-matrix $\mathcal{M}_{a,b,\gamma}$ representing the **action of** $\mathcal{A}_{a,b,\gamma}$ on this subspace;
  - locate the three **eigenvalues** of this matrix [3]

  $\Longrightarrow$ **purely imaginary if $p < 2$**

Spectral stability

Theorem

Consider the generalized KdV equation

$$u_t + (u_{xx} + u^{p+1})_x = 0, \quad p \geq 1$$

and the periodic travelling wave $q_{a,b}$ for a and b sufficiently small.

- $p < 2$
  - $k_u(\gamma) = 0,\ k_i^-(\gamma) = 2,$ *for any* $\gamma \in \left[-\frac{1}{2}, \frac{1}{2}\right) \setminus \{0\}$
  - $\sigma(\mathcal{A}_{a,b,\gamma}) \subset \mathrm{i}\mathbb{R}$
  - **the periodic wave is spectrally stable**
- $p > 2$
  - $k_u(\gamma) = 1,\ k_i^-(\gamma) = 1,$ *for sufficiently small* $\gamma = o(|a|)$
  - $\sigma(\mathcal{A}_{a,b,\gamma}) \cap \{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda > 0\} \neq \emptyset$
  - **the periodic wave is spectrally unstable**

KP equations

$$\boxed{(u_t - u_{xxx} - uu_x)_x + \sigma u_{yy} = 0} \qquad x, y \in \mathbb{R}$$

- **KP-I equation:** $\sigma = 1$ *(positive dispersion)*
- **KP-II equation:** $\sigma = -1$ *(negative dispersion)*

- *2d generalization of the KdV-equation*

$$\boxed{u_t - u_{xxx} - uu_x = 0}$$

1d periodic traveling waves

- *1d traveling waves:* $\boxed{u(x,y,t) = v(x-ct)}$; speed c
- *periodic waves:* v is periodic and

$$\boxed{v'' = -cv - \frac{1}{2}v^2 + b}$$

- Galilean and scaling invariances: $\boxed{c = 1,\ \ b = 0}$

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- Galilean and scaling invariances: $\boxed{c = 1,\ \ b = 0}$
- **small periodic waves:** $\boxed{v_a(\xi) = P_a(k_a\xi)}$**,** a **small**

$$P_a(z) = a\cos(z) + \frac{1}{4}\left(\frac{1}{3}\cos(2z) - 1\right)a^2 + O(|a|^3)$$

$$k_a^2 = 1 - \frac{5}{24}a^2 + O(a^4)$$

Linear equation

- scaling $z = k_a(x - t), \quad \tilde{y} = k_a^2 y, \quad \tilde{t} = k_a^3 t$
- *KP equation*

$$u_{tz} - u_{zzzz} - \frac{1}{k_a^2} u_{zz} - \frac{1}{k_a^2}(uu_z)_z + \sigma u_{yy} = 0$$

  - *the periodic wave $P_a(z)$ is a stationary solution*

Linear equation

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  - *the periodic wave $P_a(z)$ is a stationary solution*

- **linearized equation**

$$\boxed{w_{tz} - w_{zzzz} - \frac{1}{k_a^2} w_{zz} - \frac{1}{k_a^2}(P_a w)_{zz} + \sigma w_{yy} = 0}$$

  - *coefficients depending upon z*
  - **Ansatz** $\boxed{w(z,y,t) = e^{\lambda t + i\ell y} W(z)}$, $\lambda \in \mathbb{C}, \ \ell \in \mathbb{R}$

Spectral stability

$$\lambda W_z - W_{zzzz} - \frac{1}{k_a^2} W_{zz} - \frac{1}{k_a^2} (P_a W)_{zz} - \sigma \ell^2 W = 0$$

Spectral stability

$$\lambda W_z - W_{zzzz} - \frac{1}{k_a^2} W_{zz} - \frac{1}{k_a^2} (P_a W)_{zz} - \sigma \ell^2 W = 0$$

