Singular perturbations and heterogeneities

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Outline

1. Generalities;
2. Low Mach number limit for isentropic flows
3. Low Mach number limit for non-isentropic flows;
4. Example with strong coupling between mean flow and waves (Energetical level);
5. Pseudo-incompressible model (stratification);
6. Lake equations and heterogeneities;

Singular perturbations

Singular perturbations:

$$\partial_t \mathcal{U}^\varepsilon + \frac{1}{\varepsilon} L(\mathcal{U}^\varepsilon) + Q(U^\varepsilon, U^\varepsilon) = 0$$

with L skew symetric in H^s norm: hyperbolic structure + spectral decomposition.

Examples: Low Mach number limit for isentropic flows, rotating fluids with coriolis force independent on latitude.

If L not skew symetric in H^s norm ?

Examples: presence of heterogeneities (Low Mach number limit for non-isentropic flows, effect of bathymetry for shallow water equations, effect of stratification in meteorology....).

Singular perturbations

In the "simplest case": study of the skew symetric operator

ker$L$ define the space of well prepared data: no oscillation (mean flow).

Eigenstructure of L gives the oscillating part of the velocity.

$$\mathcal{U}^\varepsilon = \Pi\mathcal{U}^\varepsilon + (\mathrm{I} - \Pi)\mathcal{U}^\varepsilon$$

with Π the projector on the kernel (for low mach number: divergence free space, for rotating fluids: 2d horizontally incompressible flows).

$$\mathcal{U}^\varepsilon = \Pi\mathcal{U}^\varepsilon + \sum_{i\neq 0} \exp(-it\lambda_i/\varepsilon)\alpha_j^\varepsilon(t)\Phi_j$$

with λ_j eigenvalues linked to L and $\alpha_j^\varepsilon = < \mathcal{U}^\varepsilon, \Phi_j >$. The fast evolution is governed by the group $E(t) = \exp(-tL)$ and solution given by

$$\mathcal{U}^\varepsilon = E(t/\varepsilon)\mathcal{U}^\varepsilon(0) + \int_0^t E((t-s)/\varepsilon)F^\varepsilon ds.$$

Singular perturbations

Filtering method consists of studying the limit of

$$\mathcal{V}^\varepsilon = E(-t/\varepsilon)\mathcal{U}^\varepsilon = U^\varepsilon(0) + \int_0^t E(-s/\varepsilon)F^\varepsilon ds.$$

Then go back to the $\mathcal{U}^\varepsilon$ variable. The right-hand side F^ε in terms of $\mathcal{V}^\varepsilon$ gives a quadratic term linked to $E(t/\varepsilon)\mathcal{V}^\varepsilon$.

Low Mach Number limit for isentropic flows

Incompressible flows equations justification.

- Starting point: Compressible Navier Stokes or Euler equations (could be shallow-water system)
- flow velocity field small compared to sound velocity

Limit = incompressible equations.
Correction = acoustic waves.

Small parameter = Mach number, Froude number
For instance ε = Mach = fluid velocity / sound velocity

- Car: 50 km/h / 120 km/h = 1/20
- Plane = 800 km/h / 1200 km /h = 0.66

velocity motions $<$ 150 km are essentially incompressible

Difference = Noise (waves..)

Low Mach number limit for isentropic flows

Compressible barotropic Euler equations:

$$\partial_t \rho + \mathrm{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p(\rho) = 0$$

Let

$$u(t,x) = \varepsilon U(\varepsilon t, x)$$

gives

$$\partial_t \rho + \mathrm{div}(\rho U) = 0$$

$$\partial_t(\rho U) + \mathrm{div}(\rho U \otimes U) + \frac{\nabla p(\rho)}{\varepsilon^2} = 0$$

Then limit Mach $= \varepsilon \to 0$ provides

$$\nabla p(\rho) = 0.$$

Thus, using the mass equation, ρ is a constant

$$\rho = 1$$

and thus, divergence free condition $\mathrm{div} U = 0$. (U denoted u in the sequel)

Low Mach number limit for isentropic flows

Wave equation:

$$\psi = \frac{\rho - 1}{\varepsilon}$$

gives

$$\partial_t \psi + \mathrm{div}(\psi u) + \frac{\mathrm{div} u}{\varepsilon} = 0$$

$$\partial_t u + \mathrm{div}(u \otimes u) + h(\psi) + p'(1)\frac{\nabla \psi}{\varepsilon} = 0$$

Combinaison of a wave equation

$$\partial_t \sigma + \mathrm{div} v = 0$$

$$\partial_t v + p'(1)\nabla \sigma = 0$$

with a nonlinear equation (notation: $\partial_t(\psi, u) = Q(\psi, u) + \varepsilon^{-1} L(\psi, u)$).

Time scales
* $O(1)$: fluid evolution
* $O(\varepsilon)$: wave evolution (wave propagation velocity $= 1/Mach$).

Conjectured result:
If we look the incompressible part of $u \implies$ convergence to incompressible Euler

Low Mach number limit

Non-exhaustive bibliography:

- S. Klainerman, A. Majda: Existence on a time interval independent on Mach number.
- S. Klainerman, A. Majda: Convergence with well prepared data ($\psi = O(\text{Mach})$, $\text{div}u = O(\text{Mach})$).
- S. Ukai: whole space and waves going to infinity in times $O(\text{Mach})$
- S. Schochet: incompressible limit, general initial data (Filtering method).
- E. Grenier: Rotating fluids
- B. Desjardins, E. Grenier, P.-L. Lions, N. Masmoudi: incompressible viscous limit with boundaries
- B. Desjardins, E. Grenier: incompressible limit with Strichartz on weak solutions
- I. Gallagher: Oscillating limit parabolic systems
- Babin, Mahalov, B. Nikolaenko: Rotating fluids
- ......

Low Mach number limit for isentropic flows

Main ideas

Step 1: wave group

$\mathcal{L}(t)(\sigma_0, v_0)$ group solutions of

$$\partial_t \sigma + \mathrm{div} v = 0$$

$$\partial_t v + \nabla \sigma = 0$$

with initial data (σ_0, v_0).

The expression of $\mathcal{L}(t)$ is explicit in Fourier variable. The dispersion relation is fixed (Fixed spectrum):

$$\omega(k) = |k|.$$

$\mathcal{L}(t)$ is an isometry from H^s into H^s for

- periodic box
- whole space

using the explicit expression of the solution of the wave equation.

Low Mach number limit for isentropic flows

Step 2: conjugate process

Initial equations:

$$\partial_t(\psi, u) = Q(\psi, u) + \varepsilon^{-1}(\mathrm{divu}, \nabla\psi)$$

We conjugate $\mathcal{L}(t)$ posing

$$(\bar{\psi}, \bar{u}) = \mathcal{L}(-t/\varepsilon)(\psi, u)$$

and we get the equation under the form

$$\partial_t(\bar{\psi}, \bar{u}) + \mathcal{L}(-t/\varepsilon)Q(\mathcal{L}(t/\varepsilon)(\bar{\psi}, \bar{u}) = 0$$

Step 3: Compactness

$\partial_t(\bar{\rho}, \bar{u})$ is bounded (No problem with compactness in space).

