Estimates with higher angular integrability and applications

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Referencess

Joint works with Federico Cacciafesta and Renato Luca' (Rome I)

- Piero D'Ancona and Renato Luca': Stein-Weiss and Caffarelli-Kohn-Nirenberg inequalities with angular integrability (arXiv: 1105.5930) (to appear on *JMAA*)
- Piero D'Ancona and Federico Cacciafesta: Endpoint estimates and global existence for the nonlinear Dirac equation with potential (arXiv: 1103.4014)

Background

In this talk we shall mainly deal with estimates, but the ultimate goal we have in mind are improvements to local and global existence and regularity results for several nonlinear equations (Wave, Schrödinger, Dirac, Navier-Stokes).

Main theme: improvements in several classical inequalities obtained by quantifying the 'defect of symmetry' of functions

General question: given a *classical estimate*

$$\|u\|_X \leq C\|u\|_Y$$

with a sharp constant C, what can be said of the *remainder term*

$$R(u) \equiv C\|u\|_Y - \|u\|_X,$$

and what is the influence on $R(u)$ of suitable *symmetry assumptions*?

Certainly $R(u) \geq 0$, and $R(u_0) = 0$ for the *extremizers* of the inequality, or more generally $R(u_j) \to 0$ for an *extremizing sequence* u_j

Extremizers are typically highly symmetric functions; this is connected with the properties of the norms w.r.to rearrangments

Even extremizing sequences exhibit symmetry (after rescaling they tend to symmetric functions)

Trivial example: Hölder's inequality

For $p^{-1} + q^{-1} = 1$

$$\|f\|_{L^p} = \|g\|_{L^q} = 1 \qquad \Longrightarrow \qquad \|fg\|_{L^1} \leq 1$$

Using the stronger form of Young's inequality (for $1 < p \leq 2$)

$$q^{-1}(u^{p/2} - v^{q/2})^2 \leq \frac{u^p}{p} + \frac{v^q}{q} - uv \leq p^{-1}(u^{p/2} - v^{q/2})$$

the inequality can be improved as follows (for $1 < p \leq 2$, Aldaz 2007):

$$\boxed{1 - \frac{1}{p}\||f|^{p/2} - |g|^{q/2}\|_{L^2}^2 \leq \|fg\|_{L^1} \leq 1 - \frac{1}{q}\||f|^{p/2} - |g|^{q/2}\|_{L^2}^2}$$

Equality $\|fg\|_{L^1} = 1$ occurs iff $|f|^p = |g|^q$. The remainder term

$$\||f|^{p/2} - |g|^{q/2}\|_{L^2}$$

measures the distance from this "maximally symmetric" case

Example: improvement of Hardy's inequality

For $n \geq 3$, $\Omega \subseteq \mathbb{R}^n$

$$\|\nabla u\|^2_{L^2(\Omega)} \geq \frac{(n-2)^2}{4} \left\| \frac{u}{|x|} \right\|^2_{L^2(\Omega)}$$

Actually ∇u can be replaced by $\partial_r u = \frac{x}{|x|} \cdot \nabla u$
Interesting since the geometry of Ω prevents u from being symmetric

Brézis and Vazquez 1997:

$$\boxed{\|\nabla u\|^2_{L^2(\Omega)} \geq \frac{(n-2)^2}{4} \left\| \frac{u}{|x|} \right\|^2_{L^2(\Omega)} + \frac{R}{\Lambda_1} \|u\|^2_{L^2}}$$

where Λ_1= first eigenvalue of Δ on the unit disk of $\mathbb{R}^2$, while R is the radius of the ball with volume $|\Omega|$

Notice that the radial rearrangement of u decreases the norm $\|\nabla u\|$ and increases the norm $\||x|^{-2}u\|$

Example: Strauss' radial Sobolev embedding

Sobolev embedding on $\mathbb{R}^n$, $n \geq 3$

$$\|u\|_{L^{\frac{2n}{n-2}}} \lesssim \|\nabla u\|_{L^2}$$

is sharp for generic functions

If u is a radial function, W.Strauss 1977:

$$\boxed{|x|^{\frac{n-1}{2}}|u(x)| \lesssim \|\nabla u\|_{L^2}}$$

We see a dramatic improvement:

- a *pointwise* control of u
- a *decay at infinity* $u \sim |x|^{-\frac{n-1}{2}}$

Not surprising: a radial function is essentially a function of one variable (and the volume form produces the $|x|^{\frac{n-1}{2}}$ weight $\sim$ area on the $n-1$ dimensional sphere $\mathbb{S}^{n-1}$)

We can regard the general Sobolev embedding and its radial improvement as extreme cases:

- maximal symmetry $\implies$ strongest estimate
- minimal symmetry $\implies$ weakest estimate

Natural question: is it possible to bridge the gap between the two extreme cases and obtain a *family of estimates* which interpolate between them? How to measure the *distance from the symmetric case*?