- **linear operator**

$$\mathcal{M}_a(\lambda, \ell) = \lambda \partial_z - \partial_z^4 - \frac{1}{k_a^2} \partial_z^2((1 + P_a)\cdot) - \sigma \ell^2$$

- the periodic wave is **spectrally stable** if $\mathcal{M}_a(\lambda, \ell)$ *is invertible for any* $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$, and **unstable** otherwise
- the type of the perturbations is determined by *the choice of the function space and the values of* ℓ

One-dimensional perturbations

$$\boxed{\ell = 0}$$

- $\mathcal{M}_a(\lambda, 0) = \partial_z \mathcal{K}_a(\lambda), \quad \mathcal{K}_a(\lambda) = \lambda - \partial_z^3 - \frac{1}{k_a^2}\partial_z((1 + P_a)\cdot)$
  - $\mathcal{K}_a(\lambda)$ *is the linear operator in the KdV equation*
  - ∂_z **is not invertible** (in general)
- *replace* $\mathcal{M}_a(\lambda, 0)$ *by* $\mathcal{K}_a(\lambda)$

One-dimensional perturbations

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- *replace $\mathcal{M}_a(\lambda, 0)$ by $\mathcal{K}_a(\lambda)$*

Definition

The periodic wave is **spectrally stable in one dimension** *if the linear operator $\mathcal{K}_a(\lambda)$ is invertible, for any $\lambda \in \mathbb{C}$, $\boxed{\mathrm{Re}\,\lambda > 0}$,*

- *in $L^2(0, 2\pi)$, for 2π-periodic perturbations;*
- *in $L^2(\mathbb{R})$ or $C_b(\mathbb{R})$, for localized or bounded perturbations.*

Two-dimensional perturbations

$\boxed{\ell \neq 0}$

Definition

The periodic wave is **transversely spectrally stable** *if*

- *it is spectrally stable in one dimension*
- *the linear operator* $\underline{\mathcal{M}_a(\lambda,\ell)}$ *is invertible, for any* $\lambda \in \mathbb{C}$, $\boxed{\mathrm{Re}\,\lambda > 0}$, *and any* $\ell \in \mathbb{R}$, $\boxed{\ell \neq 0}$,
  - *in* $L^2(0,2\pi)$, *for perturbations which are* 2π*-periodic in* z;
  - *in* $L^2(\mathbb{R})$ *or* $C_b(\mathbb{R})$, *for perturbations which are localized or bounded in* z.

Previous results

- *spectral stability in one dimension for periodic, localized, bounded perturbations*

  [H. & Kapitula, 2008; Bottman & Deconinck, 2009]

- *transverse spectral (in)stability for perturbations which are periodic in z, when ℓ is small*

  [Johnson & Zumbrun, 2009]

  - *long wavelength transverse perturbations*, when $\ell \ll 1$
  - *short wavelength transverse perturbations*, when $\ell \gg 1$
  - *finite wavelength transverse perturbations*, otherwise

Spectral stability problem

- *study the invertibility of the operator* $\mathcal{M}_a(\lambda, \ell)$ *in* $L^2(0, 2\pi)$, *for* $\boxed{\mathrm{Re}\,\lambda > 0}$, *and* $\boxed{\ell \neq 0}$

Spectral stability problem

- *study the invertibility of the operator* $\mathcal{M}_a(\lambda, \ell)$ *in* $L^2(0, 2\pi)$, *for* $\boxed{\mathrm{Re}\,\lambda > 0}$, *and* $\boxed{\ell \neq 0}$

Lemma

Assume that $\lambda \in \mathbb{C}$ *and* $\ell \in \mathbb{R}$, $\ell \neq 0$.