Low Mach number limit for isentropic flows

Step 4: Limit equation

a) The projection $\Pi\overline{u}$ on the divergence free fields satisfies the incompressible Euler equations.
b) $(\mathrm{Id}-\Pi)\overline{u}$ and $\overline{\psi}$ satisfiy an equation describing the acoustic mode evolution:
– non-linear coupling between resonant modes $\omega(k_1)+\omega(k_2)=\omega(k_3)$ with $k_1+k_2=k_3$.
– Interaction with $\Pi\overline{u}$.

Physically: incompressible limit + interaction of acoustic waves.

Low Mach number limit for isentropic flows

Whole space:
– Physically: dispersion of acoustic waves at speed $1/\varepsilon$
– Mathematically: on all compact $\mathcal{L}(\varepsilon^{-1}t)(\psi_0, u_0) \to 0$ for all reasonable norm.
– consequently: $(\psi, u) = (0, \Pi u_0) +$ (initial boundary layer) $+ o(1)$.

Periodic case:
– Physically: confined waves.
– Mathematically $\mathcal{L}$ does not converge to 0.

Bounded domain with viscosity:
– Physically: boundary layers with strong dissipation (viscous damping process).
– $\mathcal{L}$ tends to 0 as in the whole space case.

Low Mach number limit for non-isentropic flows

$$\partial_t \rho + \mathrm{div}(\rho u) = 0,$$

$$\rho(\partial_t u + u \cdot \nabla u) + \nabla p = 0,$$

with

$$\partial_t S + u \cdot \nabla S = 0.$$

where S entropy, p given by the state law $\rho = R(p, S)$.
Example:

$$\rho = p^{1/\gamma} e^{-S/\gamma}.$$

Change of variable (see Métivier-Schochet): Let (p, u) then denoting $p = \bar{p}\exp^{\varepsilon q}$, we get

$$a(\partial_t q + u \cdot \nabla q) + \frac{1}{\varepsilon}\mathrm{div} u = 0,$$

$$r(\partial_t u + u \cdot \nabla u) + \frac{1}{\varepsilon}\nabla q = 0$$

$$\partial_t S + u \cdot \nabla S = 0$$

Low Mach number limit for non-isentropic flows

Formal limit

$\mathrm{div} u = 0$ and $\nabla q = 0$ then

$$\mathrm{div} u = 0,$$
$$r(\partial_t u + u \cdot \nabla u) + \nabla \Pi = 0$$
$$\partial_t S + u \cdot \nabla S = 0$$

with $\rho = R(\bar{p}, S)$ and $r(S)$.

Wave equation:

$$\partial_t(\sigma, v) = \frac{1}{\varepsilon}\mathcal{A}(\sigma, v)$$

with

$$\mathcal{A} = \begin{pmatrix} 0 & a^{-1}(S)\nabla . \\ r^{-1}(S)\nabla & 0 \end{pmatrix}.$$

which gives

$$\varepsilon^2 \partial_{tt}\sigma - \mathrm{div}(S(t,x)^{-1}\nabla\sigma) = 0.$$

Remark: $\partial_t S$ is bounded but wave equation with variable coefficients (in space and time).

Low Mach number limit for non-isentropic flows

Step 1: Wave equation

$$\partial_{tt}\sigma - \varepsilon^{-2}\mathrm{div}(S(t,x)^{-1}\nabla\sigma) = 0.$$

Let $\mathcal{L}(t,t',\varepsilon)$ the resolvent. We want that $\mathcal{L}(t,t',\varepsilon)$ is bounded uniformly from H^s into H^s.

Energy estimates:

* L^2: Energy gives uniform bound in L^2.
* H^1: $\partial_t\sigma$ satisfies a wave equation with unbounded source term with respect to ε.

Spectral decomposition

Problem: Variable coefficients with respect to time !

Problem: Crossing eigenvalues possibility !

$\Longrightarrow$ **bad behavior possibility ....** Energy exchange between modes.

Generic results: "for almost all initial data"

Low Mach number limit for non-isentropic flows

Two questions:

- Can we solve equations on some time interval which is independent of Mach number?
- Can we characterize the limit when Mach goes to zero?

First question : Métivier et Schochet.
Second question: Métivier Schochet (whole space by using dispersion for wave equation with non-constant coefficients). T. Alazard (exterior domain and for full CNS eqs).
Relies upon a Theorem of G. Métivier and S. Schochet proved using H measures

$$\varepsilon^2 \partial_t (a^\varepsilon(t,x)\partial_t \phi^\varepsilon) - \mathrm{div}(b^\varepsilon(t,x)\nabla\phi^\varepsilon) = \varepsilon f^\varepsilon(t,x)$$

where

$$\phi^\varepsilon \text{ is bounded in } \mathcal{C}^0([0,T];H^2(\mathbb{R}^d)), \quad f^\varepsilon \text{ is bounded in } L^2([0,T];L^2(\mathbb{R}^d)),$$

$$a^\varepsilon \text{ and } b^\varepsilon \text{ decay to zero at spatial infinity in same similar manner :}$$

$$a^\varepsilon(t,x) \geq c, \qquad |a^\varepsilon(t,x) - \underline{a}| = \mathcal{O}(|x|^{-1-\delta}), \qquad |\nabla a^\varepsilon(t,x)| = \mathcal{O}(|x|^{-2-\delta}),$$

Then ϕ^ε converges strongly to 0 in $L^2_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ to $(0,0)$.

Low Mach number limit for non-isentropic flows

Singular limit and nonisentropic Euler or NS systems.

- T. Alazard. Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions. Adv. Differential Equations 10 (2005), no. 1, 19-44.
- T. Alazard. Low Mach number limit of the full Navier-Stokes equations, Arch. Ration. Mech. Anal. 180 (2006), no. 1, 1-73.
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- D. Bresch, B. Desjardins, E. Grenier, K. Lin. Low Mach number limit of viscous polytropic flows: formal asymptotics in the periodic case. Studies in Applied Math., 109, (2002), 125–149.

Low Mach number limit for non-isentropic flows

For discussions on the problem, see:

D. Bresch, B. Desjardins, E. Grenier. Oscillatory limit with changing eigenvalues: A formal study, p. 91–105. New Directions in Mathematical Fluid Mechanics. The Alexander V. Kazhikhov Memorial Volume. Series: *Advances in Mathematical Fluid Mechanics.* Fursikov, Andrei V.; Galdi, Giovanni P.; Pukhnachev, Vladislav V. (Eds.) (2010).

Low Mach number limit for non-isentropic flows

Averaged equation for non-isentropic NS equations (from D.B., B. Desjardins, E. Grenier, C.K. Lin, 2002):

$$\partial_t \bar\rho + \mathrm{div}(\overline{\rho u}) = 0, \qquad \mathrm{div}\,\overline{u} = 0, \qquad \bar\rho\,\bar a = 1,$$

$$\partial_t(\overline{\rho u}) + \mathrm{div}(\overline{\rho u} \otimes \overline{u}) + \nabla \bar P - \mu \Delta \overline{u}$$

$$= \sum_{\substack{\ell, m \\ \varphi_\ell = \varphi_m}} \frac{\alpha_\ell^+ \alpha_m^- + \alpha_\ell^- \alpha_m^+}{2} \left(\nabla(\Psi_m \Psi_\ell) - \frac{\bar a}{\lambda_\ell^2} \nabla(\nabla \Psi_\ell \cdot \nabla \Psi_m) \right)$$

with (λ_j^2, Ψ_j) denote the eigenvectors of the nonlinear wave equation

$$-\mathrm{div}(\bar a \nabla \Psi_j) = \lambda_j^2 \Psi_j \quad \text{and} \quad \varphi_j(t) = \int_0^t \lambda_j(s)\, ds.$$

