The Muskat Problem

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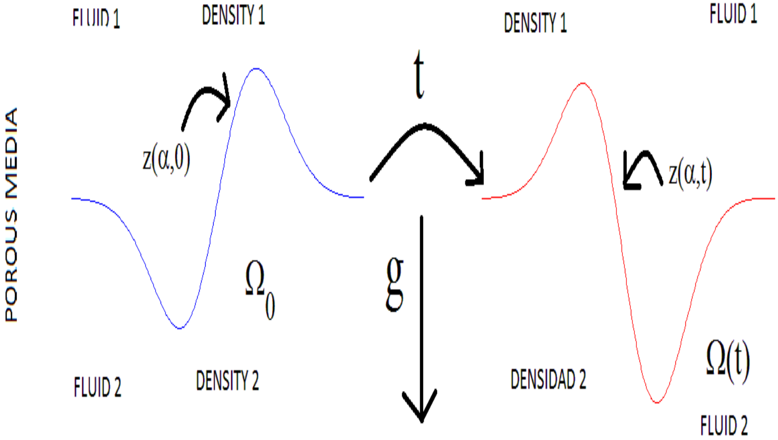
Basque Meeting on PDE's

The Muskat problem



The Muskat problem

The evolution of an interface between two fluids in porous media is modeled by the following relations:

- Darcy's law

$$u = -\nabla p - (0, \rho).$$

- Incompresibility condition

$$\nabla \cdot u = 0.$$

- Mass coservation equation

$$\rho_t + u \cdot \nabla \rho = 0.$$

Darcy's Law

Darcy's law

$$u = -\nabla p - (0, \rho)$$
$$\text{VELOCITY} = \text{FORCES}$$

Netow's law

$$\text{ACCELERATION} = \text{ FORCES}$$

We can think of the more famous Ohm's law for the electricity:

$$J = \sigma E$$
$$\text{ELECTRIC CURRENT} = \text{ELECTRIC FIELD}$$
$$\text{"VELOCITY of the ELECTRONS"} = \text{"FORCES on the ELECTRONS"}.$$

The electrons are moving in a conductor. The higher the velocity, the higher the crash against the cores of the conductor. We can model the motion of the electrons (on average) by the ODE

$$m_e \frac{dv_e}{dt} = E - \frac{1}{\sigma} v_e$$
$$v_e = \sigma E - \sigma \exp\left(-\frac{t}{\sigma m_e}\right) \sim \sigma E$$

Incompressibility condition

Let us consider a portion of the fluid, $\Omega(t)$, which is moving with velocity u. That means

$$\begin{aligned} \Omega(t) =& X(\Omega(0), t) \\ \frac{dX(x,t)}{dt} =& u(X(x,t), t) \\ X(x,0) =& x \end{aligned}$$

Then $J = det(\nabla X)$ satisfies

$$\left. \begin{array}{c} \frac{d}{dt}J = (\nabla \cdot u)\, J = 0 \\ J(0) = 1 \end{array} \right\} \Rightarrow J(t) = 1$$

Therefore the volume occupied by the fluid is constant

$$V(t) = \int_{\Omega(t)} dx = \int_{\Omega(0)} J dx = \int_{\Omega(0)} dx = V(0)$$

The mass conservation equation

The mass of the fluid is constant

$$\int_{\Omega(t)} \rho(x,t)dx = \int_{\Omega(0)} \rho(x,0)dx$$

Then

$$0 = \frac{d}{dt}\int_{\Omega(t)} \rho(x,t)dx = \frac{d}{dt}\int_{\Omega(0)} \rho(X(x,t),t)dx$$

$$= \int_{\Omega(0)} (\partial_t \rho)(X(x,t),t) + \left[(u\cdot\nabla)\rho\right](X(x,t),t)dx$$

$$= \int_{\Omega(t)} (\partial_t \rho)(x,t) + \left[(u\cdot\nabla)\rho\right](x,t)dx$$

$$\Rightarrow \partial_t \rho(x,t) + (u(x,t)\cdot\nabla)\rho(x,t) = 0$$

The Muskat equation

We shall obtain a equation for the interface (S) $z(\alpha, t)$ between the two fluids. We introduce the ansatz

$$\rho(x,t) = \begin{cases} \rho^1 & x \in \Omega^1(t) \\ \rho^2 & x \in \Omega^2(t) \end{cases}$$

where $\Omega^1(t) = \mathbb{R}^2 \setminus \Omega^2(t)$.

- Incompresibility Condition.

$$\nabla \cdot u = 0 \Rightarrow u = \nabla^\perp \psi \equiv (-\partial_y \psi, \partial_x \psi)$$

- Darcy law's.

$$u = -\nabla p - (0, \rho) \Rightarrow \nabla^\perp \psi = -\nabla p - (0, \rho)$$

Taking the curl (in a weak sense)

$$\Delta \psi = \nabla^\perp \cdot \nabla^\perp \psi = -\partial_x \rho$$

In order to compute $-\partial_x \rho$ we define the sets

$$\Omega^1_\varepsilon(t) = \{x \in \Omega^1(t) \ : \ \mathrm{dist}(x, S) \geq \varepsilon\}$$
$$\Omega^2_\varepsilon(t) = \{x \in \Omega^2(t) \ : \ \mathrm{dist}(x, S) \geq \varepsilon\}$$

for $\varepsilon > 0$.

Let $\phi \in C_c^\infty(\mathbb{R}^2)$. Then

$$(-\partial_x \rho, \phi) = (\rho, \partial_x \phi) = \int_{\mathbb{R}^2} \rho(x,t)\partial_x\phi(x,t)dxdy$$
$$= \lim_{\varepsilon \to 0} \left(\int_{\Omega_\varepsilon^1(t)} \rho^1 \partial_x \phi(x,t)dxdy + \int_{\Omega_\varepsilon^2(t)} \rho^2 \partial_x \phi(x,t)dxdy \right)$$

An integration by parts yields

$$(-\partial_x \rho, \phi) = \lim_{\varepsilon \to 0} \left(\int_{\partial\Omega_\varepsilon^1(t)} \rho^1 \phi n_1^{\Omega_\varepsilon^1(t)} d\sigma + \int_{\partial\Omega_\varepsilon^1(t)} \rho^2 \phi n_1^{\Omega_\varepsilon^2(t)} d\sigma \right)$$
$$= (\rho^1 - \rho^2) \int_{-\infty}^{\infty} \phi(z(\alpha,t)) z_\alpha^2(\alpha,t) d\alpha$$

Therefore

$$-\partial_x \rho = (\rho^1 - \rho^2) z_\alpha^2 \delta((x,y) - (z^1, z^2))$$

We recall the fundamental solution of the Laplace equation

$$\Delta F = f$$
$$F(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|) f(y) dy$$

Then we have that for $x \notin S$

$$\psi(x) = \frac{\rho^1 - \rho^2}{2\pi} \int_{-\infty}^{\infty} \log(|x - z(\alpha,t)|) z_\alpha^2(\alpha,t) d\alpha$$
$$u(x) = \frac{\rho^1 - \rho^2}{2\pi} \int_{-\infty}^{\infty} \frac{(x - z(\alpha,t))^\perp}{|x - z(\alpha,t)|^2} z_\alpha^2(\alpha,t) d\alpha$$

Now we take the limits to the S of the velocity

$$\lim_{\varepsilon \to 0} u\left(z(\alpha,t) + \varepsilon z_\alpha^\perp(\alpha,t),t)|,t\right) = u^1(\alpha,t)$$
$$\lim_{\varepsilon \to 0} u\left(z(\alpha,t) - \varepsilon z_\alpha^\perp(\alpha,t),t\right) = u^2(\alpha,t)$$

We obtain

$$u^1(\alpha,t) = (\rho^1-\rho^2)BR(z,z_\alpha^2) - (\rho^1-\rho^2)\frac{z_\alpha^2(\alpha,t)}{2|z_\alpha(\alpha,t)|}z_\alpha(\alpha,t)$$

$$u^2(\alpha,t) = (\rho^1-\rho^2)BR(z,z_\alpha^2) + (\rho^1-\rho^2)\frac{z_\alpha^2(\alpha,t)}{2|z_\alpha(\alpha,t)|}z_\alpha(\alpha,t)$$

where $BR(z,z_\alpha^2)$ is the Birkhoff-Rott integral,

$$BR(z,z_\alpha^2) = \frac{1}{2\pi}P.V.\int_{-\infty}^{\infty}\frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}z_\beta^2(\beta,t)d\beta$$

Therefore

- The velocity $u(x,t)$ is discontinuous in the tangential direction on the interface.
- The normal component of the velocity is continuous:

  $$u^1(\alpha,t)\cdot z_\alpha^\perp(\alpha,t) = u^2(\alpha,t)\cdot z_\alpha^\perp(\alpha,t) = (\rho^1-\rho^2)BR(z,z_\alpha^2)\cdot z_\alpha^\perp(\alpha,t).$$

Why these limits?

$$u^2(\alpha,t)=\lim_{\varepsilon\to 0}\frac{\rho^1-\rho^2}{2\pi}\int_{-\infty}^{\infty}\frac{(z(\alpha,t)-z(\beta,t))^{\perp}+\varepsilon z_{\alpha}^{\perp}(\alpha,t)}{|(z(\alpha,t)-z(\beta,t))-\varepsilon z_{\alpha}^{\perp}(\alpha,t)|^2}z_{\beta}(\beta,t)d\beta$$

$$\lim_{\varepsilon\to 0}\left(\int_{|\alpha-\beta|\leq\delta}+\int_{|\alpha-\beta|>\delta}\right)\equiv\lim_{\varepsilon\to 0}(I_1^{\delta,\varepsilon}+I_2^{\delta\varepsilon})$$

We can easily compute the limit for $I_2^{\delta,\varepsilon}$,

$$\lim_{\varepsilon\to 0}I_2^{\delta,\varepsilon}=I_2^{\delta}=\frac{\rho^1-\rho^2}{2\pi}\int_{|\alpha-\beta|>\delta}\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}z_{\beta}^2(\beta,t)d\beta$$

and

$$\lim_{\delta\to 0}I_2^{\delta}=(\rho^1-\rho^2)BR(z,z_{\alpha}^2).$$

For $I_1^{\delta,\varepsilon}$ we proceed as follow

$$I_1^{\delta,\varepsilon}=\frac{\rho^1-\rho^2}{2\pi}\int_{|\alpha-\beta|<\delta}\frac{(z(\alpha,t)-z(\beta,t))^{\perp}+\varepsilon z_{\alpha}(\alpha,t)}{|(z(\alpha,t)-z(\beta,t))-\varepsilon z_{\alpha}^{\perp}(\alpha,t)|^2}z_{\beta}^2(\beta,t)d\beta$$

$$= \frac{\rho^1-\rho^2}{2\pi}\int_{|\alpha-\beta|<\delta}\left(\frac{(z(\alpha,t)-z(\beta,t))^{\perp}+\varepsilon z_\alpha(\alpha,t)}{\left|(z(\alpha,t)-z(\beta,t))-\varepsilon z_\alpha^{\perp}(\alpha,t)\right|^2}-\frac{z_\alpha^{\perp}(\alpha,t)(\alpha-\beta)+\varepsilon z_\alpha(\alpha,t)}{\left|z_\alpha(\alpha,t)(\alpha-\beta)-\varepsilon z_\alpha^{\perp}(\alpha,t)\right|^2}\right)z_\beta^2(\beta,t)d\beta$$

$$+\frac{\rho^1-\rho^2}{2\pi}\int_{|\alpha-\beta|<\delta}\frac{z_\alpha^{\perp}(\alpha,t)(\alpha-\beta)+\varepsilon z_\alpha(\alpha,t)}{\left|z_\alpha(\alpha,t)(\alpha-\beta)-\varepsilon z_\alpha^{\perp}(\alpha,t)\right|^2}z_\beta^2(\beta,t)d\beta$$

$$= J_1^{\delta,\varepsilon}+J_2^{\delta,\varepsilon}$$

We have that

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}J_1^{\delta,\varepsilon}=0$$

and

$$J_2^{\delta,\varepsilon}=\frac{\rho^1-\rho^2}{2\pi|z_\alpha(\alpha,t)|^2}\int_{|\alpha-\beta|<\delta}\frac{z_\alpha^{\perp}(\alpha,t)(\alpha-\beta)+\varepsilon z_\alpha(\alpha,t)}{(\alpha-\beta)^2+\varepsilon^2}z_\beta^2(\beta,t)d\beta,$$

with

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}J_2^{\delta,\varepsilon}=(\rho^1-\rho^2)\frac{z_\alpha^2(\alpha,t)}{2|z_\alpha(\alpha,t)|^2}z_\alpha(\alpha,t)$$

- Mass conservation equation.

  We understand the mass conservation equation in a weak sense:

$$\int_0^T \int_{\mathbb{R}^2} \rho(x,t)\left(\partial_t \phi(x,t) + \nabla \cdot (u(x,t)\phi(x,t))\right) dx_1 dx_2 dt = 0$$
$$\forall \phi \in C_c^\infty(\mathbb{R}^2 \times (0,T))$$

Then

$$\lim_{\varepsilon \to 0} \left(\int_0^T \int_{\Omega_\varepsilon^1(t)} \rho^1 (\partial_t \phi(x,t) + \nabla \cdot (u(x,t)\phi(x,t))) dx_1 dx_2 dt \right.$$
$$\left. + \int_0^T \int_{\Omega_\varepsilon^2(t)} \rho^2 (\partial_t \phi(x,t) + \nabla \cdot (u(x,t)\phi(x,t))) dx_1 dx_2 dt \right)$$

We parameterize $\partial\Omega_\varepsilon^2(t)$ by the curve $z^\varepsilon(\alpha,t)$. Then

$$\int_{\Omega_\varepsilon^2(t)} \rho^2 \nabla \cdot (u(x,t)\phi(x,t)) dx_1 dx_2$$
$$= \rho^2 \int_{-\infty}^{\infty} \phi(z^\varepsilon(\alpha,t),t) u(z^\varepsilon(\alpha,t),t) \cdot z_\alpha^{\varepsilon\perp}(\alpha,t) d\alpha dt$$

We consider the surface

$$O = (z^{\varepsilon}(\alpha, t), t).$$

The normal unit vector to O is given by

$$n^{\varepsilon} = \frac{(z_{\alpha}^{\varepsilon\perp}, -z_t^{\varepsilon}(\alpha, t) \cdot z_{\alpha}^{\varepsilon\perp}(\alpha, t))}{|z_t^{\varepsilon}(\alpha, t) \times z_{\alpha}^{\varepsilon}(\alpha, t)|}.$$

Therefore

$$\int_0^T \int_{\Omega_1^{\varepsilon}(t)} \rho^2 \partial_t \phi(x, t) dx_1 dx_2 dt$$
$$= -\rho^2 \int_0^T \int_{-\infty}^{\infty} z_t^{\varepsilon}(\alpha, t) \cdot z_{\alpha}^{\varepsilon\perp}(\alpha, t) \phi(z^{\varepsilon}(\alpha, t), t) d\alpha dt$$

And we get

$$\lim_{\varepsilon \to 0} \int_{\Omega_{\varepsilon}^2(t)} \partial_t \phi(x, t) + \nabla \cdot (u(x, t)\phi(x, t)) dx_1 dx_2 dt$$
$$= -\rho^2 \int_0^T \int_{-\infty}^{\infty} (z_t(\alpha, t) - (\rho^1 - \rho^2) BR(z, z_{\alpha}^2)) \cdot z_{\alpha}^{\perp}(\alpha, t) \phi(z(\alpha, t), t) d\alpha dt$$