In the following we focus on *radial symmetry*

Quantifying the defect of radiality

We shall use the following type of norm, with different integrability in the radial and in the angular directions:

$$\|f\|_{L^p_{|x|}L^{\widetilde{p}}_\theta} = \left(\int_0^{+\infty} \|f(\rho\,\cdot\,)\|^p_{L^{\widetilde{p}}(\mathbb{S}^{n-1})}\rho^{n-1}d\rho\right)^{\frac{1}{p}}$$

$$\|f\|_{L^\infty_{|x|}L^{\widetilde{p}}_\theta} = \sup_{\rho>0}\|f(\rho\,\cdot\,)\|_{L^{\widetilde{p}}(\mathbb{S}^{n-1})}.$$

Clearly

$$\|u\|_{L^p_{|x|}L^p_\theta} \equiv \|u\|_{L^p(\mathbb{R}^n)}$$

and

$$u \text{ radial} \quad \Longrightarrow \quad \|u\|_{L^pL^{\widetilde{p}}} \simeq \|u\|_{L^p(\mathbb{R}^n)} \quad \forall p, \widetilde{p} \in [1,\infty]$$

Norms with different radial vs. angular properties are not new in the context of dispersive equations

A few examples

Hoshiro 1997: smoothing estimates with angular regularity

$$\| |x|^{\alpha-1} |D|^{\alpha} e^{it\Delta} \Lambda_\theta^{\frac{1}{2}-\alpha} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \qquad 0 \le \alpha < \frac{1}{2}$$

$$\| |x|^{-\alpha-\frac{1}{2}} \Lambda_\theta^{\alpha} f \|_{L^2(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \qquad 0 < \alpha < \frac{n-1}{2}$$

Here

$$\Lambda_\theta = (1 - \Delta_{\mathbb{S}^{n-1}})^{\frac{1}{2}}$$

Sterbenz-Rodnianski 2004: Strichartz estimates with angular regularity

Strichartz estimates combined with Sobolev embedding give for all $n \geq 3$ (Kato, Yajima, Ginibre-Velo, Keel-Tao)

$$\boxed{\| |D|^{\frac{n}{r}+\frac{1}{p}-\frac{n}{2}} e^{it|D|} f \|_{L^p_t L^r_x} \lesssim \|f\|_{L^2}}$$

for all p, r such that

$$p \in [2, \infty] \qquad 0 < \frac{1}{r} \leq \frac{1}{2} - \frac{2}{(n-1)p}.$$

The endpoint case $r = \infty$, for general data f,

$$\|e^{it|D|} f\|_{L^2_t L^\infty_x} \lesssim \||D| f\|_{L^2} \quad (n = 3) \quad \text{is false}$$

An analogous situation for the Schrödinger equation: similar inequalities are available, and the endpoint case in general

$$\|e^{it\Delta} f\|_{L^2_t L^\infty_x} \lesssim \|f\|_{L^2} \quad (n = 2) \quad \text{is false}$$

These estimates are a fundamentale important tool in the study of local and global existence for NLS, NLWE, NLDirac...

The failure of the endpoint case ($r = \infty$) is a major obstruction in dealing with several critical cases

No improvement for BMO data, or for data with localized spectrum in Fourier variables

Klainerman-Machedon 1993: the estimate

$$\|e^{it|D|}f\|_{L^2_t L^\infty_x} \lesssim \||D||f\|_{L^2} \quad (n=3)$$

is true for radial data f

Machihara-Nakamura-Nakanishi-Ozawa 2005 found a way to bridge the gap in the case of non-radial data: for generic f and for all $p < \infty$

$$\|e^{it|D|}f\|_{L^2_t L^\infty_{|x|} L^p_\theta} \lesssim \sqrt{p} \cdot \||D|f\|_{L^2} \quad (n=3)$$