Low Mach number limit for non-isentropic flows

The coefficients $\alpha_k^{\sigma_k}$ with $\sigma_k \in \{+,-\}$ denote the components of the acoustic waves on a basis depending on $\{\Psi_j\}_{j\in N}$. They are governed by the dynamical system

$$\frac{d\alpha_k^{\sigma_k}}{dt} + \frac{\lambda_k^2(\lambda+2\mu)}{2}\alpha_k^{\sigma_k} + \sum_{\substack{\ell \\ \varphi_k=\varphi_\ell}} \mu \frac{\alpha_\ell^{\sigma_k}}{2\lambda_k^2}\int \mathrm{curl}(\bar a \nabla \Psi_k)\cdot \mathrm{curl}(\bar a \nabla \Psi_\ell)\,dx$$

$$= \sum_{\substack{\ell \\ \lambda_k=\lambda_\ell}} \frac{\alpha_\ell^{\sigma_k}}{2}\int \left\{ \Psi_\ell \partial_t \Psi_k + \frac{\nabla \Psi_\ell}{\lambda_k}\partial_t\left(\frac{\bar a \nabla \Psi_k}{\lambda_k}\right)\right\} dx$$

$$+\frac{(\gamma-1)}{4\sqrt{2}} \sum_{\substack{\ell,m,\sigma_\ell,\sigma_m \\ \sigma_\ell\varphi_\ell+\sigma_m\varphi_m=\sigma_k\varphi_k}} i\sigma_k\lambda_k\alpha_\ell^{\sigma_\ell}\alpha_m^{\sigma_m}\int \Psi_k\Psi_m\Psi_\ell\,dx$$

$$-\sum_{\substack{\ell \\ \varphi_\ell=\varphi_k}} \frac{\alpha_\ell^{\sigma_k}}{2\lambda_k^2}\int \bar a\ \mathrm{div}(\bar u\otimes\nabla\Psi_\ell + \nabla\Psi_\ell\otimes\bar u)\cdot\nabla\Psi_k\,dx$$

$$-\sum_{\substack{\ell,m,\sigma_\ell,\sigma_m \\ \sigma_\ell\varphi_\ell+\sigma_m\varphi_m=\sigma_k\varphi_k}} \frac{i\alpha_\ell^{\sigma_\ell}\alpha_m^{\sigma_m}}{2\sqrt{2}}\,\frac{1}{\sigma_k\lambda_k\sigma_\ell\lambda_\ell\sigma_m\lambda_m}\int \bar a\ \mathrm{div}(\bar a\nabla\Psi_\ell\otimes\nabla\Psi_m)\cdot\nabla\Psi_k\,dx.$$

Low Mach number limit for non-isentropic flows

Transversality and crossing of eigenvalues.

- R. Abraham, J. Robbin. Transversal mappings and flows. W.A. Ben- jamin, New-york, Amsterdam 1967.
- P. Gérard et F. Golse. Averaging Regularity Results for PDEs under Transversality Assumptions. *Comm. on Pure and Appl. Math.* 45, (1992), 126
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- K. Uhlenbeck. Generic properties of eigenfunctions. *Amer. J. Math.*, 98, (1976), 1059-1078.

  Several papers..... Clothilde Fermanian, Patrick Gérard, Y. Colin de Verdière.... etc..

Low Mach number limit for non-isentropic flows

Spectral decomposition

$$\partial_{tt}\sigma - \varepsilon^{-2}\mathrm{div}(S(x)^{-1}\nabla\sigma) = 0$$

forgetting time dependency

Spectrum:

$-\mathrm{div}(S(x)^{-1}\nabla\cdot)$ is a self-adjoint operator

Eigenvalues λ_j (with eventual multiplicity)

Π_j its corresponding eigenspace and ψ_j orthonormal basis.

Eigenspaces geometry:

Double eigenvalues

$$\Sigma_{j,k} = \Big\{\lambda_j(S) = \lambda_k(S)\Big\}.$$

In a neighborhood of a double eigenvalue,

$$\Pi_j + \Pi_k$$

is continuous, but not ψ_j, nor ψ_k.

Low Mach number limit for non-isentropic flows

Is $\Sigma_{j,k}$ of codimension 2?

A matrix model

Symetric matrices with eigenvalue at least double are of co-dimension 2 in the symmetric matrices set.

In dimension 2 ....

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Characteristic polynomial

$$X^2 - (a+c)X + ac - b^2$$

Eigenvalues:

$$\frac{a+c}{2} \pm \frac{\sqrt{(a-c)^2 + b^2}}{2}$$

Then

$$\Sigma_{j,k} = \{b = 0, a = c\}$$

line in a three dimensional space.
The eigenvectors do not depend on $x - \Pi x$ where Π is the projection on $\Sigma_{j,k}$.
Eigenvectors make one round when we make one round of $\Sigma_{j,k}$.

Low Mach number limit for non-isentropic flows

Is $\Sigma_{j,k}$ of codimension 2?

Pure Maths litterature ....

It seems that all have to be done !!

Question:

$$\mu\Big(\{S \quad | \quad |\lambda_j(S) - \lambda_k(S)| \leq \varepsilon\}\Big) \leq C\varepsilon^2$$

Difficulties:

- Definition of the measure μ in infinite dimension space ?
- Uniformity with respect to the approximation ?

Let Π_N projection on finite dimension space (Galerkin)
Let

$$\Sigma_{j,k}^{N,\varepsilon} = \{S = \Pi_N(S) \quad | \quad |\lambda_j(S) - \lambda_k(S)| < \varepsilon\}$$

On R^N the measure of Besov type

$$\mu_N = \otimes_{k=1}^N \frac{k^s}{2} 1_{[-1/k^s, 1/k^s]}$$

μ_∞ does not see the Besov $\{|\hat{u}(k)| < 1/k^s\}$.

Low Mach number limit for non-isentropic flows

Measure of neighborhoods of $\Sigma_{j,k}$

$$\Sigma_{j,k}^{N,\varepsilon} = \{S = \Pi_N(S) \quad | \quad |\lambda_j(S) - \lambda_k(S)| < \varepsilon\}$$

$$\mu_N = \otimes k^s 1_{[-1/k^s, 1/k^s]}$$

Theorem. Under hypothesis of non degeneracy, there exists a constant C_0 such that

$$\mu_N(\Sigma_{j,k}^{N,\varepsilon}) \leq C_0 \varepsilon^2$$

for all N and all ε.
Proof

Effect of regularity: $\Sigma_{j,k}$ is a graph with respect to the first components $\Pi_N x$.

Remarks:

- Codimension 2 notion "in the measure μ_N sense".
- $\Sigma_{j,k}$ has a null measure too, but what is important is its approximation.