Making similar computations with the integral on $\Omega^1_\varepsilon(t)$ we obtain the equation

$$\big(z_t(\alpha,t)-(\rho^1-\rho^2)BR(z,z^2_\alpha)\big)\cdot z^\perp_\alpha(\alpha,t)=0$$

or equivalently

$$\begin{aligned} z_t(\alpha,t)=&-\frac{\rho^2-\rho^1}{2\pi}\int_{-\infty}^{\infty}\frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}z^2_\beta(\beta,t)d\beta\\ &+c(\alpha,t)z_\alpha(\alpha,t) \end{aligned} \qquad (1)$$

where c is a free quantity which arises because of the parametrization freedom. Indeed, if $z(\alpha,t)$ satisfies the equation (1) then $\tilde{z}(\gamma,t)=z(\alpha(\gamma,t),t)$ satisfies

$$\begin{aligned} \tilde{z}_t(\gamma,t)=&(\rho^1-\rho^2)BR(\tilde{z},\tilde{z}^2_\gamma)\\ &+\frac{(c(\alpha(\gamma,t),t)+\partial_t\alpha(\gamma,t))}{\partial_\gamma\alpha(\gamma,t)}\tilde{z}_\gamma(\gamma,t) \end{aligned}$$

The Periodic Case

We shall study the case in which the curve is periodic in the horizontal variable. That means

$$z(\alpha+2\pi,t)=z(\alpha,t)+(2\pi,0)$$

If we consider the curve in the complex plane $z(\alpha,t)=z^1(\alpha,t)+iz^2(\alpha,t)$, then

$$\bar{z}_t(\alpha,t)=-\frac{(\rho^2-\rho^1)}{2\pi i}PV\int_{\mathbb{R}}\frac{\partial_\alpha z_2(\beta,t)}{z(\alpha,t)-z(\beta,t)}d\beta+c(\alpha,t)\bar{z}_\alpha(\alpha,t).$$

Using that $z(\alpha+2k\pi,t)=z(\alpha,t)+2k\pi$, for $k\in\mathbb{Z}$, and the identity

$$\left(\frac{1}{z}+\sum_{k\geq 1}\frac{z}{z^2-(2\pi k)^2}\right)=\frac{1}{2\tan(z/2)}$$

we find that

$$\bar{z}_t(\alpha,t)=\frac{1}{4\pi i}P.V.\int_{-\pi}^{\pi}\frac{z^2_\beta(\beta,t)}{\tan\left(\frac{z(\alpha,t)-z(\beta,t)}{2}\right)}d\beta+c(\alpha,t)\bar{z}_\alpha(\alpha,t)$$

And we can write

$$z_t(\alpha) = \frac{(\rho^2-\rho^1)}{4\pi}\int_{\mathbb{T}} \frac{(\sinh(z_2(\alpha)-z_2(\beta)), -\sin(z_1(\alpha)-z_1(\beta)))}{\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))}\partial_\alpha z_2(\beta)d\beta,$$

Now, we notice that

$$P.V.\int_{\mathbb{T}} \partial_\beta \left(\log\left(\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))\right)\right)d\beta = 0,$$

thus

$$-\frac{(\rho^2-\rho^1)}{4\pi}PV\int_{\mathbb{R}} \frac{\sinh(z_2(\alpha)-z_2(\beta))}{\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))}\partial_\alpha z_2(\beta)d\beta$$

$$= \frac{(\rho^2-\rho^1)}{4\pi}PV\int_{\mathbb{R}} \frac{\sin(z_1(\alpha)-z_1(\beta))}{\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))}\partial_\alpha z_1(\beta)d\beta$$

Then, we can write

$$\partial_t z = -\frac{\rho^2-\rho^1}{4\pi} P.V. \int_{\mathbb{T}} \frac{\sin(z_1(\alpha)-z_1(\beta))}{\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))} z_\beta(\beta) d\beta$$
$$+ c(\alpha,t) z_\alpha(\alpha,t)$$

Finally, by choosing

$$c(\alpha,t) = \frac{\rho^2-\rho^1}{4\pi} P.V. \int_{\mathbb{T}} \frac{\sin(z_1(\alpha)-z_1(\beta))}{\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))} d\beta$$

yields

$$\partial_t z = \frac{\rho^2-\rho^1}{4\pi} P.V. \int_{\mathbb{T}} \frac{\sin(z_1(\alpha)-z_1(\beta))(z_\alpha(\alpha)-z_\beta(\beta))}{\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))} d\beta$$

The Muskat equation

$$\partial_t z = \frac{\rho^2 - \rho^1}{4\pi} P.V. \int_{\mathbb{T}} \frac{\sin(z_1(\alpha) - z_1(\beta))(z_\alpha(\alpha) - z_\beta(\beta))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))} d\beta \quad (2)$$
$$z(\alpha, 0) = z_0(\alpha)$$

Solving the Muskat equation we find a solution of the Muskat problem. Indeed, if $z(\alpha, t)$ is a solution of (2), then the velocity

$$u(x,t) = \frac{\rho^1 - \rho^2}{2\pi} \int_{-\infty}^{\infty} \frac{(x - z(\alpha,t))^\perp}{|x - z(\alpha,t)|^2} z_\alpha^2(\alpha,t) d\alpha$$

and the density

$$\rho(x,t) = \begin{cases} \rho^1 & x \in \Omega^1(t) \\ \rho^2 & x \in \Omega^2(t) \end{cases}$$

solve

$$\begin{aligned} \partial_t \rho + u \cdot \nabla \rho &= 0 \\ \nabla \cdot u &= 0 \\ u + (0, \rho) &= \nabla p \end{aligned}$$

Summary

General Properties of the Muskat equation

- The equation for a graph $(\alpha, f(\alpha,t))$.

$$\partial_t \alpha = \frac{\rho^2-\rho^1}{4\pi}\int_{\mathbb{T}} \frac{\sin(\alpha-\beta)(\partial_\alpha \alpha - \partial_\beta \beta)}{\cosh(f(\alpha)-f(\beta))-\cos(\alpha-\beta)} d\beta$$

$$(0 = \quad 0)$$

$$\partial_t f(\alpha) = \frac{\rho^2-\rho^1}{4\pi}\int_{\mathbb{T}} \frac{\sin(\alpha-\beta)(\partial_\alpha f(\alpha) - \partial_\beta f(\beta))}{\cosh(f(\alpha)-f(\beta))-\cos(\alpha-\beta)} d\beta$$

with initial data

$$z^1(\alpha,0) = \alpha$$

$$z^2(\alpha,0) = f(\alpha,0) = f_0(\alpha)$$

- The straight line $(\alpha, 0)$ is a solution
- Linearization around $(\alpha, 0)$.

$$
\begin{aligned}
&\cosh(f(\alpha) - f(\beta)) - \cos(\alpha - \beta) \\
&= (\cosh(f(\alpha) - f(\beta)) - 1) + (1 - \cos(\alpha - \beta)) \\
&= (\cosh(f(\alpha) - f(\beta)) - 1) + 2\sin^2\left(\frac{\alpha - \beta}{2}\right) \\
&= 2\sin^2\left(\frac{\alpha - \beta}{2}\right)\left(1 + \frac{\cosh(f(\alpha) - f(\beta)) - 1}{2\sin^2\left(\frac{\alpha - \beta}{2}\right)}\right) \\
&\sim 2\sin^2\left(\frac{\alpha - \beta}{2}\right)
\end{aligned}
$$

We are going to use

$$
\frac{\sin(\alpha)}{2\sin^2\left(\frac{\alpha}{2}\right)} = \frac{1}{\tan\left(\frac{\alpha}{2}\right)}
$$

Then

$$\begin{aligned}\partial_t f(\alpha) =& \frac{\rho^2-\rho^1}{4\pi}\int_{\mathbb{T}} \frac{f_\alpha(\alpha)-f_\beta(\beta)}{\tan\left(\frac{\alpha-\beta}{2}\right)} \\ =& -\frac{\rho^2-\rho^1}{4\pi} P.V. \int_{\mathbb{T}} \frac{f_\beta(\beta)}{\tan\left(\frac{\alpha-\beta}{2}\right)} = -\frac{\rho^2-\rho^1}{2}\partial_\alpha Hf\end{aligned}$$

where H is the Hilbert transform and

$$\partial_\alpha H = \Lambda = (-\Delta)^{\frac{1}{2}}.$$

For asymptotic flat curve

$$Hf(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{f(\beta)}{\alpha-\beta} d\beta$$

We distinguish two regimes:

1. The stable regime $\rho^2 > \rho^1$.

$$\partial_t f = -\frac{\rho^2 - \rho^1}{2}\Lambda f \Rightarrow \hat{f}(\xi) = \hat{f}_0(\xi)e^{-\frac{\rho^2-\rho^1}{2}|\xi|t}$$

$$f(\alpha,t) = \left(P_{\frac{\rho^2-\rho^1}{2}t} * f_0\right)(\alpha,t) \quad \text{with } P_t(\alpha) = \frac{1}{2\pi}\frac{t}{\alpha^2+t^2}$$

2. The unstable regime $\rho^2 < \rho^1$.

$$\hat{f}(\xi,t) = \hat{f}_0(\xi)e^{\frac{\rho^2-\rho^1}{2}|\xi|t}$$

Summary

Local Existence

Theorem (D. Córdoba and F. Gancedo)

Let $f_0(\alpha) \in H^k$ for $k \geq 3$ and $\rho_2 > \rho_1$. Then there exists a time $T > 0$ so that there is a unique solution of the Muskat equation in H^k with $f(\alpha, 0) = f_0(\alpha)$.

Ill-Posedness.

Theorem (D. Córdoba and F. Gancedo)

Let $s > 3/2$, then for any $\varepsilon > 0$ there exists a solution f of the Muskat equation with $\rho_1 > \rho_2$ and $0 < \delta < \varepsilon$ such that $\|f\|_{H^s}(0) \leq \varepsilon$ and $\|f\|_{H^s}(\delta) = \infty$.

Proof of Local Existence

We use energy estimate in $H^3(\mathbb{T})$,

$$||f||^2_{H^3(\mathbb{T})} = \int_{\mathbb{T}} |f(\alpha)|^2 d\alpha + \int_{\mathbb{T}} |\partial^3_\alpha f(\alpha)|^2 d\alpha$$

We need to control $\frac{d}{dt}||f||_{H^3(\mathbb{T})}$ in terms of $||f||_{H^3(\mathbb{T})}$. In order to do it the most singular quantity is

$$\frac{d}{dt}\int_{\mathbb{T}} |\partial^3_\alpha f(\alpha)|^2 d\alpha = 2\int_{\mathbb{T}} \partial^3_\alpha f(\alpha)\partial^3_\alpha \partial_t f(\alpha) d\alpha.$$

where

$$\begin{aligned}\partial^3_\alpha \partial_t f(\alpha) =& \partial^3_\alpha \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sin(\alpha-\beta)(\partial_\alpha f(\alpha)-\partial_\beta f(\beta))}{\cosh(f(\alpha)-f(\beta))-\cos(\alpha-\beta)} d\beta\\ & \partial^3_\alpha \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sin(\beta)(\partial_\alpha f(\alpha)-\partial_\alpha f(\alpha-\beta))}{\cosh(f(\alpha)-f(\alpha-\beta))-\cos(\beta)} d\beta\end{aligned}$$

(we take $\frac{\rho_2-\rho_1}{2} = 1$)

Therefore the most dangerous term that we have to control is the following

$$\text{Danger} = \int_{\mathbb{T}} \partial_\alpha^3 f(\alpha) \text{Dangerous}(\alpha) d\alpha$$

where

$$\text{Dangerous}(\alpha) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\sin(\beta)(\partial_\alpha^4 f(\alpha) - \partial_\alpha^4 f(\alpha-\beta))}{\cosh(f(\alpha) - f(\alpha-\beta)) - \cos(\beta)} d\beta$$

We will assume that

$$\frac{d}{dt}||f||^2_{H^3(\mathbb{T})} = \text{Danger} + \text{Controlled Quantity}$$

We will say that a term $I(\alpha)$ in $\frac{d}{dt}||f||^2_{H^3(\mathbb{T})}$ is "Nice" if

$$\left| \int_{\mathbb{T}} \partial_\alpha^3 f(\alpha) I(\alpha) d\alpha \right| \leq C||f||^m_{H^3(\mathbb{T})},$$

for some integer m.

We deal with Dangerous(α).

$$f(\alpha)-f(\alpha-\beta)=f_\alpha(\alpha)\beta+O(\beta^2)$$
$$\cosh(f(\alpha)-f(\alpha-\beta))-\cos(\beta)=\frac{1}{2}(f_\alpha(\alpha)^2+1)\beta^2+O(\beta^3)$$

Then

$$\begin{aligned}\text{Dangerous}(\alpha)=&\frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{\sin(\beta)}{\cosh(f(\alpha)-f(\alpha-\beta))-\cos(\beta)}\right.\\&\left.-\frac{\sin(\beta)}{(1+f_\alpha(\alpha)^2)2\sin^2\left(\frac{\beta}{2}\right)}\right)(\partial_\alpha^4 f(\alpha)-\partial_\alpha^4 f(\alpha-\beta))d\beta\\&+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\sin(\beta)}{(1+f_\alpha(\alpha)^2)2\sin^2\left(\frac{\beta}{2}\right)}(\partial_\alpha^4 f(\alpha)-\partial_\alpha^4 f(\alpha-\beta))d\beta\\&=\frac{1}{2\pi}\int_{\mathbb{T}}a(\alpha,\beta)(\partial_\alpha^4 f(\alpha)-\partial_\alpha^4 f(\alpha-\beta))d\beta\\&+\frac{1}{(1+f_\alpha(\alpha)^2)}\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha^4 f(\alpha)-\partial_\alpha^4 f(\alpha-\beta))}{\tan\left(\frac{\beta}{2}\right)}d\beta\end{aligned}$$

$$\begin{aligned}\text{Dangerous}(\alpha) =& \partial_\alpha^4 f(\alpha)\frac{1}{2\pi}\int_{\mathbb{T}} a(\alpha,\beta)d\beta - \frac{1}{2\pi}\int_{\mathbb{T}} \partial_\beta a(\alpha,\beta)\partial_\beta^3 f(\alpha-\beta)d\beta \\ &- \sigma(\alpha)\Lambda\partial_\alpha^3 f(\alpha),\end{aligned}$$

where

$$\sigma(\alpha) = \frac{1}{1+f_\alpha(\alpha)^2}$$

is called the Rayleigh-Taylor function.
Then

$$\text{Dangerous}(\alpha) = \text{Nice} - \sigma(\alpha)\Lambda\partial_\alpha^3 f(\alpha),$$

and therefore

$$\text{Danger} = -\int_{\mathbb{T}} \sigma(\alpha)\partial_\alpha^3 f(\alpha)\Lambda\partial_\alpha^3 f(\alpha)d\alpha + \text{Controlled Quantities}$$

The Lambda Operator

$$\Lambda f(\alpha) = \frac{1}{2\pi} P.V. \int_{\mathbb{T}} \frac{\partial_\beta f(\beta)}{\tan\left(\frac{\alpha-\beta}{2}\right)} d\beta = \frac{1}{4\pi} P.V. \int_{\mathbb{T}} \frac{f(\alpha) - f(\beta)}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} d\beta$$

Lemma (A. Córdoba and D. Córdoba)

The following pointwise inequality holds

$$f(\alpha)\Lambda f(\alpha) \geq \Lambda(f^2)(\alpha)$$

Proof:

$$\frac{P.V.}{4\pi} \int_{\mathbb{T}} \frac{2f(\alpha)(f(\alpha) - f(\beta))}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} d\beta = \frac{P.V.}{4\pi} \int_{\mathbb{T}} \frac{(f(\alpha) - f(\beta))^2 + f^2(\alpha) - f^2(\beta)}{\sin^2\left(\frac{\alpha-\beta}{2}\right)} d\beta$$

Then, since $\sigma > 0$, the previous inequality yields

$$
\begin{aligned}
-\int_{\mathbb{T}} \sigma(\alpha)\partial_\alpha^3 f(\alpha)\Lambda\partial_\alpha^3 f(\alpha) &\leq -\frac{1}{2}\int_{\mathbb{T}} \sigma(\alpha)\Lambda\left(\partial_\alpha^3 f(\alpha)\right)^2 d\alpha \\
&\leq -\frac{1}{2}\int_{\mathbb{T}} \Lambda\sigma(\alpha)\left(\partial_\alpha^3 f(\alpha)\right)^2
\end{aligned}
$$

Finally we obtain

$$
\frac{d}{dt}||f||^2_{H^3(\mathbb{T})} \leq C||f||^m_{H^3(\mathbb{T})}
$$

Galerkin approximation

$$\Pi_N : \sum_{-\infty}^{\infty} A_k e^{ik\zeta} \mapsto \sum_{-N}^{N} A_k e^{ik\zeta}.$$

We define $f_0^{[N]}(\alpha)$ by stipulating that

$$f_0^{[N]}(\alpha) = \Pi_N f_0(\alpha)$$

In place of the Muskat equation, we will solve its Galerkin approximation. That is, we will solve the equation

$$\partial_t f^{[N]}(\alpha, t) = \Pi_N J^N(\zeta, t),$$

where

$$J^N(\zeta, t) =$$

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\sin(\alpha - \beta)(\partial_\alpha f^{[N]}(\alpha) - \partial_\beta f^{[N]}(\beta))}{\cosh(f^{[N]}(\alpha) - f^{[N]}(\beta)) - \cos(\alpha - \beta)} d\beta$$

We impose the initial condition

$$f^{[N]}(\alpha, 0) = f_0^{[N]}(\alpha).$$

Then

$$\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^k f^{[N]}(\alpha)|^2 = \int_{\mathbb{T}} \partial_\alpha^k f^{[N]}(\alpha)\Pi_N J(\alpha)d\alpha$$
$$= \int_{\mathbb{T}} \partial_\alpha^k f^{[N]}(\alpha)J^N(\alpha)d\alpha$$

We get bound of the H^3- norm of $f^{[N]}$ which do not depend on N.