Tao 1998: the following estimate holds for general f

$$\|e^{it\Delta}f\|_{L^2_t L^\infty_{|x|} L^2_\theta} \lesssim \|f\|_{L^2} \quad (n=2)$$

thus in particular the endpoint estimate for the Schrödinger equation is true for radial data f

The critical Dirac equation

A clever application of the estimate

$$\|e^{it|D|}f\|_{L^2_t L^\infty_{|x|} L^p_\theta} \lesssim \sqrt{p} \cdot \||D|f\|_{L^2} \quad (n=3)$$

to the cubic Dirac equation

$$iu_t = \mathcal{D}u + P_3(u, \overline{u}), \qquad u(0,x) = f(x)$$

where $P_3(u, \overline{u})$ is a cubic homogeneous polynomial

Here $u : \mathbb{R}_t \times \mathbb{R}^3 \to \mathbb{C}^4$, and the Dirac operator $\mathcal{D}$ is the selfadjoint operator on $L^2(\mathbb{R}^3; \mathbb{C}^4)$

$$\mathcal{D} = i^{-1}\left(\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_2 \frac{\partial}{\partial x_3}\right)$$

where α_j are the Dirac matrices

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$\mathcal{D}$ is not positive, and $\sigma(\mathcal{D}) = \mathbb{R}$

The main open problem is well posedness in the energy space H^1

Using the relation

$$(\mathcal{D} + i\partial_t)(\mathcal{D} - i\partial_t) = (-\Delta + \partial_{tt}^2)I_4$$

we can espress the Dirac flow $e^{it\mathcal{D}}$ in terms of the wave flow $e^{it|D|}$:

$$e^{it\mathcal{D}}f = \cos(t|D|)f + i\frac{\sin(t|D|)}{|D|}\mathcal{D}f, \qquad |D| = (-\Delta)^{1/2}$$

Hence Strichartz estimates for $e^{it\mathcal{D}}$ and for $e^{it|D|}$ are identical

In particular the following endpoint estimate fails:

$$\|e^{it\mathcal{D}}f\|_{L^2L^\infty} \lesssim \||D|f\|_{L^2} \qquad (n = 3)$$

If the endpoint estimate were true one could write for the map

$$v \mapsto \Phi(v) = e^{it\mathcal{D}} f + i \int_0^t e^{i(t-t')\mathcal{D}} v(t')^3 dt'.$$

an estimate like

$$\left\| \int_0^t e^{i(t-t')\mathcal{D}} v(t')^3 dt \right\|_{L^2 L^\infty} \lesssim \int_{-\infty}^{+\infty} \| e^{it\mathcal{D}} e^{-it'\mathcal{D}} v(t')^3 \|_{L^2 L^\infty} dt' \lesssim \| v^3 \|_{L^1 H^1}$$

and in conjuction with the conservation of H^1 energy this would imply

$$\|\Phi(v)\|_{L_t^\infty H_x^1} + \|\Phi(v)\|_{L_t^2 L_x^\infty} \lesssim \|f\|_{H^1} + \|v\|_{L^\infty H^1} \|v\|_{L^2 L^\infty}^2.$$

For small data, Φ would be a contraction in the norm

$$\| \cdot \|_{L^2 L^\infty} + \| \cdot \|_{L^\infty H^1}$$

yielding the existence of global small H^1 solutions to the cubic NLD

The fact that the endpoint estimate is true for radial data unfortunately does not help

Indeed, the tempting argument

$$\textit{radial symmetry} \implies \textit{endpoint estimate} \implies \textit{global existence}$$

does not work for the cubic Dirac equation

Radial data do not produce a radial solution since the Dirac operator does not commute with rotations

M-N-N-O 2005: if the data are in H^1 and have an additional angular regularity (i.e. $\Lambda_\theta^\epsilon f \in H^1$ for some $\epsilon > 0$), it is possible to use the almost-endpoint estimate

$$\|e^{it\mathcal{D}} f\|_{L^2_t L^\infty_{|x|} L^p_\theta} \lesssim \sqrt{p} \cdot \||D|f\|_{L^2}$$

to prove global existence with small data

This method can be extended to general cubic Dirac equations in presence of potential perturbations

Indeed, we considered an equation of the form

$$\boxed{iu_t = \mathcal{D}u + V(x)u + P_3(u,\overline{u}), \qquad u(0,x) = f(x)} \tag{1}$$