Low Mach number limit for non-isentropic flows

Measure of neighborhoods of $\Sigma_{j,k}$

- Approximate diagonalisation
- Ansatz on (ψ_j, ψ_k):
  If $S_0 \in \Sigma_{j,k}$ then $\lambda_j(S_0+S)$ and $\lambda_k(S_0+S)$ are given by

$$\frac{\lambda_j(S_0)+\lambda_k(S_0)}{2} + \frac{1}{2}\Big(\int S|\nabla\psi_j|^2 + \int S|\nabla\psi_k|^2\Big)$$

$$\pm\frac{1}{2}\sqrt{\Big(\int S|\nabla\psi_j|^2 - \int S|\nabla\psi_l|^2\Big)^2 + 4\Big(\int S\nabla\psi_j\cdot\nabla\psi_k\Big)^2}$$

$$+O(|S|^2_{H^s}).$$

  gives informations locally.
- Simple eigenvalues are Lipschitzian

$$\nabla_S\lambda_j(S_0).S = -\int S|\nabla\psi_j|^2.$$

- Eigenvalues cannot be too fast closed.
- When they are closed ... Above ansatz.

Low Mach number limit for non-isentropic flows

Measure of neighborhoods of $\Sigma_{j,k}$

If

$$|\nabla\psi_j|^2 - |\nabla\psi_l|^2 \text{ and } \nabla\psi_j\nabla\psi_k$$

are linearly independent, $\Sigma_{j,k}$ is locally of codimension 2.

Low Mach number limit for non-isentropic flows

Outside $\Sigma_{j,k}$

$$\partial_{tt}\sigma - \varepsilon^{-2}\mathrm{div}(S(t,x)^{-1}\nabla\sigma) = 0$$

We decompose

$$\sigma(t) = \sum_j \alpha_j(t)\psi_j(S(t))\exp\Big(\varepsilon^{-2}\int_0^t \lambda_j(S(t))\Big).$$

We get

$$\partial_t\alpha_j = -\Big(\sum_k \alpha_k(t)\nabla\psi_k(S(t)).S'(t) \quad | \quad \psi_j(S(t))\Big).$$

This is correctly bounded from above (far from $\Sigma_{j,k}$!

As soon as $S(t)$ avoids double eigenvalues, $\mathcal{L}$ is bounded.
We introduce $\widetilde{q}, \widetilde{u}) = \mathcal{L}(\varepsilon^{-1}t)(q,u)$ for which all derivatives are bounded $\Longrightarrow$ compactness $\Longrightarrow$ convergence.
Limit equation: take care of resonances.

Low Mach number limit for non-isentropic flows

Is it possible to avoid $\Sigma_{j,k}$?

Geometry of the problem:

Find initial data which avoid a codimension 2 subset.

Regular flow case in finite dimension

$\Theta(t_1, t_2)$ flow, Σ of codimension 2 to be avoided
We have to evaluate

$$A_\varepsilon = \{x \quad | \quad \exists 0 \le t \le T \quad \Theta(0,t)x \in \Sigma_\varepsilon\}.$$

$$= \cup_t \{x \quad | \quad \Theta(0,t)x \in \Sigma_\varepsilon\}.$$

Two hypothesis:

- Flow with bounded divergence
- Bounded flow

$$\mu(A_\varepsilon) \le C\varepsilon T.$$

Problem: The flow is not regular!!!

Low Mach number limit for non-isentropic flows

Limit equation

Well prepared data:

Waves with $O(Mach)$ size. Limit = incompressible non-homogeneous Euler equations

Ill prepared data:

- Waves with $O(1)$ size.
- Limit = Euler with a source term: wave interactions.
- Source term= combinaison of terms involving $\psi_j(S)$ which is singular around to $\Sigma_{j,k}$.

Type equation

ODE of the form

$$\partial_t \phi + Q(\phi) = R\Big(\frac{x - \Pi x}{\|x - \Pi x\|}\Big)$$

with Π projection on a codimension 2 variety.

Low Mach number limit for non-isentropic flows

Dimension 2 example

$$\dot{x} = \phi\Big(\frac{x}{|x|}\Big)$$

with ϕ continuous defined from the unit circle to R^2.

Polar coordinates:

$$x(t) = \rho(t)e^{i\theta(t)}$$

with

$$\rho\dot{\theta} = \chi(\theta)$$

$$\dot{\rho} = \psi(\theta)$$

with $\chi(\theta) = \mathrm{Im}(\phi(e^{i\theta})e^{-i\theta})$. Change of time gives

$$\dot{\theta} = \chi(\theta)$$

$$\dot{\rho} = \psi(\theta)\rho.$$

Low Mach number limit for non-isentropic flows

Discussion

- Possible asymptots: θ with $\chi(\theta) = \theta$.
- Stability depends on χ'.
- Multiple possibility in function of sign of ψ.

Not proved:
Flow:

- The flow is discontinuous: We pass on the left or on the right of the singularity
- or we enter directly in the singularity in finite time.

Divergence:

Through calculation, if A set

$$\mu(\Theta(t)(A)) \leq C\mu(A)$$

with C independent on t and on A.

Low Mach number limit for non-isentropic flows

Vector field with a homogeneous degre 0 singularity near a codimension 2 set.

$$\dot{x} = \phi\Big(x, \frac{x_h}{|x_h|}\Big)$$

with $x_h = (x_1, x_2)$.

- Perturbative arguments with respect to the dimension 2.
- Under geometrical hypothesis: Existence except for a codimension 1 subset.

See D.B., B. Desjardins, E. Grenier. *Proc AMS* (2011).

Low Mach number limit for non-isentropic flows

Limit Equation =
incompressible nonhomogeneous equations + source term (nonlinear interaction of waves).
Source term = combination of terms $\Psi_j(S)$ which are singular on $\Sigma_{j,k}$.

Simple model:

$$\partial_t u = f(u, |v|, arg(v))$$

$$\partial_t v = g(u, v)$$

with f and g regular.
On some geometrical hypothesis on $\Sigma_{j,k}$, existence of a regular flot for the limit equation.

Expected result: Under geometrical hypothesis on $\Sigma_{j,k}$, existence of a regular map for limit equation.

Low Mach number limit for non-isentropic flows

Resonances

$$\Sigma_{j,k,l} = \{S \quad | \quad \lambda_j(S) + \lambda_k(S) = \lambda_l(S)\}.$$

- Heuristically $\Sigma_{j,k,l}$ is of codimension 1.
- Codimension 1 in the measure sense

$$\mu\{S \quad | \quad |\lambda_j(S) + \lambda_k(S) - \lambda_l(S)| < \varepsilon\} \leq C\varepsilon.$$

More precisely
Theorem. Under non degeneracy hypothesis,

$$\mu_N\left(\Sigma_{j,k,l}^{N,\varepsilon}\right) \leq C\varepsilon.$$

Low Mach number limit for non-isentropic flows

Proof of resonance theorem

- Differential calculus

$$d(\lambda_j + \lambda_k - \lambda_l) = \left(|\nabla\psi_j|^2 + |\nabla\psi_k|^2 - |\nabla\psi_l|^2\right)$$

- The differential does not vanished if

$$|\nabla\psi_j|^2 + |\nabla\psi_k|^2 - |\nabla\psi_l|^2 \neq 0.$$

- The differential belongs to all H^s: eigenvalues vary slowly when we perturbate high frequencies.
- Differential depends essentially of the first N components...
- $\Sigma_{j,k,l}$ is a graph with respect to its first N components is N is large enough.

Low Mach number limit for non-isentropic flows

In progress: non-homogeneous incompressible limit

- First step: Check that the limit system has a solution for almost all initial data.
- Check that almost all initial data avoids $\Sigma_{j,k}$.
- Conjugate nonhomogeneous incompressible NS equation with $\mathcal{L}$.
- Pass to the limit
- Pass to the limit in the resonances.