Energy Estimates for a Curve

For a periodic curve $z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t))$ we study de evolution of the norms

$$||z_1(\cdot,t) - \cdot||^2_{H^4(\mathbb{T})} = \int_{\mathbb{T}} |z_1(\alpha,t) - \alpha|^2 d\alpha + \int_{\mathbb{T}} |\partial^4_\alpha z_1(\alpha,t)|^2 d\alpha$$

$$||z_2(\cdot,t)||^2_{H^4(\mathbb{T})} = \int_{\mathbb{T}} |z_2(\alpha,t)|^2 d\alpha + \int_{\mathbb{T}} |\partial^4_\alpha z_2(\alpha,t)|^2 d\alpha$$

We will assume that $z(\alpha,t)$ satisfies the chord-arc condition

$$|\cosh(z_2(\alpha) - z_2(\alpha-\beta),t) - \cos(z_1(\alpha,t) - z_1(\alpha-\beta,t)) \geq c_{CA}\beta^2$$

$$\left(|z(\alpha,t) - z(\alpha-\beta,t)|^2 \geq c_{CA}\beta^2\right)$$

for $\beta \in [-\pi,\pi]$.
$H^4(\mathbb{T})+$ chord-arc condition $\Rightarrow z$ is locally $C^{3.5}$.
We also need the chord-arc condition to perform the estimates.

Again

$$\text{Danger} = \int_{\mathbb{T}} \partial_\alpha^4 z(\alpha,t) \cdot \text{Dangerous}(\alpha,t) d\alpha,$$

$$\text{Dangerous}(\alpha) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\sin(z_1(\alpha) - z_1(\alpha-\beta))(\partial_\alpha^5 z(\alpha) - \partial_\alpha^5 z(\alpha-\beta))}{\cosh(z_2(\alpha) - z_2(\alpha-\beta)) - \cos(z_1(\alpha) - z_1(\alpha-\beta))} d\beta$$

Now we take

$$a(\alpha,\beta) = \frac{\sin(z_1(\alpha) - z_1(\alpha-\beta))}{\cosh(z_2(\alpha) - z_2(\alpha-\beta)) - \cos(z_1(\alpha) - z_1(\alpha-\beta))} - \frac{\partial_\alpha z_1(\alpha)}{\partial_\alpha z_1(\alpha)^2 + \partial_\alpha z_2(\alpha)^2} \frac{1}{\tan\left(\frac{\beta}{2}\right)}$$

$$\sigma(\alpha) = \frac{\partial_\alpha z_1(\alpha)}{\partial_\alpha z_1(\alpha)^2 + \partial_\alpha z_2(\alpha)^2}$$

Thus

$$\text{Dangerous}(\alpha) = \partial_\alpha^5 z(\alpha,t) \frac{1}{2\pi}\int_{\mathbb{T}} a(\alpha,\beta)d\beta - \frac{1}{2\pi}\int_{\mathbb{T}} \partial_\beta a(\alpha,\beta)\partial_\alpha^4 z(\alpha-\beta,t)d\beta$$
$$- \sigma(\alpha)\Lambda\partial_\alpha^4 z(\alpha,t)$$

Therefore, if the Rayleigh-Taylor function $\sigma(\alpha,t)$ is strictly positive we can show local existence.

$$\sigma(\alpha) = \frac{\partial_\alpha z_1(\alpha)}{\partial_\alpha z_1(\alpha)^2 + \partial_\alpha z_2(\alpha)^2} > 0 \quad \forall\alpha \in \mathbb{T}$$

Let us define

- Stable regime: $\partial_\alpha z_1(\alpha) > 0\ \forall\alpha \in \mathbb{T}$
- Unstable regime: there is a point $\alpha_0 \in \mathbb{T}$ such that $\partial_\alpha z_1(\alpha_0) < 0$.

Summary

Global Existence?

- Maximum principle for the L^2-norm

$$||f(\cdot,t)||^2_{L^2(\mathbb{R})}+\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\log\left(1+\left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right)^2\right)d\alpha d\beta=||f_0||^2_{L^2(\mathbb{R})}$$

Proof:

$$f_t=\int_{\mathbb{R}}\frac{\beta(f_\alpha(\alpha)-f_\alpha(\alpha-\beta))}{\beta^2+(f(\alpha)-f(\alpha-\beta))^2}d\beta=\int_{\mathbb{R}}\partial_\alpha\arctan\left(\frac{f(\alpha)-f(\alpha-\beta)}{\beta}\right)d\beta$$

Then

$$\frac{1}{2}\frac{d}{dt}||f(\cdot,t)||^2_{L^2(\mathbb{R})}=\int_{\mathbb{R}}f(\alpha)\partial_t f(\alpha)d\alpha=$$

$$\int_{\mathbb{R}}f(\alpha)\int_{\mathbb{R}}\partial_\alpha\arctan\left(\frac{f(\alpha)-f(\alpha-\beta)}{\beta}\right)d\beta d\alpha$$

$$=-\int_{\mathbb{R}}f_\alpha(\alpha)\int_{\mathbb{R}}\arctan\left(\frac{f(\alpha)-f(\alpha-\beta)}{\beta}\right)d\beta d\alpha$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}} f_\alpha(\alpha) \int_{\mathbb{R}} \arctan\left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right) d\beta d\alpha \\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f_\alpha(\alpha)(\alpha-\beta)-(f(\alpha)-f(\beta))}{\alpha-\beta} \arctan\left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right) d\beta d\alpha \\
&- \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(f(\alpha)-f(\beta))}{\alpha-\beta} \arctan\left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right) d\beta \equiv I_1 + I_2
\end{aligned}
$$

Notice that

$$
\partial_\alpha \frac{f(\alpha)-f(\beta)}{\alpha-\beta} = \frac{f_\alpha(\alpha)(\alpha-\beta)-(f(\alpha)-f(\beta))}{(\alpha-\beta)^2}
$$

Also

$$
G(x) = \int_0^x \arctan(y)dy = x\arctan(x) - \frac{1}{2}\log(1+x^2)
$$

Then

$$
I_1 = -\int_{\mathbb{R}}\int_{\mathbb{R}} (\alpha-\beta)\partial_\alpha G\left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right) d\beta d\alpha = \int_{\mathbb{R}}\int_{\mathbb{R}} G\left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right) d\beta d\alpha
$$

$$I_1 = -I_2 - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \log\left(1 + \left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right)^2\right) d\alpha d\beta$$

Finally,

$$\frac{1}{2}\frac{d}{dt}||f(\cdot,t)||^2_{L^2(\mathbb{R})} = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \log\left(1 + \left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right)^2\right) d\alpha d\beta$$

But

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \log\left(1 + \left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right)^2\right) d\alpha d\beta \leq C||f(\cdot,t)||_{L^1}$$

Compare with the linear case

$$||f(\cdot,t)||_{L^2(\mathbb{R})} + \int_0^T \underbrace{\int_{\mathbb{R}} \left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right)^2 d\alpha d\beta}_{=\int_{\mathbb{R}} f(x)\Lambda f(x)dx = ||\Lambda^{\frac{1}{2}} f(\cdot,t)||^2_{L^2(\mathbb{R})}} dt = ||f_0||^2_{L^2(\mathbb{R})}$$

- Maximum principle for $L^\infty-$ norm.

  Let $x(t)$ be the point where the solution $f(x,t)$ reaches its maximum:

$$\max_{\alpha} f(\alpha,t) = f(x(t),t)$$

  Then

$$\partial_t f(x(t),t) = P.V.\int_{\mathbb{R}} \frac{\beta(f_\alpha(x(t)) - f_\alpha(x(t)-\beta))}{\beta^2 + (f(x(t)) - f(x(t)-\beta))^2} d\beta$$

$$= P.V.\int_{\mathbb{R}} \frac{\beta f_\beta(x(t)-\beta)}{\beta^2 + (f(x(t)) - f(x(t)-\beta))^2} d\beta$$

$$= -P.V.\int_{\mathbb{R}} \frac{\beta \partial_\beta(f(x(t)) - f(x(t)-\beta))}{\beta^2 + (f(x(t)) - f(x(t)-\beta))^2} d\beta$$

$$= -P.V.\int_{\mathbb{R}} \frac{\partial_\beta(f(x(t)) - f(x(t)-\beta))}{\beta} \frac{d\beta}{1 + \left(\frac{f(x(t)) - f(x(t)-\beta)}{\beta}\right)^2}$$

$$= -P.V. \int_{\mathbb{R}} \frac{f(x(t)) - f(x(t)-\beta)}{\beta^2} \frac{1}{1 + \left(\frac{f(x(t))-f(x(t)-\beta)}{\beta}\right)^2} d\beta$$

$$- P.V. \int_{\mathbb{R}} 2 \frac{\left(\frac{f(x(t))-f(x(t)-\beta)}{\beta}\right)^2}{1 + \left(\frac{f(x(t))-f(x(t)-\beta)}{\beta}\right)^2} \partial_\beta \left(\frac{f(x(t)) - f(x(t)-\beta)}{\alpha}\right)$$

$$= -P.V. \int_{\mathbb{R}} \frac{f(x(t)) - f(x(t)-\beta)}{\beta^2} \frac{1}{1 + \left(\frac{f(x(t))-f(x(t)-\beta)}{\beta}\right)^2} d\beta$$

$$- P.V. \int_{\mathbb{R}} \partial_\beta G\left(\frac{f(x(t)) - f(x(t)-\beta)}{\beta}\right) d\beta$$

with

$$G(x) = -\frac{x}{1+x^2} + \arctan(x)$$

Then

$$(\partial_t f)(x(t), t) \leq 0$$

$$\frac{d}{dt}\max_{\alpha\in\mathbb{R}} f(\alpha,t) = (\partial_t f)(x(t),t) + \underbrace{\frac{dx(t)}{dt}}_{\text{differentiable?}} \cdot \underbrace{f_\alpha(x(t),t)}_{=0}$$

If the solution is small enough ($f \in H^{\frac{3}{2}^+}$), by using Radamacher's theorem we have that

$$\frac{d}{dt}\max_{\alpha\in\mathbb{R}} f(\alpha,t) = (\partial_t f)(x(t),t) \leq 0$$

Similar computations yields

$$\frac{d}{dt}\min_{\alpha\in\mathbb{R}} f(\alpha,t) = (\partial_t f)(x(t),t) \geq 0$$

- Maximum principle for $||f_\alpha(\cdot,t)||_{L^\infty(\mathbb{R})}$ for a initial data satisfying $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})} \leq 1$.

  Let $x(t)$ be the point where $f_\alpha(\alpha,t)$ reaches the maximum:

$$\max_{\alpha\in\mathbb{R}} f_\alpha(\alpha,t) = f_\alpha(x(t),t)$$

  Let us assume that $||\partial f_\alpha(\cdot,t)||_{L^\infty(\mathbb{R})} \leq 1$.

$$\partial_t f_x(x(t),t) = -\int_{\mathbb{R}} \frac{\partial_\alpha f(x(t)) - \frac{f(x(t))-f(x(t)-\beta)}{\beta}}{(x(t)-\beta)^2} Q(x(t),\beta)d\beta$$

  where

$$Q(x(t),\beta) = 2\frac{1+f_\alpha(x(t))\frac{f(x(t))-f(x(t)-\beta)}{\beta}}{\left(1+\left(\frac{f(x(t))-f(x(t)-\beta)}{\beta}\right)^2\right)^2}$$

  Then

$$\partial_t f_x(x(t),t) \leq 0$$

Global Existence for Small Initial Data

- Global existence of weak solutions for $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})}$

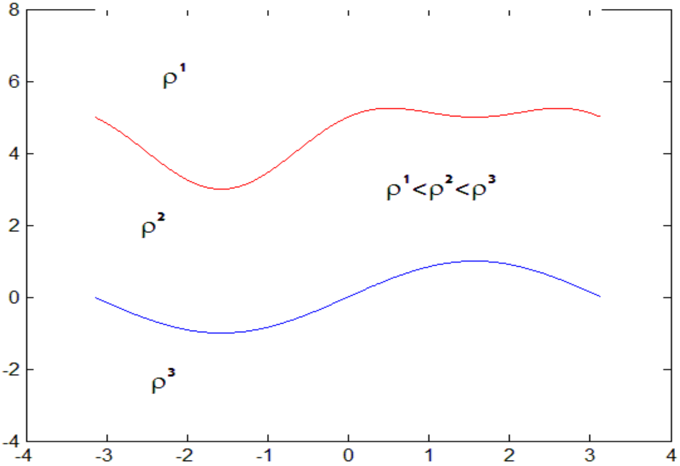
Theorem (D. Córdoba, P. Constantin, F. Gancedo and R. Strain)

Suppose that $||f_0||_{L^\infty(\mathbb{R})}$ *and* $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})} < 1$ *.Then there exists a global in time weak solution of the Muskat equation that satisfies*

$$f(\alpha, t) \in C([0, T] \times \mathbb{R}) \cap L^\infty([0, T]; W^{1,\infty}(\mathbb{R}))$$

In particular f *is Lipschitz continuous.*

Numerical Simulation



The equations for the two interfaces problem can be written as follows:

$$
\begin{aligned}
f_t(x,t) =& \frac{\rho^2-\rho^1}{4\pi} PV \int_{\mathbb{R}} \frac{(f_x(x,t)-f_x(x-y,t))y}{[|y|^2+(f(x,t)-f(x-y,t))^2]} dy \\
&+ \frac{\rho^3-\rho^2}{4\pi} PV \int_{\mathbb{R}} \frac{(f_x(x,t)-g_x(x-y,t))y}{[|y|^2+(f(x,t)-g(x-y,t))^2]} dy
\end{aligned}
$$

$$
\begin{aligned}
g_t(x,t) =& \frac{\rho^3-\rho^2}{4\pi} PV \int_{\mathbb{R}} \frac{(g_x(x,t)-g_x(x-y,t))y}{[|y|^2+(g(x,t)-g(x-y,t))^2]} dy \\
&+ \frac{\rho^2-\rho^1}{4\pi} PV \int_{\mathbb{R}} \frac{(g_x(x,t)-f_x(x-y,t))y}{[|y|^2+(g(x,t)-f(x-y,t))^2]} dy
\end{aligned}
$$

Breakdown for the Muskat Problem

Theorem

There exists a non-empty open set of initial data in H^4, satisfying the R-T (strictly positive $\sigma > 0$) for which the solution of the Muskat problem becomes immediately analytic and then pass to the unstable regime in finite time.

Theorem

There exist a class of solutions of the Muskat equation composed of curves which initially are real-analytic and are in the stable regime and then pass to the unstable regime where they lose its regularity (they are not C^4).