The linear operator $\mathcal{M}_a(\lambda, \ell)$ *acting in* $L^2(0, 2\pi)$ *is invertible* *if and only if* λ *belongs to the spectrum of the operator*

$$\boxed{\mathcal{A}_a(\ell) = \partial_z^3 + \frac{1}{k_a^2}\partial_z((1 + P_a)\cdot) + \ell^2\partial_z^{-1}}$$

acting in $L_0^2(0, 2\pi)$ *(square-integrable functions with zero-mean).*

Proof

- *$\mathcal{M}_a(\lambda,\ell)$ is invertible in $L^2(0,2\pi)$ if and only if its restriction to the subspace $L^2_0(0,2\pi)$ is invertible*
  - elements in the kernel have zero mean when $\ell \neq 0$
- $\boxed{\mathcal{M}_a(\lambda,\ell) = \partial_z\left(\lambda - \mathcal{A}_a(\ell)\right)}$ and ∂_z *is invertible in* $L^2_0(0,2\pi)$

Spectrum of $\mathcal{A}_a(\ell)$

- **Properties of the spectrum**
  - consists of isolated eigenvalues with finite algebraic multiplicity
  - is symmetric with respect to the real and the imaginary axis
- **We rely on**
  - *the decomposition* $\boxed{\mathcal{A}_a(\ell) = -\partial_z \mathcal{L}_a(\ell)}$
    $\mathcal{L}_a(\ell) = -\partial_z^2 - \frac{1}{k_a^2}((1+P_a)\cdot) - \ell^2\partial_z^{-2}$ self-adjoint
    and the property: $\mathcal{A}_a(\ell)$ has no unstable eigenvalues, if $\mathcal{L}_a(\ell)$ has positive spectrum
  - *perturbation arguments for linear operators: $\mathcal{A}_a(\ell)$ is a small perturbation, for small a, of the operator with constant coefficients* $\boxed{\mathcal{A}_0(\ell) = \partial_z^3 + \partial_z + \ell^2\partial_z^{-1}}$

Transverse instability

Theorem

For any a sufficiently small, there exists $\boxed{\ell_a^2 = \frac{1}{12}a^2 + O(a^4)}$*, such that*

1. *for any* $\ell^2 \geq \ell_a^2$*, the spectrum of* $\mathcal{A}_a(\ell)$ *is purely imaginary;*
2. *for any* $\ell^2 < \ell_a^2$*, the spectrum of* $\mathcal{A}_a(\ell)$ *is purely imaginary, except for a pair of simple real eigenvalues, with opposite signs.*

- *small periodic waves of the KP-I equation are transversely unstable*
- *the instability occurs in the transverse long-wave regime,* $\ell^2 = O(a^2)$

Proof

- **spectrum of the unperturbed operator $\mathcal{L}_0(\ell)$**
  - *strictly positive if* $\ell \neq 0$
  - 0 *is a simple eigenvalue if* $\ell = 0$

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$|\ell| \geq \ell_*$

- *use the decomposition* $\mathcal{A}_a(\ell) = -\partial_z \mathcal{L}_a(\ell)$
- *perturbation arguments show that* $\mathcal{L}_a(\ell)$ *has no negative eigenvalues*

ℓ *small*

Proof

ℓ small

- **decompose** $\boxed{\sigma(\mathcal{A}_a(\ell)) = \sigma_0(\mathcal{A}_a(\ell)) \cup \sigma_1(\mathcal{A}_a(\ell))}$

$\boxed{\sigma_1(\mathcal{A}_a(\ell))}$

- *use the decomposition* $\boxed{\mathcal{A}_a(\ell) = -\partial_z \mathcal{L}_a(\ell)}$
- *positivity of the restriction of $\mathcal{L}_a(\ell)$ to the corresponding spectral subspace*

$\boxed{\sigma_0(\mathcal{A}_a(\ell))}$ *contains two eigenvalues*

- *direct computation of the eigenvalues:* compute successively a basis of the spectral subspace, the 2×2 matrix representing the action of $\mathcal{A}_a(\ell)$ on this basis, and the eigenvalues of this matrix

Localized or bounded perturbations

- *study invertibility of the operator* $\mathcal{M}_a(\lambda, \ell)$ *acting in* $L^2(\mathbb{R})$ *or* $C_b(\mathbb{R})$, *for* $\boxed{\mathrm{Re}\,\lambda > 0}$ *and* $\boxed{\ell \neq 0}$

Localized or bounded perturbations

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Lemma

The linear operator $\mathcal{M}_a(\lambda, \ell)$ *is invertible, in either* $L^2(\mathbb{R})$ *or* $C_b(\mathbb{R})$, *if and only if the linear operators*

$$\boxed{\mathcal{M}_a(\lambda, \ell, \gamma) = \lambda(\partial_z + i\gamma) - (\partial_z + i\gamma)^4 - \frac{1}{k_a^2}(\partial_z + i\gamma)^2((1 + P_a)\cdot) - \ell^2}$$

acting in $\boxed{L^2(0, 2\pi)}$ *are invertible, for any* $\boxed{\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]}$.