Objective: convergence for almost all initial data convergence....

Massless limit

In progress: E. Grenier, Y. Guo, B. Pausader

Work in progress where all the steps are possible to check. The solution avoids $\Sigma_{j,k}$ and cross $\Sigma_{j,k,l}$ transversally.

Anelastic limit for Euler equations

Paper: D.B., G. MÉTIVIER. Anelastic limit for Euler type systems. *AMRX* (2010).

The goal is to find an example where we get a strong coupling between mean flows and waves at the limit from an energetical point of view.
Let us consider the two following systems

$$\partial_t h + \mathrm{div}(hv) = 0$$

$$\partial_t(hv) + \mathrm{div}(hv \otimes v) + h\frac{\nabla(h-b)}{\varepsilon^2} = 0$$

and

$$\partial_t \rho + \mathrm{div}(\rho v) = 0$$

$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \frac{\nabla(c(x)\rho^\gamma)}{\varepsilon^2} = 0.$$

Two usual questions:

- Can we solve the equations on some time interval which is independent of ε?
- Can we characterize the limit when ε goes to zero?

Anelastic limit for Euler equations

First system: Defining

$$\psi = \frac{h-b}{\varepsilon}, \qquad q = \frac{1}{\varepsilon}\ln(1+\varepsilon\psi/b).$$

The system may be written under the form

$$b(\partial_t q + v\cdot\nabla q) + \frac{\mathrm{div}(bv)}{\varepsilon} = 0.$$

$$\partial_t v + v\cdot\nabla v + \frac{\nabla\psi}{\varepsilon} = 0.$$

Anelastic limit for Euler equations

Second system: Defining

$$\psi = \frac{\gamma}{\gamma - 1} \frac{'c^{1/\gamma}\rho)^{\gamma-1} - 1}{\varepsilon}, \qquad q = \frac{1}{\varepsilon(\gamma-1)} \ln(1 + \varepsilon(\gamma-1)\psi/\gamma).$$

The system may be written under the form

$$c^{-1/\gamma}(\partial_t q + v \cdot \nabla q) + \frac{\mathrm{div}(c^{-1/\gamma} v)}{\varepsilon} = 0,$$

$$c^{-1/\gamma}(\partial_t v + v \cdot \nabla v) + \frac{\nabla \psi}{\varepsilon} = 0.$$

Anelastic limit for viscous equations / weak solutions

To the author's knowledge, from a mathematical point of view:

First answer: (degenerate viscosity – "modulated energy")
D.B., M. GISCLON, C.K. LIN. An example of low mach (Froude) number limit for compressible flows with nonconstant density (height) limit. *M2AN*, 39, 477-486 (2005).
Second answer: (constant viscosities)
N. MASMOUDI. Rigorous derivation of the anelastic approximation.
J. Math. Pures et Appl. 230-240, (2007).
Third answer: (constant viscosities)
E. FEIREISL, J. MALEK, A. NOVOTNY, I. STRASKRABA. Anelastic approximation as a singular limit of the compressible Navier-Stokes system. *Comm. Partial Diff. Equations*, 157–176, 33, 1 (2008).

All concern : global weak solutions and systems with viscosities.

Anelastic limit for Euler equations

What about strong solution?

All these systems may be written under the form

$$a(\partial_t q + v \cdot \nabla q) + \frac{1}{\varepsilon}\mathrm{div} u = 0$$

$$b(\partial_t m + v \cdot \nabla m) + \frac{1}{\varepsilon}\nabla \psi = 0$$

with $a(t,x)$, $b(t,x)$ known and positive, and

$$q = \frac{1}{\varepsilon}Q(t,x,\varepsilon\psi), \qquad m = \mu(t,x)u, \qquad v = V(t,x,u,q)$$

where Q, μ and V are smooth, $Q(t,x,0) = 0$, $\partial_\theta Q > 0$ and $\mu > 0$.

Anelastic limit for Euler equations

Existence of solutions on a time interval independent on ε ?

Uniform bounds on $\|(u,\psi)\|_{H^s}, \qquad s > d/2+1$?

Main idea by G. Métivier and S. Schochet + T. Alazard.

Use estimate on $(\varepsilon\partial_t)^k$ derivatives and control of $\mathrm{curl}(bu)$ and $\mathrm{div} u$ (elliptic estimates) to decrease time derivative and increase space derivative.

Anelastic limit for Euler equations

Sketch of proof:
We define $K := \sup_{t\in[0,T]} \|(u,\psi)\|_{H^s}, \qquad s > d/2+1$

First step:
The constant K controls various other derivatives of the unknowns which will be present in the analysis of commutators:

$\widetilde{K} := \sup_{t\in[0,T]} \sum_{k=0}^{s} \|(\varepsilon\partial_t)^k(u,\psi)\|_{H^{s-k}} \le C(K)$.

For all $s \le k$, $\sup_{t\in[0,T]} \sum_{k=0}^{s} \|(\varepsilon\partial_t)^k(q,m,\psi)\|_{H^{s-k}} \le C(\widetilde{K})$.

Ingredient: Use equation directly

$$\varepsilon\partial_t(u,\psi) = \Phi_\varepsilon(t,x,u,\psi)\nabla(u,\psi) + \Psi_\varepsilon(t,x,u,\psi)$$

and induction.

Anelastic limit for Euler equations

Second step: Control on $(u_k, \psi_k) = (\varepsilon\partial_t)^k (u, \psi)$:

bound $(\varepsilon\partial_t)^k (u, \psi)$: linked to linearized system (L^2 estimate):

$\|(\dot u, \dot\psi)\|_{L^2}\| \le C_0(1 + tC(K))\|(\dot u, \dot\psi)(0)\|_{L^2} + c(K) \int_0^t \|(\dot f, \dot g)(t)\|_{L^2}\, dt'$
and $\sup_{t\in[0,T]} \|(f_k, g_k)\|_{L^2} \le C(\widetilde K)$

This implies
$\|(u_k, \psi_k)\|_{L^2} \le C_0 + tC(K)$

Third step: Control on quantities linked to curl and div.

bound $(\varepsilon\partial_t)^\ell ((\partial_t + v\cdot\nabla)\omega)$ with $\omega := \mathrm{curl}(b\mu u)$ and $\ell \le s-1$
and thus $\|(\varepsilon\partial_t)^\ell \omega\|_{H^{s-1-\ell}} \le C_0 + tC(K)$

Anelastic limit for Euler equations

Fourth step:
Prove that

$$\|(u_{s-k},\psi_{s-k})\|_{H^l} \le C_0 + (t+\varepsilon)C(K) + C_1\|(u_{s-k},\psi_{s-k})\|_{H^{l-1}}$$

for $0 \le \ell \le k \le s$.

Ideas: Induction to prove the inequality in the begining of this slide. $k=0$ is the estimate on ε time derivatives. Assume ok for $k-1$. when $l=0$, this is again time derivatives estimates. Use $1 \le l \le k \le s$.
Use equations to get bounds $\|\mathrm{div}(\varepsilon\partial_\mathrm{t})^{\mathrm{s-k}}\mathrm{u}\|_{H^{\ell-1}}$ and $\|\nabla(\varepsilon\partial_t)^{s-k}\psi\|_{H^{\ell-1}}$. More precisely

$$\|\mathrm{div}(\varepsilon\partial_\mathrm{t})^{\mathrm{s-k}}\mathrm{u}\|_{H^{\ell-1}} + \|\nabla(\varepsilon\partial_t)^{s-k}\psi\|_{H^{\ell-1}} \le C_0 + (t+\varepsilon)C(K)$$

using equations and induction hypothesis.