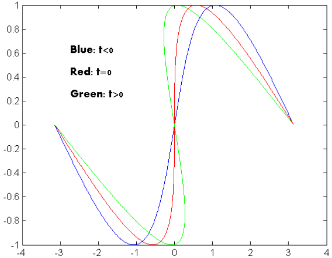
Joint work with D. Córdoba, C. Fefferman, F. Gancedo and María López-Fernández.

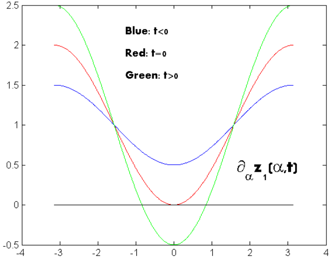
Summary

The "Guide" Unperturbed Solution

We shall construct a solution satisfying the following:

- $z_1(\alpha, t) - \alpha$, $z_2(\alpha, t)$ are $2\pi-$periodic real analytic functions for fixed $t \in [-T, T]$
- $z(\alpha, t)$ satisfies the chord-arc condition for fixed $t \in [-T, T]$
- $\partial_\alpha z_1(0, 0) = 0$ $(\partial_\alpha z_2(\alpha_0, 0) > 0)$
- $z(\alpha, t)$ is in the stable regime for $t \in [-T, 0)$
- $z(\alpha, t)$ is in the unstable regime for $t \in (0, T]$.





Main steps to construct such a solution:

- 1. We use the equation for the curve:

$$\begin{aligned}\partial_t z(\alpha,t) =& \frac{1}{2\pi} P.V. \int_{\mathbb{T}} \frac{\sin(z_1(\alpha)-z_1(\beta))(z_\alpha(\alpha)-z_\beta(\beta))}{\cosh(z_2(\alpha)-z_2(\beta))-\cos(z_1(\alpha)-z_1(\beta))} d\beta \\ \equiv& \left(v^1(\alpha,t), v^2(\alpha,t)\right), \\ z(\alpha,0) =& z_0(\alpha)\end{aligned}$$

  with $z_0(\alpha)$ satisfying:

  - $\partial_\alpha z_0^1(0) = 0$, otherwise $\partial_\alpha z_0^1(\alpha) > 0$ $(\partial_\alpha z_2(0) > 0)$
  - $(\partial_\alpha v^1)(0,0) < 0$ (strictly negative) Non trivial condition

Why?
Let us assume there exist a solution $z(\alpha,t)$ for the Muskat equation for this initial data in the time interval $t \in [-T,T]$. Then

$$\partial_t \partial_\alpha z^1(0,0) = \partial_\alpha v^1(0,0) < 0 \Rightarrow \begin{cases} \partial_\alpha z_1(\alpha,t) > 0 & \forall \alpha \in \mathbb{T}\, t \in [-T,0) \\ \partial_\alpha z_1(0,t) < 0 & t \in (0,T] \end{cases}$$

- 2. Proving local existence for this initial data.

  Big problem: The R-T condition fails,

$$\sigma(0,0) = \frac{\partial_\alpha z_2^1(0)}{\partial_\alpha z_2^1(0)^2 + \partial_\alpha z_2^2(0)^2} = 0$$

  In order to overcome this problem:

  - We will take $z_0(\alpha, t)$ (satisfying the previous conditions) analytic
  - We will apply a C-K theorem (which does not need the positivity of the R-T function).

Abstract Cauchy-Kowalewski theorem

Definition

Let $S = \{X_r\}_r$ be a scale of Banach spaces, and let all X_r for $\rho > 0$ linear subspaces of X_0. It is assumed that:

- $X_r \subset X_{r'}$ for $r' \leq r$.
- $||\cdot||_{r'} \leq ||\cdot||_r$ for $r' \leq r$.

Consider the initial value problem

$$\frac{du}{dt} = F(u(t), t), \quad |t| < \delta, \tag{3}$$

$$u(0) = 0 \tag{4}$$

Assume the following conditions on F:

- $R > 0$, $\eta > 0$, $r_0 > 0$ the map

  $$(u, t) \to F(u, t)$$

  is continuous of

  $$\{u \in X_r;\ ||u||_r < R\} \times \{t;\ |t| < \eta\} \quad \text{into} \quad X_{r'}$$

  for any $0 \leq r' \leq r \leq r_0$.

- $u, v \in X_r$, $||u||_{X_r}, ||v||_{X_r} < R$, $|t| < \eta$

$$||F(u) - F(v)||_{X_{r'}} \leq C\frac{||u - v||_{X_r}}{r - r'}$$

for any $0 \leq r' \leq r \leq r_0$.

- $F(u_0, t)$ is a continuous function of t, $|t| < \eta$ such that

$$||F(u_0, t)||_{X_r} \leq \frac{K}{r_0 - r}$$

for $0 \leq r \leq r_0$ (we do not need to consider this since $F(u, t) = F(u)$ in the Muskat equation).

Theorem

Under the preceding hypotheses there is a positive constant A such that there exists a unique function $u(t)$ *which, for every positive* $r < r_0$ *and* $|t| < A(r_0 - r)$, *is a continuously differentiable function of* t *with values in* X_r, $||u(t)||_r < R$, *and* (3) *and* (4).

L. Nirenberg. An abstract form of the nonlinear Cauchy-Kowalewski theorem. *J. Differential Geometry*, 6, (1972), 561-576

T. Nishida. A note on a theorem of Nirenberg. *J. Differential Geometry*, 12, (1977), 629-633

An example

Consider

$$\begin{aligned} \frac{du}{dt} =& \partial_\alpha u, \\ u(\alpha, 0) =& u_0(\alpha) \in X_{r_0} \end{aligned}$$

Let X_r be the Banach space consisting of all the $2\pi-$periodic functions, analytic in the complex strip

$$S_r = \{z \in \mathbb{C}/(2\pi\mathbb{Z}) \ : \ |\Im z| < r\}$$

and whose restriction to ∂S_r belongs to L^2.

$$||u||_{X_r}^2 = \sum_{\pm} \int_{\mathbb{T}} |u(\alpha \pm ir)|^2 d\alpha$$

Then $F(u) = \partial_\alpha u$, and we can write

$$
\begin{aligned}
u(\alpha) &= \sum_{-\infty}^{\infty} \hat{u}(n) \exp(in\alpha) \\
\partial_\alpha u(\alpha) &= \sum_{-\infty}^{\infty} in\hat{u}(n) \exp(in\alpha) \\
\partial_\alpha u(\alpha \pm ir) &= \sum_{-\infty}^{\infty} in\hat{u}(n) \exp(\pm nr) \\
||\partial_\alpha u(\cdot \pm ir)||^2_{L^2(\mathbb{T})} &= \frac{1}{2\pi} \sum_{-\infty}^{\infty} n^2 |\hat{u}|^2(n) \exp(\pm 2nr) \\
\sum_{\pm} ||\partial_\alpha u(\cdot \pm ir)||^2_{L^2(\mathbb{T})} &= \frac{1}{2\pi} \sum_{-\infty}^{\infty} n^2 |\hat{u}|^2(n) 2\cosh(2nr) \\
&= \frac{1}{2\pi} \sum_{-\infty}^{\infty} \frac{n^2 2\cosh(2nr)}{2\cosh(2nr')} |\hat{u}|^2(n) 2\cosh(2nr) \\
&\leq \frac{C}{r' - r} \sum_{\pm} ||u(\cdot \pm ir)||^2_{L^2(\mathbb{T})}.
\end{aligned}
$$

C-K theorem for the Muskat equation

The scale of Banach spaces: Let X_r be the Banach space consisting of all the $2\pi-$periodic functions, analytic in the strip S_r whose restriction to ∂S_r and the restriction to ∂S_r of its derivatives up to order 4 belong to L^2.

$$||f||_{X_r}^2 = \sum_{\pm} \int_{\mathbb{T}} |f(\alpha \pm ir)|^2 d\alpha + \sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^4 f(\alpha \pm ir)|^2 d\alpha$$

The extended equation

$$\frac{1}{2\pi} P.V. \int_{\mathbb{T}} \frac{\sin(z_1(\alpha) - z_1(\alpha-\beta))(z_\alpha(\alpha) - z_\beta(\alpha-\beta))}{\cosh(z_2(\alpha) - z_2(\alpha-\beta)) - \cos(z_1(\alpha) - z_1(\alpha-\beta))} d\beta$$

$$\downarrow$$

$$\frac{1}{2\pi} P.V. \int_{\mathbb{T}} \frac{\sin(z_1(\alpha+ir) - z_1(\alpha+ir-\beta))(z_\alpha(\alpha+ir) - z_\beta(\alpha+ir-\beta))d\beta}{\cosh(z_2(\alpha+ir) - z_2(\alpha+ir-\beta)) - \cos(z_1(\alpha+ir) - z_1(\alpha+ir-\beta))}$$

$$\equiv F(z(\alpha+ir))$$

We need the estimate

$$||F(z)||_{X_r} \leq C(R)\frac{||z(\cdot)-(\cdot,0)||_{X_{r'}}}{r'-r}, \tag{5}$$

assuming

$$||z(\cdot)-(\alpha,0)||_{X_{r'}} \leq R$$

for $r < r' < r_0$.

$$\begin{aligned}
&\text{Dangerous}(\alpha+ir)\\
&=\partial_\alpha^5 z(\alpha+ir,t)\frac{1}{2\pi}\int_{\mathbb{T}} a(\alpha+ir,\beta)d\beta\\
&-\frac{1}{2\pi}\int_{\mathbb{T}}\partial_\beta a(\alpha+ir,\beta)\partial_\alpha^4 z(\alpha+ir-\beta,t)d\beta\\
&-\sigma(\alpha+ir)\Lambda\partial_\alpha^4 z(\alpha+ir)
\end{aligned}$$

Using the complex chord-arc condition

$$|\cosh(z_2(\alpha+ir)-z_2(\alpha+i\zeta-\beta))-\cos(z_1(\alpha+i\zeta)-z_1(\alpha+i\zeta-\beta))| \geq c_{CA}\beta^2,$$

for $\beta\in(-\pi,\pi]$ and for $-r_0\leq\zeta\leq r_0$, we can prove (5).

Dicussion about the chord-arc condition

- Let $z(\alpha)$ be such that

$$\cosh(z_2(\alpha) - z_2(\alpha - \beta)) - \cos(z_1(\alpha) - z_1(\alpha - \beta)) \geq c_{CA}\beta^2$$

for all $\beta \in (-\pi, \pi]$. Then

$$\begin{aligned}&|\cosh(z_2(\alpha + i\zeta) - z_2(\alpha + i\zeta - \beta)) \\ &- \cos(z_1(\alpha + i\zeta) - z_1(\alpha + i\zeta - \beta))| \geq c'_{CA}\beta^2\end{aligned}$$

for all $\beta \in (-\pi, \pi]$, $-r_0 \leq \zeta \leq r_0$ and r_0 small enough.

- Since we need to control the chord-arc condition to control $||F(z)||_{X_r}$ we have to adapt the proof of the C-K theorem to include the evolution of the chord-arc condition. This proof is based in an iteration method:

$$\begin{aligned}z^{n+1}(\alpha, T) =& z_0(\alpha) + \int_0^T F(z^n)(\alpha, t)dt \\ z^0(\alpha) =& z_0(\alpha)\end{aligned}$$

The chord-arc constant must be estimated for each iteration z^n.

The initial data

Remains to be constructed an analytic initial data such that

$a.\ \partial_\alpha z_1(\alpha) > 0$ if $\alpha \neq 0.$ $\quad b.\ \partial_\alpha z_1(0) = 0.$

$c.\ \partial_\alpha z_2(0) > 0.$ $\quad d.\ \partial_\alpha v_1(0) < 0.$

Also $z_1(\alpha) - \alpha$ and $z_2(\alpha)$ are $2\pi-$periodic.
Here $v_\mu(\alpha, t)$, with $\mu = 1, 2$, are the velocities given by

$$v_\mu(\alpha, t) = \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\beta))(\partial_\alpha z_\mu(\alpha) - \partial_\alpha z_\mu(\beta))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))} d\beta.$$

Lemma

There exists a curve $z(\alpha) = (z_1(\alpha), z_2(\alpha))$ *with the following properties:*

1. $z_1(\alpha) - \alpha$ *and* $z_2(\alpha)$ *are analytic* $2\pi -$ *periodic functions and* $z(\alpha)$ *satisfies the arc-chord condition,*
2. $z(\alpha)$ *is odd and*
3. $\partial_\alpha z_1(\alpha) > 0$ *if* $\alpha \neq 0$, $\partial_\alpha z_1(0) = 0$ *and* $\partial_\alpha z_2(0) > 0$,

such that

$$\begin{aligned}(\partial_\alpha v_1)(0) &= \left(\partial_\alpha \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\beta))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\beta))}{\cosh(z_2(\alpha) - z_2(\beta)) - \cos(z_1(\alpha) - z_1(\beta))} d\beta\right)\Bigg|_{\alpha=0} \\ &< 0. \end{aligned} \tag{6}$$

Proof:

$$(\partial_\alpha v_1)(\alpha) = \partial_\alpha \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\alpha-\beta))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha-\beta))}{\cosh(z_2(\alpha) - z_2(\alpha-\beta)) - \cos(z_1(\alpha) - z_1(\alpha-\beta))} d\beta$$

$$= \int_{-\pi}^{\pi} \frac{\cos(z_1(\alpha) - z_1(\alpha-\beta))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha-\beta))^2}{\cosh(z_2(\alpha) - z_2(\alpha-\beta)) - \cos(z_1(\alpha) - z_1(\alpha-\beta))} d\beta$$

$$+ \int_{-\pi}^{\pi} \frac{\sin(z_1(\alpha) - z_1(\alpha-\beta))(\partial_\alpha^2 z_1(\alpha) - \partial_\alpha^2 z_1(\alpha-\beta))}{\cosh(z_2(\alpha) - z_2(\alpha-\beta)) - \cos(z_1(\alpha) - z_1(\alpha-\beta))} d\beta$$

$$- \int_{-\pi}^{\pi} \sin((z_1(\alpha) - z_1(\alpha-\beta)))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha-\beta))$$

$$\times \frac{\sinh(z_2(\alpha) - z_2(\alpha-\beta))(\partial_\alpha z_2(\alpha) - \partial_\alpha z_2(\alpha-\beta))}{(\cosh(z_2(\alpha) - z_2(\alpha-\beta)) - \cos(z_1(\alpha) - z_1(\alpha-\beta)))^2} d\beta$$

$$- \int_{-\pi}^{\pi} \sin((z_1(\alpha) - z_1(\alpha-\beta)))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha-\beta))$$

$$\times \frac{\sin(z_2(\alpha) - z_2(\alpha-\beta))(\partial_\alpha z_1(\alpha) - \partial_\alpha z_1(\alpha-\beta))}{(\cosh(z_2(\alpha) - z_2(\alpha-\beta)) - \cos(z_1(\alpha) - z_1(\alpha-\beta)))^2} d\beta.$$

Evaluating this expression at $\alpha = 0$ we have that

$$(\partial_\alpha v_1)(0) = \int_{-\pi}^{\pi} \frac{\cos(z_1(\beta))(\partial_\alpha z_1(\beta))^2 + \sin(z_1(\beta))\partial_\alpha^2 z_1(\beta)}{\cosh(z_2(\beta)) - \cos(z_1(\beta))} d\beta$$

$$- \int_{-\pi}^{\pi} \sin(z_1(\beta))\partial_\alpha z_1(\beta) \frac{\sin(z_1(\beta))\partial_\alpha z_1(\beta) - \sinh(z_2(\beta))(\partial_\alpha z_2(0) - \partial_\alpha z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2} d\beta.$$

Integration by parts yields

$$\int_{-\pi}^{\pi} \frac{\sin(z_1(\beta))\partial_\alpha^2 z_1(\beta)}{\cosh(z_2(\beta)) - \cos(z_1(\beta))} d\beta$$

$$= - \int_{-\pi}^{\pi} \cos(z_1(\beta)) \frac{(\partial_\alpha z_1(\beta))^2}{\cosh(z_2(\beta)) - \cos(z_1(\beta))} d\beta$$

$$+ \int_{-\pi}^{\pi} \sin(z_1(\beta))\partial_\alpha z_1(\beta) \frac{\sin(z_1(\beta))\partial_\alpha z_1(\beta) + \sinh(z_2(\beta))\partial_\alpha z_2(\beta)}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2} d\beta.$$

Therefore we obtain that

$$(\partial_\alpha v_1)(0) = \partial_\alpha z_2(0) \int_{-\pi}^{\pi} \frac{\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2} \partial_\alpha z_1(\beta) d\beta$$

$$= 2\partial_\alpha z_2(0) \int_{0}^{\pi} \frac{\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2} \partial_\alpha z_1(\beta) d\beta \qquad (7)$$

From the expression (7) we can control the sign of $(\partial_\alpha v_1)(0)$. In order to clarify the proof we shall take

$$z_1(\beta) = -\sin(\beta) + \beta.$$

We construct the function $z_2(\beta)$ in the following way:
Let β_1 and β_2 be real numbers satisfying $0 < \beta_1 < \beta_2 < \pi$, and let $z^*(\beta)$ be a smooth function on $[-\pi, \pi]$, with the following properties,

a. $z^*(\beta)$ is odd. $\quad$ b. $(\partial_\beta z^*)(0) > 0$.

c. $z^*(\beta) > 0$ if $\beta \in (0, \beta_1)$. $\quad$ d. $z^*(\beta) < 0$ if $\beta \in (\beta_1, \beta_2]$

e. $z^*(\beta) \leq 0$ if $\beta \in [\beta_2, \pi]$.