Assumptions on the potential: $V(x)$ is 4×4 hermitian and for some $s > 1$, $C > 0$, δ small enough

$$\|\Lambda_\omega^s V(|x|\,\cdot\,)\|_{L^2(\mathbb{S}^2)} \leq \frac{\delta}{v(x)}, \qquad \|\Lambda_\omega^s \nabla V(|x|\,\cdot\,)\|_{L^2(\mathbb{S}^2)} \leq \frac{C}{v(x)}, \tag{2}$$

where $v(x) = |x|^{\frac{1}{2}} |\log|x||^{\frac{1}{2}+} + |x|^{1+}$

Theorem (D'A-Cacciafesta 2011)

Let $P_3(u,\overline{u})$ be a $\mathbb{C}^4$-valued homogeneous cubic polynomial, V as in (2). Then there exists ϵ_0 such that for all initial data satisfying ($s > 1$)

$$\|\Lambda_\theta^s f\|_{H^1} < \epsilon_0 \tag{3}$$

the Cauchy problem (1) admits a unique global solution $u \in CH^1$, such that $u \in L^2 L^\infty$ and $\Lambda_\theta^s u \in L^\infty H^1$.

$a_b c_d e_f$

Main results: weighted estimates for fractional integrals

A large number of inequalities cah be reduced to the study of the fractional integral

$$T_\gamma f(x) = \int \frac{f(y)}{|x-y|^\gamma} dy, \qquad 0 < \gamma < n$$

thanks to the relation

$$|D|^{\gamma-n} f = c_{n,\gamma} T_\gamma f$$

For example

$$\begin{aligned} &\textit{Sobolev embeddings} \iff T_\gamma : L^p \to L^q \ \textit{bounded} \\ &\textit{Hardy inequality} \iff T_1 : L^2(|x|^{-1}dx) \to L^2 \ \textit{bounded} \end{aligned}$$

The problem of weighted L^p estimates for T_γ is a classical problem of harmonic analysis

The possible weights such that

$$T_\gamma : L^p(w(x)dx) \to L^q(v(x)dx)$$

have been completely characterized (Muckenhoupt weights)

Here we shall be concerned with power weigths $|x|^\alpha$

Stein-Weiss 1958 gave necessary and sufficient conditions for

$$\||x|^{-\beta} T_\gamma \phi\|_{L^q} \leq C(\alpha, \beta, p, q) \cdot \||x|^{\alpha}\phi\|_{L^p} \tag{4}$$

in the range $n \geq 1$, $1 < p \leq q < \infty$.

Obvious necessary conditions:

local integrability $\qquad \alpha < \dfrac{n}{p'} \qquad \beta < \dfrac{n}{q}$

scaling invariance $\qquad \alpha + \beta + \gamma = n + \dfrac{n}{q} - \dfrac{n}{p}$

Theorem (Stein-Weiss 1958)

Assume the necessary conditions. Estimate (4) *holds iff* $\alpha + \beta \geq 0$.

In the radial case, a substantial improvement is possible and negative values of $\alpha + \beta$ are admissible

Theorem (Rubin 1983, De Napoli-Dreichman-Duran 2009)

For radial functions, estimate (4) *holds provided*

$$\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p}\right)$$

Our first result bridges the gap between these extreme cases. In the range $n \geq 2$, $1 < p \leq q < \infty$, $1 \leq \widetilde{p} \leq \widetilde{q} \leq \infty$, we consider the inequality

$$\boxed{\| |x|^{-\beta} T_\gamma \phi \|_{L^q_{|x|} L^{\widetilde{q}}_\theta} \lesssim \| |x|^\alpha \phi \|_{L^p_{|x|} L^{\widetilde{p}}_\theta}} \tag{5}$$

Obvious necessary conditions, as above:

local integrability $\qquad \alpha < \frac{n}{p'} \qquad \beta < \frac{n}{q}$

scaling invariance $\qquad \alpha + \beta + \gamma = n + \frac{n}{q} - \frac{n}{p}$

Theorem (D-Luca' 2011)

Assume the necessary conditions. Estimate (5) *holds provided*

$$\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}\right)$$

Remark: When the last condition is strict, or when $\widehat{\phi}$ has compact support, we can cover the range $1 \leq p \leq q \leq \infty$

In particular:

- when ϕ is radial, we obtain Rubin's-DDD's result since we can choose $\widetilde{q} = \widetilde{p} = \infty$
- if we choose $p = \widetilde{p}$ and $q = \widetilde{q}$ we reobtain the result of Stein and Weiss