Proof: *Floquet theory*

Spectral stability problem

$\boxed{\gamma = 0}$ corresponds to periodic perturbations

$\boxed{\gamma \neq 0}$ *the operator $\partial_z + i\gamma$ is invertible*

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$\boxed{\gamma = 0}$ corresponds to periodic perturbations

$\boxed{\gamma \neq 0}$ *the operator* $\partial_z + i\gamma$ *is invertible*

Lemma

Assume that $\gamma \in (-\frac{1}{2}, \frac{1}{2}]$ *and* $\boxed{\gamma \neq 0}$.

The linear operator $\mathcal{M}_a(\lambda, \ell, \gamma)$ *is invertible in* $L^2(0, 2\pi)$

if and only if λ *belongs to the spectrum of the operator*

$$\boxed{\mathcal{A}_a(\ell, \gamma) = (\partial_z + i\gamma)^3 + \frac{1}{k_a^2}(\partial_z + i\gamma)((1 + P_a)\cdot) + \ell^2(\partial_z + i\gamma)^{-1}}$$

acting in $L^2(0, 2\pi)$.

Spectrum of $\mathcal{A}_a(\ell, \gamma)$

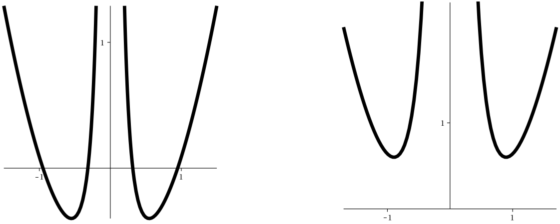
- **Properties of the spectrum**
  - consists of isolated eigenvalues with finite algebraic multiplicity
  - is symmetric with respect to *the imaginary axis*
  - $\sigma(\mathcal{A}_a(\ell,\gamma)) = \sigma(-\mathcal{A}_a(\ell,-\gamma)) \longrightarrow$ restrict to $\boxed{\gamma \in \left(0, \frac{1}{2}\right]}$
- **We rely on**
  - *the decomposition* $\boxed{\mathcal{A}_a(\ell,\gamma) = -(\partial_z + i\gamma)\mathcal{L}_a(\ell,\gamma)}$ *and the property:* $\mathcal{A}_a(\ell,\gamma)$ *has no unstable eigenvalues, if* $\mathcal{L}_a(\ell,\gamma)$ *has positive spectrum*
  - *perturbation arguments:* $\mathcal{A}_a(\ell,\gamma)$ *is a small perturbation, for small a, of the operator with constant coefficients*

$$\boxed{\mathcal{A}_0(\ell,\gamma) = (\partial_z + i\gamma)^3 + (\partial_z + i\gamma) + \ell^2(\partial_z + i\gamma)^{-1}}$$

Unperturbed operators

- **spectrum of $\mathcal{L}_0(\ell,\gamma)$:**

eigenvalues $\boxed{k = n+\gamma,\ n\in\mathbb{Z}}$ in $\boxed{k \mapsto k^2 - 1 + \dfrac{\ell^2}{k^2}}$



$\ell = 0.3$ $\qquad$ $\ell = 0.8$

Spectrum of $\mathcal{L}_0(\ell, \gamma)$

- positive spectrum for $\ell^2 > \ell_-^2$
- one negative eigenvalue if $\ell_0^2 < \ell^2 < \ell_-^2$
- two negative eigenvalues if $0 < \ell^2 < \ell_0^2$

$$\boxed{0 < \ell_0^2 = \gamma^2(1-\gamma^2) < \ell_-^2 = \gamma(1-\gamma)^2(2-\gamma)}$$