For a positive real number b to be fixed later, we define a piecewise smooth function $\tilde{z}(\beta)$ on $[-\pi, \pi]$, by setting

$$\begin{aligned} \tilde{z}(\beta) &= bz^*(\beta) \quad \text{if } |\beta| \leq \beta_1, \\ \tilde{z}(\beta) &= z^*(\beta) \quad \text{if } \beta_1 < |\beta| < \pi. \end{aligned}$$

Then

$$\int_{\beta_1}^{\pi} \frac{\sin(z_1(\beta)) \sinh(\tilde{z}(\beta))}{(\cosh(\tilde{z}(\beta)) - \cos(z_1(\beta)))^2} \partial_\alpha z_1(\beta) d\beta$$

is negative and independent of b, while

$$\int_{0}^{\beta_1} \frac{\sin(z_1(\beta)) \sinh(\tilde{z}(\beta))}{(\cosh(\tilde{z}(\beta)) - \cos(z_1(\beta)))^2} \partial_\alpha z_1(\beta) d\beta$$

tends to zero as $b \to \infty$.

Therefore, we can fix b large enough so that

$$\int_0^{\pi} \frac{\sin(z_1(\beta))\sinh(\tilde{z}(\beta))}{(\cosh(\tilde{z}(\beta)) - \cos(z_1(\beta)))^2} \partial_\alpha z_1(\beta) d\beta < 0.$$

It is now easy to approximate $\tilde{z}(\beta)$ in $L^2[-\pi, \pi]$ by an odd, real-analytic $2\pi-$periodic function such that

$$\int_0^{\pi} \frac{\sin(z_1(\beta))\sinh(z_2(\beta))}{(\cosh(z_2(\beta)) - \cos(z_1(\beta)))^2} \partial_\alpha z_1(\beta) d\beta < 0,$$

and $\partial_\alpha z_2(0) > 0$.

Summary

Theorem

There exists a non-empty open set of initial data in H^4*, satisfying the R-T (strictly positive* $\sigma > 0$*) for which the solution of the Muskat problem becomes immediately analytic and then pass to the unstable regime in finite time.*

Instant Analyticity

Theorem

Let $z(\alpha,0)-(\alpha,0)=z_0(\alpha)-(\alpha,0)\in H^4(\mathbb{T})$, satisfying the chord-arc condition and $\partial_\alpha z_1(\alpha,0)>0$ (R-T). Then there is a solution of the Muskat problem $z(\alpha,t)$ defined for $0<t\leq T$ that continues analytically into the strip $S(t)=\{\alpha+i\zeta : |\zeta|<ct\}$ for each t. Here, c and T are determined by upper bounds of the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of $\partial_\alpha z_1(\alpha,0)$. Moreover, for $0<t\leq T$, the quantity

$$\sum_{\pm}\int(|z(\alpha\pm ict)-(\alpha\pm ict,0)|^2+|\partial_\alpha^4 z(\alpha\pm ict)|^2)d\alpha$$

is bounded by a constant determined by upper bounds for the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of $\partial_\alpha z_1(\alpha,0)$. Above $|\cdot|$ is the modulus of a complex number or a vector in $\mathbb{C}^2$.

Again we use energy estimates. We need to control

$$\frac{d}{dt}\sum_{\pm}\int (|z(\alpha \pm ict) - (\alpha \pm ict, 0)|^2 + |\partial_\alpha^4 z(\alpha \pm ict)|^2)d\alpha$$

in terms of

$$\sum_{\pm}\int (|z(\alpha \pm ict) - (\alpha \pm ict, 0)|^2 + |\partial_\alpha^4 z(\alpha \pm ict)|^2)d\alpha$$

The most singular term cames from

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z(\alpha \pm ict, t)|^2 d\alpha = \sum_{j=1,2}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z_j(\alpha \pm ict, t)|^2 d\alpha$$

where

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z_j(\alpha \pm ict, t)|^2 d\alpha \\ &= \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z_j(\alpha \pm ict, t)}(\partial_t(\partial_\alpha^4 z_j)(\alpha \pm ict, t) \pm \underbrace{ic\partial_\alpha^5 z_j(\alpha \pm ict, t)}_{\text{Extra term}})d\alpha.\end{aligned}$$

Let us denote

$$(x_1, x_2) \cdot (x_3, x_4) = x_1 x_3 + x_2 x_4$$

for $x_j \in \mathbb{C}$,

$$\gamma \equiv \alpha \pm ict$$

and

$$Q(\gamma, \beta) = \cosh(z_2(\gamma) - z_2(\gamma - \beta)) - \cos(z_1(\gamma) - z_1(\gamma - \beta)).$$

Again the most singular quantity (coming from z_t) we have to control is

$$\text{Danger} = \Re \int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot \text{Dangerous}(\gamma) d\alpha.$$

where

$$\text{Dangerous}(\gamma) = \int_{\mathbb{T}} \frac{\sin(z_1(\gamma) - z_1(\gamma - \beta))}{Q(\gamma, \beta)} (\partial_\alpha^5 z(\gamma) - \partial_\alpha^5 z(\gamma - \beta)) d\beta$$

As we did before

$$\begin{aligned}\text{Dangerous}(\gamma) =& \partial_\alpha^5 z(\alpha)\int_{\mathbb{T}} a(\gamma,\beta)d\beta - \int_{\mathbb{T}} a(\gamma,\beta)\partial_\alpha^5 z(\gamma-\beta)d\beta \\ & - \frac{\partial_\alpha z_1(\gamma)}{\partial_\alpha z_1(\gamma)^2+\partial_\alpha z_2(\gamma)^2}\Lambda\partial_\alpha^5 f(\gamma) \\ =& \partial_\alpha^5 z(\alpha)\tilde{a}(\gamma) - \int_{\mathbb{T}} a(\gamma,\beta)\partial_\alpha^5 z(\gamma-\beta)d\beta \\ & -\sigma(\gamma)\Lambda\partial_\alpha^5 f(\gamma) \\ =& \partial_\alpha^5 z(\alpha)\tilde{a}(\gamma) - \sigma(\gamma)\Lambda\partial_\alpha^5 f(\gamma) + \text{Nice}\end{aligned}$$

First, we look at

$$K_3 = \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^5 z(\gamma)\tilde{a}(\gamma)d\alpha = L_1 + L_2$$

$$L_1 = \int_{\mathbb{T}} \Re(\tilde{a})(\Re(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z) + \Im(\partial_\alpha^4 z)\Im(\partial_\alpha^5 z))d\alpha,$$

$$L_2 = \int_{\mathbb{T}} \Im(\tilde{a})(-\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^5 z) + \Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z))d\alpha.$$

An easy integration by parts allows us to get

$$L_1 = -\frac{1}{2}\int_{\mathbb{T}} \Re(\partial_\alpha \tilde{a})|\partial_\alpha^4 z|^2 d\alpha = \text{Controlled Quantity}$$

For L_2 we find

$$L_2 = \int_{\mathbb{T}} \Im(\partial_\alpha \tilde{a})\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^4 z) d\alpha + \underbrace{2\int_{\mathbb{T}} \Im(\tilde{a})\Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z)\, d\alpha}_{=M_1}.$$

Now we notice that

$$\begin{aligned} M_1 = & -2\int_{\mathbb{T}} \Im(\tilde{a})\Im(\partial_\alpha^4 z)\Re(\Lambda(H(\partial_\alpha^4 z)))d\alpha \\ & -2\int_{\mathbb{T}} \Lambda^{1/2}(\Im(\tilde{a})\Im(\partial_\alpha^4 z))\Re(\Lambda^{1/2}(H(\partial_\alpha^4 z)))d\alpha \end{aligned}$$

and therefore

$$\begin{aligned} M_1 \leq & 2\|\Lambda^{1/2}(\Im(\tilde{a})\Im(\partial_\alpha^4 z))\|_{L^2(\partial S)}\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(\partial S)} \\ \leq & C\|\Im(\tilde{a})\|_{H^2(\partial S)}(\|\partial_\alpha^4 z\|_{L^2(\partial S)} + \|\Lambda^{1/2}(\partial_\alpha^4 z)\|_{L^2(\partial S)})\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(\partial S)} \\ & \text{Controlled Quantity} + K\|\Im(\tilde{a})\|_{H^2(\partial S)}\|\Lambda^{1/2}\partial_\alpha^4 z\|_{L^2(\partial S)}^2. \end{aligned}$$

Next we estimate the R-T term

$$\text{R-Tterm} = -\Re\int_{\mathbb{T}} \sigma(\gamma)\overline{\partial_\alpha^4 z(\gamma)} \cdot \Lambda\partial_\alpha^4 z(\gamma) d\alpha = M_2 + M_3$$

where

$$M_2 = \int_{\mathbb{T}} \Im(\sigma(\gamma))(-\Re(\partial_\alpha^4 z) \cdot \Im(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z) \cdot \Re(\Lambda(\partial_\alpha^4 z)))d\alpha,$$

$$M_3 = -\int_{\mathbb{T}} \Re(\sigma(\gamma))(\Re\partial_\alpha^4 z \cdot \Re(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z) \cdot \Im(\Lambda(\partial_\alpha^4 z)))d\alpha.$$

In M_2 it is easy to find a commutator formula:

$$M_2 = \int_{\mathbb{T}} [-\Lambda\big(\Im(\sigma(\gamma))\Re(\partial_\alpha^4 z)\big) + \Im(\sigma(\gamma))\Re(\Lambda(\partial_\alpha^4 z))] \cdot \Im(\partial_\alpha^4 z) d\alpha$$
$$= \text{Controlled Quantity}$$

We split $M_3 = N_1 + \text{Key-term}$

$$N_1 = -\int_{\mathbb{T}} [\Re(\sigma(\gamma)) - m(t)](\Re(\partial_\alpha^4 z) \cdot \Re(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z) \cdot \Im(\Lambda(\partial_\alpha^4 z)))d\alpha,$$

$$\text{Key-term} = -m(t)\|\Lambda^{1/2}(\partial_\alpha^4 z)\|^2_{L^2(S)},$$

where

$$m(t) = \min_{\gamma} \Re(\sigma(\gamma)).$$

By the pointwise estimate

$$2g\Lambda(g) - \Lambda(g^2) \geq 0,$$

we have that

$$N_1 \leq \frac{1}{2}\|\Lambda(\Re(\sigma(\gamma))\|_{L^\infty(\partial S)}\|\partial_\alpha^4 z\|^2_{L^2(\partial S)} = \text{Controlled Quantity}$$

Finally we estimate the extra term coming from the constant c

$$\begin{aligned}|\text{Extra-term}| =& c\left|\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)} \cdot ic\partial_\alpha^5 z(\gamma) d\alpha\right| \\ \leq& c\|\Lambda^{1/2}(\partial_\alpha^4 z)\|^2_{L^2(S)}\end{aligned}$$

Putting all these things together

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}} |\partial_\alpha^4 z(\alpha \pm ict)|^2 d\alpha \leq (c + K\|\Im(\tilde{a})\|_{H^2(S)}(t) - m(t))\|\Lambda^{1/2}(\partial_\alpha^4 z)\|^2_{L^2(S)}(t)$$
$$+ \text{Controlled Quantity}$$

Analyticity at $\sigma \geq 0$

Theorem

Let $z(\alpha,0) = z^0(\alpha)$ *be an analytic curve in the strip*

$$S = \{\alpha + i\zeta \in \mathbb{C} \,:\, |\zeta| < h(0)\},$$

with $h(0) > 0$ *and satisfying:*

- *The arc-chord condition,*
- *The Rayleigh-Taylor condition,* $\partial_\alpha z_1^0(\alpha) > 0$.
- *The curve* $z^0(\alpha)$ *is real for real* α.
- *The functions* $z_1^0(\alpha) - \alpha$ *and* $z_2^0(\alpha)$ *are periodic with period* 2π.
- *The functions* $z_1^0(\alpha) - \alpha$ *and* $z_2^0(\alpha)$ *belong to* $H^4(\partial S)$.

Then there exist a time T *and a solution of the Muskat problem* $z(\alpha,t)$ *defined for* $0 < t \leq T$ *that continues analytically into some complex strip for each fixed* $t \in [0,T]$. *Here* T *is either a small constant depending only on* $||z_0 - (\alpha,0)||_{H^4(\partial S)}$ *or it is the first time a vertical tangent appears, whichever occurs first.*

Lemma

Let $\varphi(\alpha \pm i\zeta) = \sum_{k=-N}^{N} A_k e^{ik\alpha \mp k\zeta}$. *Then, for* $\zeta > 0$, *we have*

$$\underbrace{\frac{\partial}{\partial\zeta}\sum_{\pm}\int_{\mathbb{T}} |\varphi(\alpha \pm i\zeta)|^2 d\alpha}_{i\partial_\alpha\varphi \text{ arises}} \geq \frac{1}{10}\sum_{\pm}\int_{\mathbb{T}} \Lambda(\varphi)(\alpha \pm i\zeta)\overline{\varphi(\alpha \pm i\zeta)}d\alpha$$

$$- 10\int_{\mathbb{T}} \Lambda(\varphi)(\alpha)\overline{\varphi}(\alpha)d\alpha,$$

where $\Lambda\varphi(\alpha \pm i\zeta) = \sum_{k=-N}^{N} |k| A_k e^{ik\alpha} e^{\mp k\zeta}$.

Proof:

$$\sum_{\pm}\int_{\mathbb{T}} |\varphi(\alpha \pm i\zeta)|^2 dx = 4\pi \sum_{k=-N}^{N} |A_k|^2 \cosh(2|k|\zeta)$$

$$\sum_{\pm}\int_{\mathbb{T}} \Lambda\varphi(\alpha \pm i\zeta)\overline{\varphi(\alpha \pm i\zeta)}dx = 4\pi \sum_{k=-N}^{N} |k||A_k|^2 \cosh(2|k|\zeta)$$

$$\int_{\mathbb{T}} \Lambda\varphi(\alpha)\overline{\varphi(\alpha)}d\alpha = 2\pi \sum_{k=-N}^{N} |k||A_k|^2$$

Differentiating in ζ we obtain

$$\frac{\partial}{\partial \zeta} \int_{\mathbb{T}} |\varphi(\alpha \pm i\zeta)|^2 d\alpha = 8\pi \sum_{k=-N}^{N} |k||A_k|^2 \sinh(2|k|\zeta).$$

The lemma holds since $\sinh(\zeta) \geq \cosh(\zeta) - 1$ for any $\zeta > 0$.