Anelastic limit for Euler equations

Thus elliptic estimate

$$\|u\|_{H^k} \leq C_k\big(\|\mathrm{div} u\|_{H^{k-1}} + \|\mathrm{curl}(b\mu u)\|_{H^{k-1}} + \|u\|_{H^{k-1}}\big)$$

gives the desired inequality and thus

$$\|(u,\psi)\|_{H^s} \leq C_0 + (t + \varepsilon C(K)).$$

$\Longrightarrow$

$$\|(u,\psi)\|_{H^s} \leq 2C_0$$

$\Longrightarrow$ Local existence and uniqueness of local strong solution on time intervall which does not depend on ε.

Domains: R^d, T^d, $T^{d'} \times R^{d-d'}$.

Anelastic limit for Euler equations

Three cases:

- 1) b constant and $V = d\mu u$ with d constant: decoupling between fast and slow scales.
- 2) $\Omega = R^d$ + specific decreasing assumption on coefficients: dispersion of acoustic waves.
- 3) $\Omega = T^d$ + a, b, μ do not depend on time + $V = d(t,x)u$: Energy exchange between fluid and remanent acoustic energy.

Important remark. 3) Suspected for the periodic low Mach limit problem for nonisentropic Euler equations and proved for finite dimensional models by G. MÉTIVIER and S. SCHOCHET. To the author's knowledge, here first example where strong coupling is fully mathematically justified. Only partial answer for non-isentropic Euler equations: see D.B., B. DESJARDINS, E. GRENIER (*Adv. Diff. Eqs*, 2010) $\Longrightarrow$ crossing eigenvalues (co-dimension 2 set) - singular odes homogeneous of degree 0 near a codimension 2 set on toy models.
Difficulty: time dependency.

Anelastic limit for Euler equations

Low Mach number limit in ill prepared case ? Take the curl of momentum equation and write

$$u^\varepsilon = \widetilde{u}^\varepsilon + \frac{1}{b(t,x)\mu(t,x)}\nabla G^\varepsilon$$

with

$$G^\varepsilon = (\Delta_{b\mu})^{-1}\mathrm{div}u^\varepsilon, \qquad \Delta_{b\mu} = \mathrm{div}(\frac{1}{b\mu}\nabla).$$

For some $s' < s$:

$$\widetilde{u}^\varepsilon \to u \text{ in } \mathcal{C}^0([0,T];H^{s'}(\Omega)), \qquad \nabla G^\varepsilon \text{ weakly converges to } 0$$

One has

$$\mathrm{curl}(b(\partial_t + v^\varepsilon)m^\varepsilon) = 0.$$

For G^ε we use the spectral decomposition related to spectral pb

$$-a^{-1}\mathrm{div}(\frac{1}{b\mu}\nabla\Psi_j) = \lambda_j\Psi_j.$$

Anelastic limit for Euler equations

Whole space case: Introduce microlocal defect measures of subsequences of u^ε. Assumptions on coefficients give no measures (look at defect measures supported in characteristic variety of the equation). The kernel is non trivial if and only if τ^2 is an eigenvalue of $1/a\underline{Q_1}div(1/(b\mu)\nabla_x)$.
When coefficients in some classes $\Longrightarrow$ never occurs.

Periodic case:

First system: Waves contribution has the form $b\nabla\pi$.

$$d(\nabla G^\varepsilon)\cdot\nabla(\frac{1}{b}\nabla G^\varepsilon) = \frac{d}{2b}\nabla|\nabla G^\varepsilon|^2.$$

Anelastic limit for Euler equations

Second system: Strong coupling between waves and mean velocity.

$$\partial_t u + u\cdot\nabla(c^{1/\gamma}u) + \nabla\pi + \sum_{j,\ell,\lambda_k+\lambda_j=0} \frac{\alpha_k\alpha_j}{2|\lambda_k|^2}\Big(\nabla\Psi_k\cdot\nabla(c^{1/\gamma}\nabla\Psi_j) + \nabla\Psi_j\cdot\nabla(c^{1\gamma}\nabla\Psi_k)\Big) = 0$$

$$\mathrm{div} u = 0$$

$$\partial_t\alpha_j = \sum_{\lambda_j=\lambda_\ell} \frac{\alpha_j}{2|\lambda_j|^2}\int_{T^d}(u\cdot\nabla(c^{1/\gamma}\nabla\psi_j) + \nabla\psi_j\cdot\nabla\nabla(c^{1/\gamma}u))\cdot\nabla\psi_\ell$$

$$-\sum_{\lambda_j+\lambda_k=\lambda_\ell} \frac{i\alpha_j\alpha_k}{2\sqrt{2}}\frac{1}{\lambda_\ell\lambda_k\lambda_j}\int_{T^d}((\nabla\psi_j\cdot\nabla(c^{1/\gamma}\nabla\psi_k))\cdot\nabla\psi_\ell + (\nabla\psi_k\cdot\nabla(c^{1/\gamma}\nabla\psi_j))\cdot\nabla\psi_\ell\Big).$$

$$-\sum_{\lambda_j+\lambda_k=\lambda_\ell}\Big(\frac{i(\gamma-1)}{\sqrt{2}\gamma^2}\int_{T^d} c^{-1/\gamma}\alpha_j\alpha_k\lambda_\ell\psi_j\psi_k\psi_\ell - \frac{i}{\sqrt{2}\gamma}\int_{T^d}\alpha_j\alpha_k\frac{\lambda_\ell}{\lambda_j\lambda_k}\nabla\psi_j\cdot\nabla\psi_k\,\psi_\ell\Big).$$

$\Longrightarrow$ convergence results with ill prepared data.

Pseudo-incompressible model.

Compressible Euler equations

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + P\nabla\pi = -g\rho k$$

$$\partial_t P + \operatorname{div}(P u) = 0$$

where $P = p^{1/\gamma} = \rho\theta$ the modified thermodynamic pressure variable, θ the potential temperature and $\pi = p^{\Gamma}/\Gamma$ with $\Gamma = (\gamma - 1)/\gamma$ and γ isentropic exponent.

Pseudo-incompressible model.

Assumptions by Ogura and Philipps $\Longrightarrow$
Total variations of potential temperature across troposphere of less than one Kelvin.......

$\Longrightarrow$ various generalizations have been proposed.

The anelastic model proposed by Bannon for instance:

- drop density time derivative,
- density equal to some profile
- modifying th pressure gradient and gravity terms..........

$$\mathrm{div}(\overline{\rho}v) = 0$$

$$\partial_t(\overline{\rho}v) + \mathrm{div}(\overline{\rho}v \otimes v) + \overline{\rho}\nabla\pi' = -g\overline{\rho}\frac{\theta - \overline{\theta}}{\overline{\theta}}k$$

$$\partial_t(\overline{\rho}\theta) + \mathrm{div}(\overline{\rho}\theta v) = 0$$

where $\overline{\rho} = \overline{\rho}(z)$ a prescribed density profile.

$\overline{\rho} \equiv 1 \Longrightarrow$ Boussinesq equations.

Pseudo-incompressible model.