The previous fractional integral estimate has a number of consequences. We examine a few

Weighted Sobolev embeddings

By representing $|D|^{-\alpha} f$ as $T_{n-\alpha} f$, we obtain the following general estimate

$$\boxed{\| |x|^{-\beta} u \|_{L^q_{|x|} L^{\widetilde{q}}_\theta} \lesssim \| |x|^{\alpha} |D|^{\sigma} u \|_{L^p_{|x|} L^{\widetilde{p}}_\theta}} \tag{6}$$

provided $1 < p \leq q < \infty$, $1 \leq \widetilde{p} \leq \widetilde{q} \leq \infty$ and

$$\beta < \frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad 0 < \sigma < n$$
$$\alpha + \beta = \sigma + \frac{n}{q} - \frac{n}{p}$$
$$\alpha + \beta \geq (n-1)\left(\frac{1}{q} - \frac{1}{p} + \frac{1}{\widetilde{p}} - \frac{1}{\widetilde{q}}\right).$$

As usual, if the last condition is strict we can take p, q in the full range $1 \leq p \leq q \leq \infty$.

Special cases: Strauss' estimate can be extended to

$$\boxed{|x|^{\frac{n}{p}-\sigma}|u(x)| \lesssim \||D|^{\sigma}u\|_{L^pL^{\widetilde{p}}}} \qquad \frac{n-1}{\widetilde{p}}+\frac{1}{p}<\sigma<\frac{n}{p}$$

for all $p \in (1,\infty)$, $\widetilde{p} \in [1,\infty]$ (in the radial case, choose $\widetilde{p}=\infty$)

In the weightless case we get the following (very minor) refinement of the classical Sobolev embedding

$$\|u\|_{L^qL^r} \lesssim \||D|^{\frac{n}{p}-\frac{n}{q}}u\|_{L^pL^r}$$

for all $1<p \le q<\infty$ and $r \in [1,+\infty]$

Caffarelli-Kohn-Nirenberg inequalities

Consider the family of inequalities for $n \geq 1$, $p, q, r \in [1, \infty)$, $a \in (0, 1]$

$$\boxed{\||x|^{-\gamma}u\|_{L^r} \leq C\||x|^{-\alpha}\nabla u\|_{L^p}^{a}\||x|^{-\beta}u\|_{L^q}^{1-a}} \tag{7}$$

Obvious necessary conditions:

local integrability $\quad \gamma < \frac{n}{r} \qquad \alpha < \frac{n}{p} \qquad \beta < \frac{n}{q}$

scaling invariance $\quad \gamma - \frac{n}{r} = a\left(\alpha + 1 - \frac{n}{p}\right) + (1-a)\left(\beta - \frac{n}{q}\right)$

Define the quantity

$$\boldsymbol{\Delta} := \gamma - a\alpha - (1-a)\beta$$

Theorem (C-K-N 1984)

Estimate (7) *holds iff* $\boldsymbol{\Delta} \geq 0$, *and in addition* $\boldsymbol{\Delta} \leq a$ *in the case* $\gamma - n/r = \alpha + 1 - n/p$

Our result: We consider, in the range $n \geq 2$, $p, q, r, \widetilde{p}, \widetilde{q}, \widetilde{r} \in [1, \infty)$, $a \in (0, 1]$, $\sigma \in (0, n)$, the family of inequalities

$$\boxed{\||x|^{-\gamma} u\|_{L^r_{|x|} L^{\widetilde{r}}_\theta} \leq C \||x|^{-\alpha} |D|^\sigma u\|^a_{L^p_{|x|} L^{\widetilde{p}}_\theta} \cdot \||x|^{-\beta} u\|^{1-a}_{L^q_{|x|} L^{\widetilde{q}}_\theta}} \tag{8}$$

Obvious necessary conditions, as above:

local integrability $\quad \gamma < \frac{n}{r} \qquad \alpha < \frac{n}{p} \qquad \beta < \frac{n}{q}$

scaling invariance $\quad \gamma - \frac{n}{r} = a\left(\alpha + 1 - \frac{n}{p}\right) + (1 - a)\left(\beta - \frac{n}{q}\right)$

and we define the quantities ($\boldsymbol{\Delta}$ is as above)

$$\boldsymbol{\Delta} = a\sigma + n\left(\frac{1}{r} - \frac{1-a}{q} - \frac{a}{p}\right), \qquad \widetilde{\boldsymbol{\Delta}} = a\sigma + n\left(\frac{1}{\widetilde{r}} - \frac{1-a}{\widetilde{q}} - \frac{a}{\widetilde{p}}\right)$$