Corollary

Let $\varphi(\alpha \pm i\zeta, t) = \sum_{k=-N}^{N} A_k(t) e^{ik\alpha} e^{\mp k\zeta}$ *and* $h(t) > 0$ *be a decreasing function of* t*. Then*

$$\frac{\partial}{\partial t} \sum_{\pm} \int_{\mathbb{T}} |\varphi(\alpha \pm ih(t))|^2 d\alpha \leq \frac{h'(t)}{10} \sum_{\pm} \int_{\mathbb{T}} \Lambda(\varphi)(\alpha \pm ih(t)) \overline{\varphi(\alpha \pm ih(t))} d\alpha$$

$$10h'(t) \int_{\mathbb{T}} \Lambda(\varphi)(\alpha) \overline{\varphi}(\alpha) d\alpha + 2\Re \sum_{\pm} \int_{\mathbb{T}} \varphi_t(\alpha \pm ih(t)) \overline{\varphi(\alpha \pm ih(t))} d\alpha.$$

Using the above corollary we have that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{\pm}\int_{\alpha\in\mathbb{T}} \left|\partial_\alpha^4 z_\mu(\alpha \pm ih(t),t)\right|^2 d\alpha \\
&\leq \frac{h'(t)}{10}\sum_{\pm}\int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha \\
&- 10h'(t)\int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_\mu)(\alpha) \cdot \overline{\partial_\alpha^4 z_\mu}(\alpha) d\alpha \\
&+ 2\sum_{\pm}\Re\int_{\mathbb{T}} \partial_\alpha^4(\partial_t z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha \\
&\leq \frac{h'(t)}{10}\sum_{\pm}\int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha \\
&- 10h'(t)\int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_\mu)(\alpha) \cdot \overline{\partial_\alpha^4 z_\mu}(\alpha) d\alpha \\
&- 2\Re\int_{\mathbb{T}} \sigma(\alpha \pm ih(t),t)\Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha \\
&+ \text{Controlled Quantities}
\end{aligned}
$$

$$\frac{\partial_\alpha z_1(\alpha \pm ih(t),t)}{(\partial_\alpha z_1(\alpha \pm ih(t)))^2 + (\partial_\alpha z_2(\alpha \pm ih(t)))^2} = \frac{\partial_\alpha z_1(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2} + h(t)g_\pm(\alpha,t).$$

where

$$||g_\pm||_{H^2(\mathbb{T})} = \text{Controlled Quantity}$$

One finds,

$$\begin{aligned}
\text{R-Tterm} &= -2\Re \int_{\mathbb{T}} \frac{\partial_\alpha z_1(\alpha)}{|\partial_\alpha z(\alpha)|^2} \Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha \\
&- h(t) 2\Re \int_{\mathbb{T}} g_\pm(\alpha,t) \Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t)) \cdot \overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))} d\alpha \\
&\leq \text{Controlled Quantity} + Ch(t) \|\Lambda^{1/2} \partial_\alpha^4 z\|_{L^2(\partial S)}^2
\end{aligned}$$

Then

$$\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z_\mu(\alpha \pm ih(t))|^2 d\alpha \leq \text{CQ} - 10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_\mu)(\alpha)\cdot\overline{\partial_\alpha^4 z_\mu}(\alpha)d\alpha$$

$$+\left(Ch(t)+\frac{1}{10}h'(t)\right)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_\mu)(\alpha \pm ih(t))\cdot\overline{\partial_\alpha^4 z_\mu(\alpha \pm ih(t))}d\alpha.$$

Choosing

$$h(t) = h(0)\exp(-10Ct)$$

we eliminate the most dangerous term. The other one in the expression above deals with a function on the real line and it is easily controlled. Indeed

$$\left|\int_{\mathbb{T}}\Lambda\partial_\alpha^4 z_\mu(\alpha)\cdot\partial_\alpha^4 z_\mu(\alpha)\right| \leq \frac{C}{h(t)}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z_\mu(\alpha \pm ih(t))|^2 d\alpha,$$

as one sees by examining the Fourier expansion of $\partial_\alpha^4 z_\mu(\alpha, t)$.

Summary

Breakdown of smoothness

Theorem

There exists a solution $z(x, t)$ of the Muskat equation, with the following properties:

1. *Initially at time t_0, the interface is real-analytic and satisfies the arc-chord and Rayleigh-Taylor conditions.*
2. *At time $t_1 > t_0$, the interface turns over i.e. the interface is no longer a graph.*
3. *At time $t_2 > t_1$, the interface no longer belongs to C^4, although it is real-analytic for all times $t \in [t_0, t_2)$.*

Basic ideas of the proof:

- Backward in time local existence in 'H^4' in the unstable regime.
- Stability with respect to the 'guide' unperturbed solution

Sketch of the Proof

Let $\underline{z}(x,t)$ be the 'guide' unperturbed solution. This solution satisfies the following properties properties:

4. $\underline{z}(x,t)$ is analytic in x, for $|\Im x| < \varepsilon^{00}$ and $|t| \leq \tau^{00}$.
5. For $t \in [-\tau^{00}, 0)$, $\underline{z}(x,t)$ satisfies the Rayleigh-Taylor and arc-chord conditions.
6. For $t = 0$, the curve $\underline{z}(x,t)$ has a vertical tangent at $x = 0$.
7. For $t \in (0, \tau^{00}]$, the curve $\underline{z}(x,t)$ fails to satisfy the Rayleigh-Taylor condition,

Then we will study the analytic continuation of a Muskat solution to a carefully chosen time-varying domain of the form

$$\Omega(t) = \{|\Im x| < h(\Re x, t)\}, \tag{8}$$

defined for $t \in [-\tau^2, \tau]$. Here, τ is a small enough positive number. The boundary of $\Omega(t)$ is $\Gamma_+(t) \cup \Gamma_+(t)$ where

$$\Gamma_\pm(t) = \{x \in \mathbb{C} \, : \, \Im x = \pm h(\Re x, t)\}.$$

and $h(x,t)$ is a smooth function on $\mathbb{T} \times [-\tau^2, \tau]$ satisfying:

- $h(x, \tau) > 0 \; \forall x \in \mathbb{T} \setminus \{0\}$, $h(0, \tau) = 0$,
- $h(x, t) > 0 \; \forall x \in \mathbb{T}$ and $\forall t \in [-\tau^2, \tau)$.

For $t \in [-\tau^2, \tau]$, we will work with the space $H^4(\Omega(t))$, consisting of all analytic functions $f : \Omega(t) \mapsto \mathbb{C}^2$ whose derivatives up to order 4 belong to $L^2(\partial\Omega(t))$, and endowed with the norm

$$||f||^2_{H^4(\Omega(t))} = \sum_{\pm} \int_{\zeta \in \Gamma_\pm(t)} |f(\zeta)|^2 d\Re\zeta + \sum_{\pm} \int_{\zeta \in \Gamma_\pm(t)} |\partial^4_\zeta f(\zeta)|^2 d\Re\zeta.$$

Important remark:

- The domain $\Omega(\tau)$ has 'thickness' zero at the origin.
- Consequently, $H^4(\Omega(\tau))$ is not contained in $C^4(\mathbb{T})$.

We will also take $\tau < \tau^{00}$ and $h(x,t) < \varepsilon^{00}$, so that the Muskat solution $\underline{z}(x,t)$ continues analytically to $\Omega(t)$, for each $t \in [-\tau^2, \tau]$.

We can therefore pick an 'initial' curve $z^0(x)$, such that

8. $z^0(x) - \underline{z}(x,\tau)$ belongs to $H^4(\Omega(\tau))$ and has small norm, yet
9. $z^0(x)$ does not belong to $C^4(\mathbb{T})$.

We solve the Muskat problem backwards in time, with the 'initial' condition

10. $z(x,\tau) = z^0(x)$.

We show that our Muskat solution exists and continues analytically into $\Omega(t)$, for all $t \in [t_*, \tau]$ (for a suitable time t_*); moreover,

11. $z(x,t) - \underline{z}(x,t)$ has small norm in $H^4(\Omega(\tau))$, for all $t \in [t_*, \tau]$ (section 4).

Here, either

12. $t_* = -\tau^2$ or
13. a modified Rayleigh-Taylor condition, adapted to the time-varying domain, fails at time t_*.

We can rule out 13., thanks to 11., together with our understanding of $\underline{z}(t)$ and $\Omega(t)$.

A priori energy estimates (focus on stability)

We follow the evolution of the following quantity

$$\frac{1}{2}\frac{d}{dt}\sum_{\pm}\int_{\zeta\in\Gamma_\pm(t)}\left|\partial_\zeta^4 z_\mu(\zeta,t)\right|^2 d\Re\zeta \tag{9}$$

$$= \sum_{\pm}\Re\int_{\zeta\in\Gamma_\pm(t)} \overline{[\partial_\zeta^4 z_\mu(\zeta,t)]}\cdot\left\{\partial_t[\partial_\zeta^4 z_\mu(\zeta,t)] \pm ih_t(\Re\zeta,t)\partial_\zeta^5 z_\mu(\zeta,t)\right\} d\Re\zeta.$$

$$\Gamma_\pm(t) = \{x \pm ih(x,t) \ : \ x\in\mathbb{T}\}$$

We are interested in a bound from below of (9)

$$\frac{1}{2}\frac{d}{dt}\sum_{\pm}\int_{\zeta\in\Gamma_+(t)}\left|\partial_\zeta^4 z_\mu(\zeta,t)\right|^2 d\Re\zeta \geq -C(1+||z||_{H^4(\Omega)})^k.$$

We always will assume the arc-chord condition. The most dangerous term

$$\partial_t\left[\partial_\zeta^4 z_\mu(\zeta,t)\right] =$$

$$\int_{w\in\Gamma_+(t)} \frac{\sin(z_1(\zeta,t)-z_2(w,t))(\partial_\zeta^5 z_\mu(\zeta,t)-\partial_w^5 z_\mu(w,t))}{\cosh(z_2(\zeta,t)-z_2(w,t))-\cos(z_1(\zeta,t)-z_1(w,t))}dw + \text{Safe}(\zeta,t).$$

Dealing with the dangerous term we obtain:

$$\int_{w\in\Gamma_+(t)} \frac{\sin(z_1(\zeta,t)-z_2(w,t))(\partial_\zeta^5 z_\mu(\zeta,t)-\partial_w^5 z_\mu(w,t))}{\cosh(z_2(\zeta,t)-z_2(w,t))-cos(z_1(\zeta,t)-z_1(w,t))}dw$$

$$=\sigma_+(\Re\zeta)\Lambda_{\Gamma_+(t)}F(\zeta)+\tilde{a}(\zeta,t)\partial_\zeta F(\zeta)+\text{Safe}(\zeta,t),$$

where

$$F(\zeta)=\partial_\zeta^4 z_\mu(\zeta,t),$$

$$\sigma_+(\Re(\zeta))=\frac{-2\pi\partial_\zeta z_1(\zeta,t)}{(\partial_\zeta z_1(\zeta,t))^2+(\partial_\zeta z_2(\zeta,t))^2},$$

$$\tilde{a}(\zeta,t)=\int_{w\in\Gamma_+(t)}\left\{\frac{\sin\left(z_1(\zeta,t)-z_1(w,t)\right)}{\cosh\left(z_2(\zeta,t)-z_2(w,t)\right)-\cos\left(z_1(\zeta,t)-z_1(w,t)\right)}\right.$$

$$\left.-\frac{\partial_\zeta z_1(\zeta,t)}{[\partial_\zeta z_1(\zeta,t)]^2+[\partial_\zeta z_2(\zeta,t)]^2}\cot\left(\frac{\zeta-w}{2}\right)\right\}dw,$$

$$\Lambda_{\Gamma_+}F(z)=-\frac{1}{\pi}\int_{w\in\Gamma_+}\frac{1}{2}\cot\left(\frac{\zeta-w}{2}\right)(F'(\zeta)-F'(w))dw \quad (\zeta\in\Gamma_+(t)).$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_{\zeta\in\Gamma_+(t)} \left|\partial_\zeta^4 z_\mu(\zeta,t)\right|^2 d\Re\zeta \geq$$

$$\Re \int_{\zeta\in\Gamma_+(t)} \overline{F(\zeta)}\left\{\sigma_+\Lambda_{\Gamma_+(t)}F(\zeta) + [i\partial_t h + \tilde{a}]\,\partial_\zeta F(\zeta)\right\} d\Re\zeta$$

$$- \text{Controlled}$$

We want to give a sign to this integral up to terms bounded in L^2.
We need to compare

$$\sigma_+(\Re\zeta,t)\Lambda_{\Gamma_+(t)}F(\zeta,t) \sim [i\partial_t h + \tilde{a}]\,\partial_\zeta F(\zeta)$$
$$\zeta = x + ih(x,t)$$

- We need to control the sign of

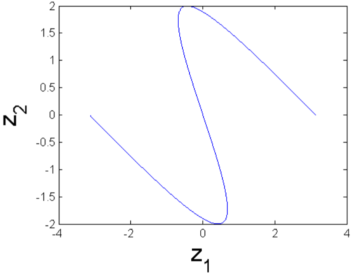
$$\sigma(x) + h_t(x,t)$$

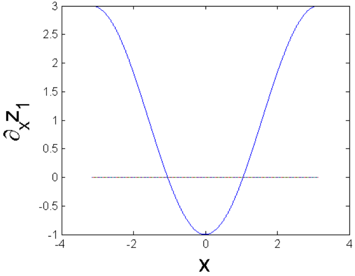
- From the discussion in the stable regime:

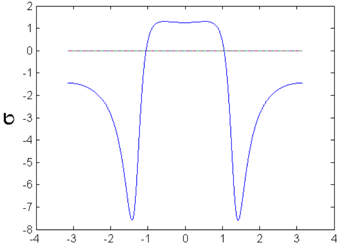
When $\sigma > 0$ we win analyticity
When σ is not positive we use a decreasing $h_t(t)$

The initial curve

We want to deal with a curve in the unstable regime. We need some suitable properties.







Rayleigh-Taylor Function

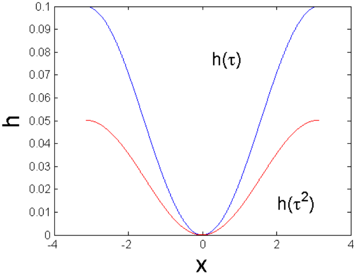
$$\sigma(x) = \frac{-2\pi\partial_x z_1(x)}{(\partial_x z_1(x))^2 + (\partial_x z_2(x))^2}$$

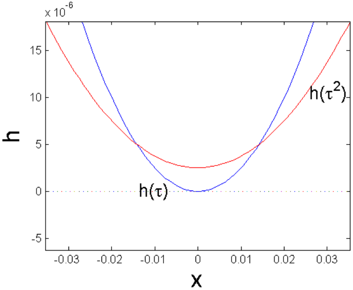
The height function $h(x,t)$

We choose a positive function $h(x,t)$ which 'measures' the analyticity of our curve.

$$h(x,t) = A^{-1}(\tau^2 - t^2) + (A^{-1} - A(\tau - t)) \sin^2\left(\frac{x}{2}\right)$$

First we pick $A >> 1$ and then $\tau << A^{-1}$.



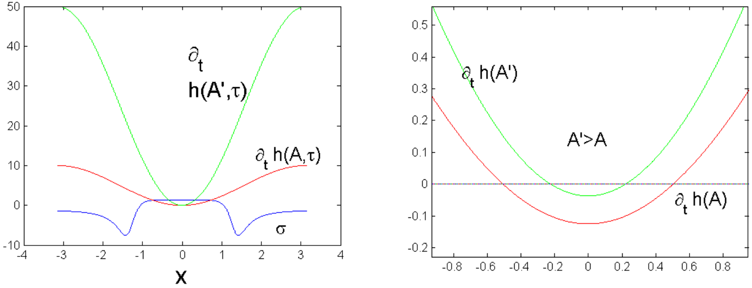


$h(x,\tau) > 0\ \forall x \in \mathbb{T} \setminus \{0\}$, $h(0,\tau) = 0$ $h(x,t) > 0\ \forall x \in \mathbb{T}$ and $\forall t \in [\tau^2, \tau)$

$$h(x,t) \leq CA^{-1}$$

The Rayleigh-Taylor function versus $\partial_t h(x,t)$

$$\partial_t h(x,t) = -2A^{-1}t + A\sin^2\left(\frac{x}{2}\right)$$



$$\partial_t h(0,\tau) < 0$$

Córdoba and Córdoba inequality

Let $A(x)$ and $f(x)$ complex valued smooth function. Let $|A'|$, $|A''| \leq C$ on $\mathbb{T}$. Then

$$\int_{x\in\mathbb{T}} A(x)\overline{f(x)}\Lambda f(x)dx$$

$$= \frac{1}{8\pi}\iint\limits_{x,y\in\mathbb{T}} A(x)\csc^2\left(\frac{x-y}{2}\right)|f(x)-f(y)|^2 dxdy + \text{Error}(x), \qquad (10)$$

where

$$|\text{Error}| \leq C'||f||^2_{L^2(\mathbb{T})}.$$

Here,

$$\Lambda f(x) = \frac{1}{4\pi}P.V.\int_{\mathbb{T}} \csc^2\left(\frac{x-y}{2}\right)(f(x)-f(y))dy.$$

Gårding inequality.