Durran proposed the pseudo-incompressible model in 1989.

$$\partial_t \rho + \mathrm{div}(\rho v) = 0$$
$$\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \overline{P}\nabla\pi = -g\rho k$$
$$\mathrm{div}(\overline{P} v) = 0$$

where $\overline{P} \equiv P(z)$ a prescribed background distribution.

$\overline{P} \equiv 1 \Longrightarrow$ zero Mach, variable density incompressible flow equations.

Pseudo-incompressible model.

A comparison of all these models (+full compressible Euler):
U. Achatz, D.B., R. Klein, O.M. Knio and P.K. Smolarkiewicz, regime of validity of sound-proof atmospheric flow models. *J. Atm. Science (AMS)*, 2010.

The first step is to analyze the internal waves vertical eigenmodes for the three flow models.

$\Longrightarrow$ Perturbed Sturm Liouville type operator (spectral perturbations).

Pseudo-incompressible model.

$$\theta = \overline{\theta}^{\varepsilon} + \varepsilon^{\nu+\mu}\widetilde{\theta}$$
$$\pi = \overline{\pi}^{\varepsilon} + \varepsilon\widetilde{\pi}$$
$$v = \varepsilon\widetilde{v}$$

with

$$\overline{\theta}^{\varepsilon} = 1 + \varepsilon^{\mu}\overline{\Theta}(z)$$
$$\frac{d\overline{\pi}^{\varepsilon}}{dz} = -\frac{1}{\overline{\theta}^{\varepsilon}}.$$

$\mu = 2, \nu = 0$: Gravity at main order.

$\mu = 0, \nu = 1$: acoustic waves and internal waves coincide.

The goal : $0 \le \mu \le 2, \quad 0 \le \nu \le 1.$

Pseudo-incompressible model.

The compressible Euler equations:

$$\partial_t \widetilde{\theta} + \widetilde{v} \cdot \nabla \widetilde{\theta} + \frac{1}{\varepsilon^\nu} \widetilde{w} \frac{d\overline{\theta}}{dz} = 0$$

$$\partial_t \widetilde{v} + \widetilde{v} \cdot \nabla \widetilde{v} + \frac{1}{\varepsilon^\nu} \frac{\widetilde{\theta}}{\overline{\theta}^\varepsilon} k + \frac{1}{\varepsilon} \overline{\theta}^\varepsilon \nabla \widetilde{\pi} = -\varepsilon^{1-\nu} \widetilde{\theta} \nabla \widetilde{\pi}$$

$$\partial_t \widetilde{\pi} + \widetilde{v} \cdot \nabla \widetilde{\pi} + \gamma \Gamma \widetilde{\pi} \mathrm{div} \widetilde{v} + \frac{1}{\varepsilon} \big(\gamma \Gamma \overline{\pi}^\varepsilon \mathrm{div} \widetilde{v} + \widetilde{w} \frac{d\overline{\pi}^\varepsilon}{dz} \big) = 0$$

Pseudo-incompressible model.

The pseudo-incompressible model:

$$\partial_t\widetilde{\theta} + \widetilde{v}\cdot\nabla\widetilde{\theta} + \frac{1}{\varepsilon^\nu}\widetilde{w}\frac{d\overline{\theta}}{dz} = 0$$

$$\partial_t\widetilde{v} + \widetilde{v}\cdot\nabla\widetilde{v} + \frac{1}{\varepsilon^\nu}\frac{\widetilde{\theta}}{\overline{\theta}^\varepsilon}k + \frac{1}{\varepsilon}(1+\varepsilon^\mu\overline{\Theta})\nabla\widetilde{\pi} = -\varepsilon^{1-\nu}\widetilde{\theta}\nabla\widetilde{\pi}$$

$$\left(\gamma\Gamma\overline{\pi}^\varepsilon \mathrm{div}\widetilde{v} + \widetilde{w}\frac{d\overline{\pi}^\varepsilon}{dz}\right) = 0$$

Pseudo-incompressible model.

The anelastic model:

$$\partial_t\widetilde{\theta} + \widetilde{v}\cdot\nabla\widetilde{\theta} + \frac{1}{\varepsilon^\nu}\widetilde{w}\frac{d\overline{\theta}}{dz} = 0$$

$$\partial_t\widetilde{v} + \widetilde{v}\cdot\nabla\widetilde{v} + \frac{1}{\varepsilon^\nu}\frac{\widetilde{\theta}}{\overline{\theta}^\varepsilon}k + \frac{1}{\varepsilon}\nabla\widetilde{\pi} = -\varepsilon^{1-\nu}\widetilde{\theta}\nabla\widetilde{\pi}$$

$$\gamma\Gamma\varepsilon^\mu\widetilde{w}\frac{d\overline{\Theta}}{dz} + \left(\gamma\Gamma\overline{\pi}^\varepsilon\mathrm{div}\widetilde{v} + \widetilde{w}\frac{d\overline{\pi}^\varepsilon}{dz}\right) = 0$$

Pseudo-incompressible model.

Internal wave time $\xi = \varepsilon^{-\nu} t$.

$$\partial_\xi \widetilde{v} + \frac{\widetilde{\theta}}{\overline{\theta}^\varepsilon} k + (1 + a\varepsilon^\mu \overline{\Theta}) \nabla \widetilde{\Pi} = 0$$

$$\partial_\xi \widetilde{\theta} + \widetilde{w} \frac{d\overline{\theta}}{dz} = 0$$

$$b\varepsilon^\mu \partial_\xi \widetilde{\Pi} + c\gamma \Gamma \varepsilon^\mu \widetilde{w} \frac{d\overline{\Theta}}{dz} + \left(\gamma \Gamma \overline{\pi}^\varepsilon \mathrm{div} \widetilde{v} + \widetilde{w} \frac{d\overline{\pi}^\varepsilon}{dz}\right) = 0$$

$a = 1, b = 1 \implies$ compressible

$a = 1, b = 0, c = 0 \implies$ pseudo-incompressible

$a = 0, b = 0, c = 1 \implies$ anelastic.

Pseudo-incompressible model.

More or less obvious that solutions will differ only by $O(\varepsilon^\mu)$ at least for times $\xi = O(1)$.
$\Longrightarrow$ solutions starting from initial data differing at most $O(\varepsilon^\mu)$ would remain close for time $\xi = O(1)$.
Flow evolutions over advective time scales $t = \varepsilon^\nu \xi$?
Over such longer time scales, differences in the internal wave eigenfrequencies of order $O(\varepsilon^\mu)$ will accumulate to phase shifs of the order $\varepsilon^\mu \xi = O(t\varepsilon^{\mu-\nu})$
As a consequence, if solutions of the three models shall remain asymptotically close over advective time scales, we must require

$$\varepsilon^{\mu-\nu} = \varepsilon^{3\mu/3-1} = o(1) \qquad \text{as } \varepsilon \to 0.$$

$\Longrightarrow \mu > 3/2.$
Conclusion: For any $d\theta/dz = O(\varepsilon^{2/3})$, OK.
Better than Ogura Philipps........ $\Delta\theta|_0^{h_{sc}} \approx 30K$ instead of $0.33K$.

Pseudo-incompressible model.