Theorem (D-Luca' 2011)

Estimate (8) *holds provided* $p > 1$, $\boldsymbol{\Delta} + (n-1)\widetilde{\boldsymbol{\Delta}} \geq 0$ *and*

$$a\left(\sigma - \frac{n}{p}\right) < \boldsymbol{\Delta} \leq a\sigma, \qquad a\left(\sigma - \frac{n}{\widetilde{p}}\right) \leq \widetilde{\boldsymbol{\Delta}} \leq a\sigma.$$

Work in progress: applications to regularity for Navier-Stokes

The C-K-N interpolation inequalities were developed to prove partial regularity of the solutions to the 3D Navier-Stokes equation

Theorem (C-K-N 1984)

The parabolic Hausdorff dimension of the singular set of a weak solution to 3DNS (satisfying suitable growth conditions) is less than $5/3$

We are investigating the consequences of our estimates for the regularity and the properties of the singular set for solutions to NS

Serrin-type regularity results: a weak solution $u(t,x)$ to NS is smooth as soon as it satisfies an a priori bound

$$\|u\|_{L^a(0,T;L^b(\mathbb{R}^3))} < \infty$$

for some a, b with

$$\frac{2}{a} + \frac{3}{b} \leq 1$$

Extended in a myriad of directions. For instance, Yong-Zhou 2009 considered $L^a L^b$ norms with power weights $|x|^c$

Easy to obtain a quite general result of this type using our estimates:

Theorem (D-Luca' 2012)

Let the initial datum $u_0 \in L^2_\sigma(\mathbb{R}^3)$ *(i.e.* $\nabla \cdot u_0 = 0$*) satisfy*

$$\sup_{y \in \mathbb{R}^3} \| |x-y|^{-\frac{1}{2}} u_0 \|_{L^2} < +\infty. \tag{9}$$

Then a weak solution (u,p) *to NS on* $[0,T] \times \mathbb{R}^3$ *is smooth as soon as* $u(t,x)$ *satisfies the bound*

$$\sup_{y \in \mathbb{R}^3} \| |x-y|^{\beta} u(x,t) \|_{L^\alpha([0,T];L^\gamma_{|x|} L^{\tilde{\gamma}}_\theta)} < +\infty,$$

for some $\alpha, \beta, \gamma, \widetilde{\gamma}$ *such that*

$$\frac{2}{\alpha} + \frac{3}{\gamma} = 1 - \beta, \quad \alpha > \frac{2}{1-\beta}, \quad \gamma > \frac{3}{1-\beta}, \quad -1 - 2(\frac{1}{\gamma} - \frac{1}{\tilde{\gamma}}) \le \beta < 1.$$

Weighted Strichartz estimates for the wave equation

We also mention that, combining smoothing estimates with our result for fractional integrals, we can obtain a family of weighted Strichartz estimates for the wave equation, of the form

$$\||x|^{-\delta}|D|^{\frac{n}{q}+\frac{1}{2}-\frac{n}{2}-\delta}e^{it|D|}f\|_{L^2_t L^q_{|x|} L^{\tilde{q}}_\theta} \lesssim \|\Lambda_\theta^{-\epsilon} f\|_{L^2}$$

for suitable ranges of the parameters

Idea of the proof

The proof of the weighted estimate for T_γ is elementary in spirit

We regard $\mathbb{R}^n$ as a product of the Lie group $(\mathbb{R}^+, \cdot)$ with the Haar measure $r^{-1}dr$, and the unit sphere $\mathbb{S}^{n-1}$

$\mathbb{S}^{n-1}$ is not a Lie group in general, but $\mathbb{S}^{n-1} \simeq SO(n)/SO(n-1)$; thus integration on $\mathbb{S}^{n-1}$ can be regarded as integration on $SO(n)$, in the corresponding Haar measure

Now

$$T_\gamma f = |x|^{-\gamma} * f$$

is a convolution of a (singular) kernel with f, and the kernel behaves nicely w.r.to the product of the two Lie groups

The proof essentially follows by two applications of Young's inequality (Lorentz space version) for convolutions, first in the Haar measure of $SO(n)$, then in the haar measure dr/r on $\mathbb{R}^+$