Let $\delta > 0$, and let a, b be real-valued functions on $\mathbb{T}$. Suppose that

1. $\left|\left(\frac{d}{dx}\right)^j a(x)\right|, \left|\left(\frac{d}{dx}\right)^j b(x)\right| \leq \delta^{1-j} \quad j = 0, 1, 2$
2. $a(x) \geq |b(x)|$

for all $x \in \mathbb{T}$. Then, for any smooth function f on $\mathbb{T}$, we have

$$\Re\left\{\int_{\mathbb{T}} \overline{f(x)}\,\{a(x)\Lambda f(x) + b(x) i f'(x)\}\,dx\right\} \geq -C\int_{\mathbb{T}} |f(x)|^2 dx,$$

where C is a universal constant.

$$\frac{1}{2}\frac{d}{dt}\int \left|\partial_\zeta^4 z_\mu(\zeta,t)\right|^2 d\Re\zeta$$
$$\geq \int \overline{F(\zeta)}\left\{\sigma_+\Lambda_{\Gamma_+(t)}F(\zeta) + [i\partial_t h + \tilde{a}]\,\partial_\zeta F(\zeta)\right\} d\Re\zeta$$

First we deal with the $\tilde{a} - term$

$$\Re \int_{\zeta\in\Gamma_+(t)} \overline{F(\zeta)}\tilde{a}(\zeta,t)\partial_\zeta F(\zeta) d\Re\zeta \qquad (11)$$

We define

$$f_+(x) = F(x+ih(x))$$
$$a_R(x) + ia_I(x) = \tilde{a}(\zeta)(1+i\partial_x h(x,t))^{-1}.$$

Then

$$(11) = -\int_{x\in\mathbb{T}} a_R'(x)\cdot\frac{1}{2}\left(|f_+(x)|^2\right)dx + \Re\int_{x\in\mathbb{T}} \overline{f_+(x)}a_I(x)if_+'(x)dx$$

Integration by parts and the Gårding yield

$$(11) \geq C\Re\int \overline{f_+(x)}|a_I(\zeta,t)|\Lambda f_+(x)dx\zeta - \text{Controlled}$$

We need to compare $\Lambda_{\Gamma_+(t)}F(\zeta)$, $\partial_\zeta F(\zeta)$ and $\Lambda f_+(x)$.

- $$\Lambda_{\Gamma_+}F(x+ih(x)) = (1+ih'(x))^{-1}\Lambda f_+(x) + \mathrm{Error}(x),$$

  where

  $$||\mathrm{Error}||_{L^2(\mathbb{T})} \leq C||f_+||_{L^2(\mathbb{T})}.$$

- $$\Lambda_{\Gamma_+}F(x+ih(x)) = iF'(x+ih(x)) + \mathrm{Erreur}(x),$$

  where

  $$||h(\cdot)\mathrm{Erreur}(\cdot)||_{L^2(\mathbb{T})} \leq C||f_-||_{L^2(\mathbb{T})}.$$

We need to compare $\Lambda_{\Gamma_+(t)}F(\zeta)$, $\partial_\zeta F(\zeta)$ and $\Lambda f_+(x)$.

- $$\Lambda_{\Gamma_+}F(x+ih(x)) = (1+ih'(x))^{-1}\Lambda f_+(x) + \mathrm{Error}(x),$$
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- $$\Lambda_{\Gamma_+}F(x+ih(x)) = iF'(x+ih(x)) + \mathrm{Erreur}(x),$$
  where
  $$||h(\cdot)\mathrm{Erreur}(\cdot)||_{L^2(\mathbb{T})} \le C||f_-||_{L^2(\mathbb{T})}.$$

- $$iF'(x+ih(x)) = (1+ih'(x))^{-1}\Lambda f_+(x) + \mathrm{Erreur}(x),$$
  where
  $$||h(\cdot)\mathrm{Erreur}(\cdot)||_{L^2(\mathbb{T})} \le C||f_-||_{L^2(\mathbb{T})}.$$

- Gårding inequality.

$$\Lambda_{\Gamma_+} F(x+ih(x)) = -\frac{1}{\pi} \int\limits_{w\in\Gamma_-} \frac{1}{2}\cot\left(\frac{z-w}{2}\right)(F'(z)-F'(w))dw$$

$$= F'(z)\left\{-\frac{1}{\pi}\int_{\Gamma_-} \frac{1}{2}\cot\left(\frac{z-w}{2}\right)dw\right\} + \frac{1}{2\pi}\int_{w\in\Gamma_-} \cot\left(\frac{z-w}{2}\right)F'(w)dw$$

Another exercise in contour integration shows that

$$\int_{\Gamma_-} \frac{1}{2}\cot\left(\frac{z-w}{2}\right)dw = -i\pi,$$

and therefore,

$$\Lambda_{\Gamma_+} F(x+ih(x)) = iF'(x+ih(x)) + \frac{1}{2\pi}\int_{w\in\Gamma_-} \cot\left(\frac{z-w}{2}\right)F'(w)dw$$

$$= iF'(x+ih(x)) - \frac{1}{4\pi}\int_{w\in\Gamma_-} \csc^2\left(\frac{z-w}{2}\right)F(w)dw. \qquad (12)$$

Next we prove that

$$\left| \int_{w \in \Gamma_-} \csc^2\left(\frac{z-w}{2}\right) F(w)dw \right| \leq C \int_{u \in \mathbb{T}} \frac{|f_-(u)|}{||x-u||^2 + h(x)^2} du$$

$$\leq Ch^{-1}(x) M[f_-](x),$$

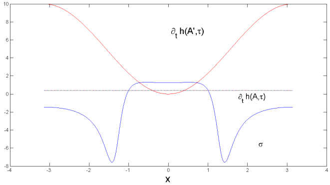
where $M[f_-]$ denotes the Hardy-Littlewood maximal function of f_-

Let θ_{in} and θ_{out} a partition of the unity,

$$\theta_{in} + \theta_{out} = 1 \quad \theta_{in\,,\,out} \geq 0$$
$$\mathrm{supp}\,\theta_{out} \subset \{x \in \mathbb{T} \,:\, ||x|| \geq \frac{1}{2}\tau^{\frac{1}{2}}A^{-1}\}$$
$$\mathrm{supp}\,\theta_{in} \subset \{x \in \mathbb{T} \,:\, ||x|| \leq \tau^{\frac{1}{2}}A^{-1}\}$$

We analyze separately,

$$\Re \int_{\zeta\in\Gamma_+(t)} (\theta_{out} + \theta_{in})(\overline{F(\zeta)}\left\{\sigma_+\Lambda_{\Gamma_+(t)}F(\zeta) + [i\partial_t h + \tilde{a}]\,\partial_\zeta F(\zeta)\right\} d\Re\zeta$$



In the outer part

$$\Re \int \theta_{out} \left[(1 + i\partial_x h(x,t))^{-1} \{\sigma_+ + \partial_t h\} - |a_I|\right] \overline{f_+(x)} \Lambda f_+(x) dx$$
$$+ \Re \int \theta_{out} \partial_t h(x,t) \overline{f_+(x)} \mathrm{Erreur}(x,t) dx$$

Using Cordaba[2]

$$\Re \int \theta_{out} \left[\Re \left\{(1 + i\partial_x h(x,t))^{-1} \{\sigma_+ + \partial_t h\} - |a_I|\right\}\right] \overline{f_+(x)} \Lambda f_+(x) dx$$
$$+ \Re \int \theta_{out} \partial_t h(x,t) \overline{f_+(x)} \mathrm{Erreur}(x,t) dx - \text{Controlled}$$

Then we have to control

- $$\Re \sigma_+ + \partial_t h - |a_I|(1 + (\partial_x h)^2) + \partial_x h \Im \sigma_+ > 0,$$

- $$||\theta_{out}(\cdot,t)\partial_t h(\cdot,t)\mathrm{Erreur}(\cdot)||_{L^2(\mathbb{T})}.$$

- $\Re\sigma_+(x,t) = \Re\sigma(x+ih(x,t)) \geq \sigma(x,t) - Ch(x,t)$.
- $|\Im\sigma_+(x,t)| = |\Im\sigma(x+ih(x,t))| \leq Ch(x,t)$.
- New assumptions: $z^0(x)$ is a odd function.

$$(1+(\partial_x h(x,\tau))^2)|a_I(x,\tau)| =$$

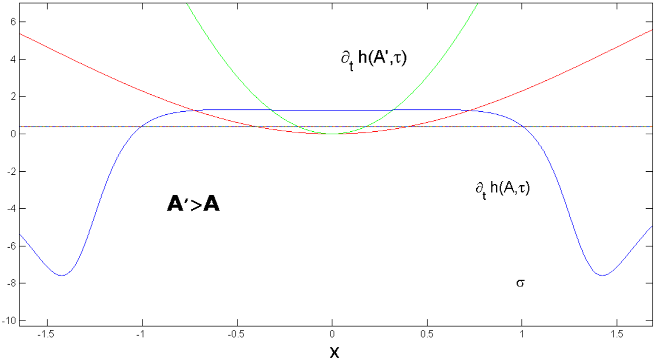
$$|\Im\{(1-i\partial_x h(x,\tau))\}\tilde{a}(\zeta,\tau)| \leq Ch(x,\tau)$$

Then

$$\begin{aligned}&\left[\Re\sigma_+ + \partial_t h - |a_I|(1+(\partial_x h)^2) + \partial_x h \Im\sigma_+\}\right]\\ &\geq \sigma(x,\tau) + \partial_t h(x,\tau) - Ch(x,\tau) > 0?.\end{aligned}$$

$$\sigma(x,\tau) + \partial_t h(x,\tau) - Ch(x,\tau) > 0.$$

for all $x \in \mathbb{T}$ if A is large enough



In the outer part

$$\{x \in \mathbb{T} \,:\, ||x|| > \frac{1}{2}\tau^{\frac{1}{2}}A^{-1}\}.$$

And we have that

$$\partial_t h(x,t) \leq CA^2 h(x,t).$$

Then

$$||\theta_{out}(\cdot)h_t(\cdot,t)\text{Erreur}(\cdot)||_{L^2} \leq C\,A^2\,||h(\cdot,t)\text{Erreur}(\cdot)||_{L^2} = \text{Controlled}$$

In the inner part

In the inner part $\{x \in \mathbb{T} \,:\, ||x|| < A^{-1}\tau^{\frac{1}{2}}\}$ we apply Gårding inequality:

$$a \geq |b|$$
$$\Re\left\{\int_{\mathbb{T}} \overline{f(x)}\,\{a(x)\Lambda f(x) + b(x) i f'(x)\}\,dx\right\} \geq -C\int_{\mathbb{T}} |f(x)|^2 dx,$$

to

$$\Re\int \theta_{in}\overline{f_+(x)}\left[\left\{\Re\left\{(1+i\partial_x h(x,t))^{-1}\sigma_+\right\} - |a_I|\right\}\Lambda f(x)\right.$$
$$\left. + i(1+i\partial_x h(x,t))^{-1}\partial_t h(x,t) f_+'(x)\right] dx$$

We need to check that

$$\theta_{in}(x)\left\{\Re\left\{(1+i\partial_x h(x,\tau))^{-1}\sigma_+\right\} - |a_I|\right\}$$
$$> \left|\theta_{in}(1+i\partial_x h(x,\tau))^{-1}\partial_t h(x,\tau)\right|.$$

This is possible since $|\partial_t h(x,\tau)| < CA^{-1}$ in the support of θ_{out}.

Stability

We take two height function:

- If $t \in [\tau^2, \tau]$

$$h(x,t) = A^{-1}(\tau^2 - t^2) + (A^{-1} - A(\tau - t)) \sin^2\left(\frac{x}{2}\right).$$

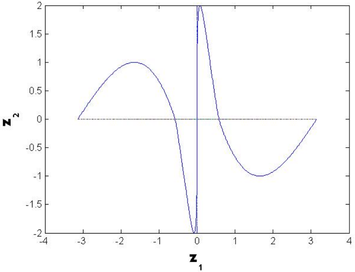
- If $t \in [-\tau^2, \tau^2]$

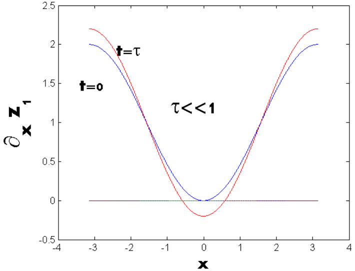
$$\hbar(x,t) = \frac{1}{4}\left(A^{-1}\tau^2 + A^{-1}\sin^2\left(\frac{x}{2}\right)\right) + A^{-2}\tau t + At\sin^2\left(\frac{x}{2}\right)$$

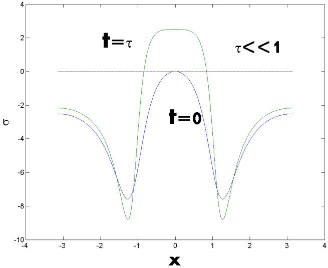
$$\Omega(t) = \begin{cases} \{\zeta \in \mathbb{T} + i\mathbb{R} \,:\, |\Im\zeta| < h(\Re\zeta, t)\} & \text{if } t \in [\tau^2, \tau] \\ \{\zeta \in \mathbb{T} + i\mathbb{R} \,:\, |\Im\zeta| < \hbar(\Re\zeta, t)\} & \text{if } t \in [-\tau^2, \tau^2] \end{cases}$$