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$$\boxed{0 < \ell_0^2 = \gamma^2(1-\gamma^2) < \ell_-^2 = \gamma(1-\gamma)^2(2-\gamma)}$$

$$\boxed{\ell^2 \geq \ell_-^2 + \varepsilon_*}$$

the spectrum of $\mathcal{A}_a(\ell, \gamma)$ is purely imaginary

$$\boxed{0 < \ell^2 < \ell_-^2 + \varepsilon_*}$$

$0 < \ell^2 < \ell_-^2 + \varepsilon_*$

- **decompose** $\sigma(\mathcal{A}_a(\ell,\gamma)) = \sigma_0(\mathcal{A}_a(\ell,\gamma)) \cup \sigma_1(\mathcal{A}_a(\ell,\gamma))$

$\sigma_1(\mathcal{A}_a(\ell,\gamma))$ **is purely imaginary**

$\sigma_0(\mathcal{A}_a(\ell,\gamma))$ *contains one or two eigenvalues*

- *one eigenvalue:* use symmetry of the spectrum
- *two eigenvalues:* direct computation (compute successively a basis of the spectral subspace, the 2×2 matrix representing the action of $\mathcal{A}_a(\ell,\gamma)$ on this basis, and the eigenvalues of this matrix)

Transverse instability

Theorem

Assume $\gamma \in \left(0, \frac{1}{2}\right]$ *and set* $\ell_c(\gamma) = \sqrt{3}\gamma(1-\gamma)$.

For any a sufficiently small, there exists $\varepsilon_a(\gamma) = \gamma^{3/2}(1-\gamma)^{3/2}|a|(1 + O(a^2)) > 0$ *such that*

1. *for* $|\ell^2 - \ell_c^2(\gamma)| \geq \varepsilon_a(\gamma)$, *the spectrum of* $\mathcal{A}_a(\ell, \gamma)$ *is purely imaginary;*
2. *for* $|\ell^2 - \ell_c^2(\gamma)| < \varepsilon_a(\gamma)$, *the spectrum of* $\mathcal{A}_a(\ell, \gamma)$ *is purely imaginary, except for a pair of complex eigenvalues with opposite nonzero real parts.*

Spectral stability problem

- **Same of formulation** *in terms of the spectra of the operators*

$$\boxed{\mathcal{A}_a(\ell) \quad \text{and} \quad \mathcal{A}_a(\ell,\gamma)}$$

Spectral stability problem

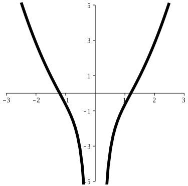
- **Same of formulation** *in terms of the spectra of the operators*

$$\boxed{\mathcal{A}_a(\ell) \quad \text{and} \quad \mathcal{A}_a(\ell,\gamma)}$$

- **Main difference:** *eigenvalues of the unperturbed operators*

$$\boxed{\mathcal{L}_0(\ell) \quad \text{and} \quad \mathcal{L}_0(\ell,\gamma)}$$

eigenvalues $\boxed{k = n+\gamma,\ n\in\mathbb{Z}}$ in $\boxed{k \mapsto k^2 - 1 - \frac{\ell^2}{k^2}}$



the number of negative eigenvalues increases with ℓ

Transverse stability result

Theorem

The spectrum of the operator $\mathcal{A}_a(\ell)$ acting in $L^2_0(0, 2\pi)$ is purely imaginary, for any ℓ and a sufficiently small.

Small periodic waves of the KP-II equation are transversely stable for perturbations

- *which are 2π-periodic in z (the direction of propagation)*
- *have long wavelength in the transverse direction*

Conclusion

- **KP-I equation** *transverse instability*
  - for periodic and non-periodic perturbations
  - instabilities occur in the transverse long-wave regime
- **KP-II equation** *transverse stability for perturbations*
  - which are periodic in the direction of propagation
  - have long wavelength in the transverse direction
- *same type of stability properties as for solitary waves*