Let $\underline{z}(x,t)$ be a turning over solution of the Muskat problem. We will say the unpertubed solution. We want to measure

$$||(z - \underline{z})(t)||_{H^4(\Omega(t))} \quad t \in [-\tau^2, \tau],$$
$$||(z - \underline{z})(\tau)||_{H^4(\Omega(\tau))} \leq \epsilon \quad \epsilon << 1.$$







$$c_1 t - c_2 x^2 \le \underline{\sigma}(x,t) \le C_1 t - C_2 x^2$$

for $||x|| \le c$

$$t \in [-\tau^2, \tau] \quad \tau << 1$$

$$\hbar(x,\tau^2) < h(x,\tau^2)$$

$$\partial_t \hbar(x,t) = A^{-2}\tau + A\sin^2\left(\frac{x}{2}\right) > 0$$

$$\frac{1}{2}\frac{d}{dt}\int_{\zeta\in\Gamma_+(t)}\left|\partial_\zeta^4(z_\mu-\underline{z}_\mu)(\zeta,t)\right|^2 d\Re\zeta \geq -C(A)\lambda^2,$$

if $t\in[\tau^2,\tau]$.
This estimate holds provided z_μ is a Muskat solutions satisfying

$$||z_\mu-\underline{z}_\mu||_{H^4(\Omega(t))}\leq\lambda$$

and

$$\lambda << \tau.$$

Here,

$$\Gamma_\pm(t)=\{x\pm ih(x,t)\ \ x\in\mathbb{T}\}.$$

$$\frac{1}{2}\frac{d}{dt}\int_{\zeta\in\Gamma_+(t)}\left|\partial_\zeta^4(z_\mu-\underline{z}_\mu)(\zeta,t)\right|^2 d\Re\zeta \geq -C(A)\tau^{-1}\lambda^2,$$

if $t\in[-\tau^2,\tau^2]$.
This estimate holds provided z_μ is a Muskat solutions satisfying

$$||z_\mu-\underline{z}_\mu||_{H^4(\Omega(t))}\leq\lambda$$

and

$$\lambda << \tau.$$

Here,

$$\Gamma_\pm(t)=\{x\pm i\hbar(x,t)\ x\in\mathbb{T}\}.$$

Between τ^2 and τ

$$\frac{1}{2}\frac{d}{dt}\int_{\zeta\in\Gamma_+(t)}\left|\partial_\zeta^4(z_\mu-\underline{z}_\mu)(\zeta,t)\right|^2 d\Re\zeta \geq$$

$$\Re\int\limits_{\zeta\in\Gamma_+(t)}\overline{F(\zeta)}\left\{\sigma_+\Lambda_{\Gamma_+(t)}F(\zeta)+[i\partial_t h+\tilde{a}]\,\partial_\zeta F(\zeta)\right\}d\Re\zeta$$

$$-\ \text{Controlled},$$

where

$$F(\zeta)=\partial_\zeta^4(z_\mu-\underline{z}_\mu)(\zeta,t)$$

$$\sigma_+(\Re(\zeta))=\frac{-2\pi\partial_\zeta z_1(\zeta,t)}{\left(\partial_\zeta z_1(\zeta,t)\right)^2+\left(\partial_\zeta z_2(\zeta,t)\right)^2}$$

$$\tilde{a}(\zeta,t)=\int\limits_{w\in\Gamma_+(t)}\left\{\frac{\sin\left(z_1(\zeta,t)-z_1(w,t)\right)}{\cosh\left(z_2(\zeta,t)-z_2(w,t)\right)-\cos\left(z_1(\zeta,t)-z_1(w,t)\right)}\right.$$

$$\left.-\frac{\partial_\zeta z_1(\zeta,t)}{[\partial_\zeta z_1(\zeta,t)]^2+[\partial_\zeta z_2(\zeta,t)]^2}\cot\left(\frac{\zeta-w}{2}\right)\right\}dw$$

Between τ^2 and τ

We use a partition of the unity $\theta_{in}(x,t)$, $\theta_{out}(x,t)$

- $\theta_{in} + \theta_{out} = 1$
- $\theta_{in}, \theta_{out} \geq 0$
- $\operatorname{supp} \theta_{in}(x,t) \subset \{||x|| \leq 20A^{-1}t^{\frac{1}{2}}\}$
- $\operatorname{supp} \theta_{out}(x,t) \subset \{||x|| \geq 10A^{-1}t^{\frac{1}{2}}\}$
- $$\left|\left(\frac{d}{dx}\right)^j \theta_{in,\,out}(x)\right| \leq C\left(A^{-1}t^{\frac{1}{2}}\right)^{-j}$$

  for $0 \leq j \leq 2$.

Between τ^2 and τ in the outer part

$$\Re \int_{\zeta\in\Gamma_+(t)} \theta_{out}(x,t)\,\overline{F(\zeta)}\left\{\sigma_+\Lambda_{\Gamma_+(t)}F(\zeta)+[i\partial_t h+\tilde{a}]\,\partial_\zeta F(\zeta)\right\}d\Re\zeta \tag{13}$$

Again we need

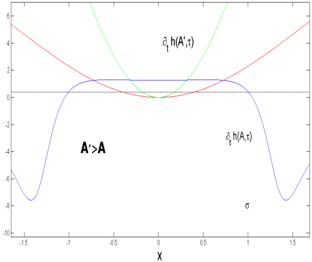
- $$\Re\sigma_+(t)+\partial_t h(t)-|a_I(t)|(1+(\partial_x h(t))^2)+\partial_x h(t)\Im\sigma_+(t)>0,$$

- $$||\theta_{out}(\cdot,t)\partial_t h(\cdot,t)\text{Erreur}(\cdot)||_{L^2(\mathbb{T})}-\mathit{Contolled}.$$

If $t\in[\tau^2,\tau]$ we can pick A and then τ small enough such that

$$\Re\underline{\sigma}_+(t)+\partial_t h(t)-|\underline{a}_I(t)|(1+(\partial_x h(t))^2)+\partial_x h(t)\Im\underline{\sigma}(t)_+>0,$$

for all $x\in\mathbb{T}$.



$$c_1 t - c_2 x^2 \leq \underline{\sigma}(x,t) \leq C_1 t - C_2 x^2$$

If $t \in [\tau^2, \tau]$ we can pick A and then τ small enough such that

$$\Re\underline{\sigma}_+(t) + \partial_t h(t) - |\underline{a}_I(t)|(1 + (\partial_x h(t))^2) + \partial_x h(t)\Im\underline{\sigma}(t)_+ > 0,$$

for all $x \in \mathbb{T}$.
Also we have than

$$\partial_t h(x,t) \leq CA^2 h(x,t)$$

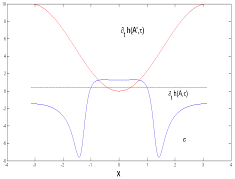
in the supp $\theta_{in}(x,t)$.

In the inner part

In the inner part we apply the Gårding inequality.
We need to check that

$$\theta_{in}(x,t)\left\{\Re\left\{(1+i\partial_x h(x,t))^{-1}\sigma_+\right\}-|a_I|\right\}$$
$$\geq\left|\theta_{in}(1+i\partial_x h(x,t))^{-1}\partial_t h(x,t)\right|.$$

with $t\in[\tau^2,\tau]$.
The unperturbed solutions satisfies the strict inequality.



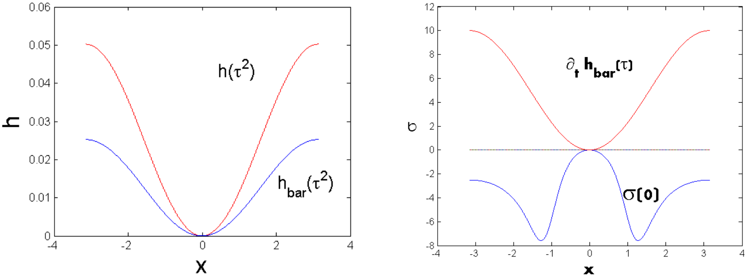
In both case, the inner and the outer one, we also can deal with the derivatives of the cut off functions.

Between $-\tau^2$ and τ^2

We use the height function $\hbar(x,t)$ We apply:

- Córdoba^2 inequality
- $\Lambda_{\Gamma_+(t)} F(\zeta) = iF'(\zeta)$ up to terms bounded in L^2

$$\partial_t \hbar(x,t) = A^{-2}\tau + A\sin^2\left(\frac{x}{2}\right)$$



Galerkin approximation

$$
\begin{aligned}
&\partial_t z^{[N]}(x,t) \\
&= \Pi_N \int_{y\in\mathbb{T}} \frac{\sin\left(z_1^{[N]}(x,t) - z_1^{[N]}(x-y)\right)\left(\partial_x z^{[N]}(x,t) - \partial_x z^{[N]}(x-y,t)\right)}{\cosh\left(z_2^{[N]}(x,t) - z_2^{[N]}(x-y)\right) - \cos\left(z_1^{[N]}(x,t) - z_1^{[N]}(x-y)\right)} dy \\
&\equiv \Pi_N J[z^{[N]}](x,t), \\
&z^{[N]}(x,t_0) = z_0^{[N]}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
&\Pi_N \ : \ \sum_{-\infty}^{\infty} A_k e^{ik\zeta} \mapsto \sum_{-N}^{N} A_k e^{ik\zeta} \\
&z_1^{[N]}(\zeta,t_0) - \zeta = \Pi_N[z_1^0(\zeta) - \zeta] \\
&z_2^{[N]}(\zeta,t_0) = \Pi_N[z_2^0(\zeta)]
\end{aligned}
$$

But

$$
\{\Pi_N J[z]\}\,(x+ih(x,t),t) \neq \Pi_N\,\{J[z](x+ih(x,t),t)\}\,.
$$

Conformal Transformation

There exist a smooth map

$$\Phi \,:\, \{(\zeta,t) \,:\, \zeta \in \overline{\Omega}(t) \times |t-t_0| \leq \delta\} \mapsto \{(\hat{\zeta},\hat{t}) \,:\, |\Im\hat{\zeta}| \leq \hat{h}(\hat{t}) \times |\hat{t}-t_0| \leq \delta\}$$

and a smooth positive function $\hat{h}(\hat{t})$ on $[t_0-\delta, t_0+\delta]$, with the following properties:

- $\Phi(\zeta,t) = (\hat{\zeta},\hat{t}) = (\phi(\zeta,t),t)$
- The map $\zeta \mapsto \phi(\zeta,t)$ is a biholomorphic map.
- $\phi(\zeta+2\pi,t) = \phi(\zeta)$
- $\phi \,:\, \Gamma_+(t) \mapsto \{\Im\hat{\zeta} = \hat{h}(t)\}$
- $\phi(\zeta,t)$ is real for real ζ.

The equation in the new coordinates

$$\partial_{\hat{t}} \hat{z}_\mu(\hat{\zeta}, \hat{t})$$

$$= \left\{ \hat{A}(\hat{\zeta}, \hat{t}) + \int_{\Im \hat{w} = \hat{h}(\hat{t})} \frac{\sin(\hat{z}_1(\hat{\zeta}, \hat{t}) - \hat{z}_1(\hat{w}, \hat{t}))\hat{B}(\hat{\zeta}, \hat{w}, \hat{t})}{\cosh(\hat{z}_2(\hat{\zeta}, \hat{t}) - \hat{z}_2(\hat{w}, \hat{t})) - \cos(\hat{z}_1(\hat{\zeta}, \hat{t}) - \hat{z}_1(\hat{w}, \hat{t}))} d\hat{w} \right\} \partial_{\hat{\zeta}} \hat{z}_\mu(\hat{\zeta}, \hat{t})$$

$$+ \int_{\Im \hat{w} = \hat{h}(\hat{t})} \frac{\sin(\hat{z}_1(\hat{\zeta}, \hat{t}) - \hat{z}_1(\hat{w}, \hat{t}))[\partial_{\hat{\zeta}} \hat{z}_\mu(\hat{\zeta}, \hat{t}) - \partial_{\hat{w}} \hat{z}_\mu(\hat{w}, \hat{t})]}{\cosh(\hat{z}_2(\hat{\zeta}, \hat{t}) - \hat{z}_2(\hat{w}, \hat{t})) - \cos(\hat{z}_1(\hat{\zeta}, \hat{t}) - \hat{z}_1(\hat{w}, \hat{t}))} d\hat{w},$$

where

$$\hat{z}(\hat{\zeta}, \hat{t}) = z(\zeta, t),$$
$$(\hat{\zeta}, t) = \Phi(\zeta, t),$$

$\hat{A}(\zeta, t)$, $\hat{B}(\zeta, w, t)$ and $\hat{h}(t)$ are smooth and they only depend on the conformal transformation. Also $B(\zeta, \zeta, t) = 0$.

RT condition

In the old coordinates

$$RT(\zeta,t) =$$

$$\Re\left(\frac{-2\pi\partial_\zeta z_1(\zeta,t)}{(\partial_\zeta z_1(\zeta,t))^2+(\partial_\zeta z_2(\zeta,t))^2}[1+i\partial_x h(\Re\zeta,t)]^{-1}\right)$$

$$\Im\left(\left\{P.V.\int_{w\in\Gamma_+(t)}\frac{\sin(z_1(\zeta,t)-z_1(w,t))}{\cosh(z_2(\zeta,t)-z_2(w,t))-\cos(z_1(\zeta,t)-z_1(w,t))})dw\right.\right.$$

$$\left.\left.+i\partial_t h(\Re\zeta,t)\right\}[1+i\partial_x h(\Re\zeta,t)]^{-1}\right)$$

The unperturbed solution $\underline{z}(x,t)$ satisfies the Rayleigh-Taylor condition for all $t\in[-\tau^2,\tau]$.

RT condition

In the new coordinates

$$\hat{RT}(\hat{\zeta},\hat{t}) \equiv$$

$$\Re\left\{\frac{-2\pi\partial_{\hat{\zeta}}\hat{z}_1(\hat{\zeta},\hat{t})}{(\partial_{\hat{\zeta}}\hat{z}_1(\hat{\zeta},\hat{t}))^2+(\partial_{\hat{\zeta}}\hat{z}_2(\hat{\zeta},\hat{t}))^2}\right\}$$

$$\Im\left\{P.V.\int_{\Im\hat{w}=\hat{h}(\hat{t})}\frac{\sin(\hat{z}_1(\hat{\zeta},\hat{t})-\hat{z}_1(\hat{w},\hat{t}))(\hat{B}(\hat{\zeta},\hat{w},\hat{t})+1)}{\cosh(\hat{z}_2(\hat{\zeta},\hat{t})-\hat{z}_2(\hat{w},\hat{t}))-\cos(\hat{z}_1(\hat{\zeta},\hat{t})-\hat{z}_1(\hat{w},\hat{t}))}d\hat{w}\right\}$$

$$+\Im\{\hat{A}(\hat{\zeta},\hat{t})\}+\hat{h}'(\hat{t}).$$

Equivalence

$$\hat{RT}(\hat{\zeta},\hat{t})>0 \Leftrightarrow RT(\zeta,t)>0$$

Energy estimate

$$\frac{d}{dt} \int\limits_{\zeta \in \Gamma_{\pm}(t)} \left| \partial_{\zeta}^{4} z_{\mu}(\zeta, t) \right|^2 d\zeta$$

Using

- Córdoba[2]
- $\Lambda_{\Gamma_+(t)} F(\zeta) = iF'(\zeta) + \text{Erreur}(\zeta)$,

we obtain

$$\frac{d}{dt} \int\limits_{\zeta \in \Gamma_{\pm}(t)} \left| \partial_{\zeta}^{4} z_{\mu}(\zeta, t) \right|^2 d\zeta \geq \int\limits_{\zeta \in \Gamma_{+}(t)} R\hat{T}(\zeta, t) \overline{F_{\mu}(\zeta)} \Lambda F_{\mu}(\zeta) d\zeta$$

$$- \text{Controlled.}$$

But the controlled term depend on a bound from below of the $h(t)$.

Closing the argument

We take

$$h_\kappa(x,t) = h(x,t) + \kappa,$$

where $\kappa > 0$ is a much smaller parameter than any other on the proof.
Let $\{z^\kappa(x,\tau)\}_\kappa)$ be a family of initial data:

- $z^\kappa(x,\tau) \in H^4(\Omega^\kappa(t))$ very close to $\underline{z}(x,\tau)$.
- $z^0(x,\tau)$ is not C^4.

We can prove

- Local existence for each $z^\kappa(x,\tau)$, with constants depending on κ.
- Stability. We lose the dependence on κ

Finally we pass two the limit $\kappa = 0$.

Let f_κ be the analytic function on $\Omega_\kappa(\tau)$ defined by setting

$$\begin{aligned}
\partial_x^4 f_\kappa(x) =& \log\left(\sin^2\left(\frac{x}{2}\right) + \sinh^2\left(\frac{\kappa}{2}\right)\right) \\
& - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\sin^2\left(\frac{s}{2}\right) + \sinh^2\left(\frac{\kappa}{2}\right)\right) ds, \\
\int_{-\pi}^{\pi} f_\kappa(x)dx =& 0,
\end{aligned}$$

for $x \in \mathbb{T}$. One sees easily that $f_\kappa(\zeta)$ can be taken to be analytic outside the union of the slits

$$\{\Re\zeta = 2\pi m, \pm\Im\zeta \geq \kappa\} \quad (m \in \mathbb{Z});$$

thus, $f_\kappa(\zeta)$ is analytic on $\Omega_\kappa(\tau)$ as claimed.
Moreover, the norm of $f_\kappa(\zeta)$ in $H^4(\Omega_\kappa(\tau))$ is bounded by a universal constant C, since the function $\log|x|$ belongs to $L^2([-\pi,\pi])$.

$$z_\kappa(\zeta,\tau) = (\underline{z}_1(\zeta,\tau) + \lambda f_\kappa(\zeta),\ \underline{z}_2(\zeta,\tau)).$$

THANK YOU FOR YOUR ATTENTION