We seek at horizontally travelling waves described by

$$(\widetilde{\theta}, \widetilde{u}, \widetilde{w}, \Pi) = (\Theta^\star, U^\star, W^\star, \Pi^\star)(z)\exp(i(\omega\xi - \lambda\cdot x)).$$

This leads to a perturbed Sturm-Liouville type second order differential equation for the vertical velocity structure function $W^\star(z)$:

$$-\frac{d}{dz}\Big(\frac{1}{1-b\varepsilon^\mu \frac{\omega^2/\lambda^2}{(\overline{c}^\varepsilon)^2}}\frac{1}{(1+a\varepsilon^\mu\overline{\Theta})\overline{P}^\varepsilon}\frac{dW^\star}{dz}\Big)+\frac{\lambda^2}{(1+a\varepsilon^\mu\overline{\Theta})\overline{P}^\varepsilon}W^\star = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{(1+a\varepsilon^\mu\overline{\theta})\overline{P}^\varepsilon}W^\star$$

where

$$\overline{c}^\varepsilon = \sqrt{\gamma\overline{P}^\varepsilon/\overline{\rho}^\varepsilon}, \qquad N^2(z) = (d\overline{\Theta}/dz)/\overline{\theta}^\varepsilon$$

and $W^\star(0) = W^\star(H) = 0$.

Pseudo-incompressible model.

$$\widetilde{\theta}_\tau + \widetilde{w}\frac{d\overline{\Theta}}{dz} = 0$$

$$\widetilde{v}_\tau - \frac{\widetilde{\theta}}{\overline{\theta}^\varepsilon}k + \overline{\theta}^\varepsilon \nabla\pi^* = 0$$

$$A\varepsilon^\mu \pi^*_\tau + \gamma\Gamma\overline{\pi}^\varepsilon \mathrm{div}\widetilde{v} + \widetilde{w}\frac{d\pi^\varepsilon}{dz} = 0$$

The system admits the following vertical mode decomposition

$$\begin{pmatrix} \widetilde{\theta} \\ \widetilde{u} \\ \widetilde{w} \\ \widetilde{\pi} \end{pmatrix}(\tau, x, z) = \begin{pmatrix} \hat{\Theta} \\ \hat{U} \\ \hat{W} \\ \hat{\Pi} \end{pmatrix}(z)\ \exp\left(i(\omega\tau - \lambda\cdot x)\right)\ , \tag{1}$$

Pseudo-incompressible model.

This gives a Sturm-Liouville system to solve

$$-\frac{d}{dz}\left(\frac{1}{1-A\varepsilon^{\mu}/\Lambda^{\lambda}\overline{c}^{\varepsilon\,2}}\,\phi\,\frac{dW}{dz}\right)+\lambda^2\,\phi\,W \quad = \quad \left(\Lambda^{\lambda}\right)^2\,\phi N^2\,W \quad , \tag{2}$$

where

$$N^2=\frac{1}{\overline{\theta}^{\varepsilon}}\frac{d\overline{\Theta}}{dz}\,, \quad \overline{P}^{\varepsilon}=(\overline{\pi}^{\varepsilon})^{\frac{1}{\gamma\Gamma}}\,, \quad \overline{c}^{\varepsilon\,2}=\gamma\Gamma\overline{\pi}^{\varepsilon}\overline{\theta}^{\varepsilon}\,,$$

$$W=\overline{P}^{\varepsilon}\,\hat{W}\,, \quad \Lambda^{\lambda}=\sqrt{\frac{\lambda^2}{\omega^2}}\,, \quad \phi=\frac{1}{\overline{P}^{\varepsilon}\overline{\theta}^{\varepsilon}}$$

Use properties on the eigenfunctions linked to the vertical velocity to derive vertical derivative control for the unknowns.

Pseudo-incompressible model.

Case $A = 0$: Sturm Liouville theory guarantees that, for each λ^2, there exists a weighted-norm Hilbert space of function of z for which the eigenfunctions $W^\lambda(z)$ form a countable, orthogonal basis under the ϕN^2 weighted L^2 scalar product.

$$\langle W_k^\lambda, W_l^\lambda \rangle_{L^2,\phi N^2} = \int_0^H W_k^\lambda W_l^\lambda \, \phi N^2 \, dz = \delta_{kl} \,. \tag{3}$$

At the same time, the eigenfunctions are also orthogonal under the ϕ–weighted H^1 scalar product,

$$\langle W_k^\lambda, W_l^\lambda \rangle_{H^1,\phi} = \int_0^H \left(\frac{dW_k^\lambda}{dz} \frac{dW_l^\lambda}{dz} + \lambda^2 W_k^\lambda W_l^\lambda \right) \phi \, dz = \left(\Lambda_k^\lambda \right)^2 \delta_{kl} \,. \tag{4}$$

The eigenvalues are simple, and form a positive, nondecreasing, unbounded sequence, $0 \leq \Lambda_0^\lambda \leq \Lambda_1^\lambda \ldots$. Moreover there is an asymptotic scaling such that $\Lambda_k^\lambda = c_1 k + c_2(\lambda) k^{-2} + o(k^{-2})$ for $k \gg 1$.
When $A \neq 0$: Sturm Liouville nontrivially perturbed.

Pseudo-incompressible model.

Need to study the large modenumber asymptotic structure of the eigenfunction $W_k^i(z)$...

Work in progress:
D. Bresch, R. Klein: Towards a rigorous justication of Durrans pseudo-incompressible model for atmospheric ows.

Highly oscillating bathymetry

Let the lake equation be:

$$\mathrm{div}(bu) = 0$$

$$\partial_t u + \varepsilon u \cdot \nabla u + \nabla p = 0$$

asking what happens for $b = b(x, x/\varepsilon)$, $b \in \mathcal{C}^0(\Omega \times T^2)$.

See: D.B., D. Gérard-Varet. Homogeneization problems from shallow water theory. *Appl. Math. Letters.* 505–510, (2007).

Use two-scale cv + Meyer elliptic estimates + defect measure characterization.

Highly oscillating bathymetry

Two scale convergence:

Let Ω a bounded domain of $\mathbb{R}^2$. Les $v^\varepsilon = v^\varepsilon(t,x)$ a sequence of functions in $L^p(\mathbb{R}_+; L^q(\Omega))$ with $1 < p, q \leq +\infty$, $(p,q) \neq (+\infty, +\infty)$ (respectively in $L^\infty(\mathbb{R}_+ \times \Omega))$. Let $v \in L^p(\mathbb{R}_+; L^q(\Omega \times T^2))$ (respectively in $L^\infty(\mathbb{R}_+ \times \Omega \times T^2)$. We say that v^ε two-scale converges to v if, for all $w \in L^{p'}(\mathbb{R}_+; \mathcal{C}^0(\overline{\Omega} \times T^2))$,

$$\lim_{\varepsilon} \int_{\mathbb{R}_+} \int_\Omega v^\varepsilon(t,x) w(t,x,x/\varepsilon)\, dxdt = \int_{\mathbb{R}_+} \int_{\Omega \times T^2} v(t,x,y) w(t,x,y)\, dx\, dy\, dt..$$

Highly oscillating bathymetry

The result. Let $q_0 > 2$, v_0^ε bounded in $L^{q_0}(\Omega)$ satisfying $\mathrm{div} v_0^\varepsilon = 0$, $v_0^\varepsilon \cdot n|_{\partial\Omega} = 0$ and $\varepsilon \mathrm{curl} v_0^\varepsilon$ bounded in $L^\infty(\Omega)$. Assume moreover that v_0^ε two scale converges to v_0 on Ω with

$$\|v_0^\varepsilon\|_{L^2(\Omega)} \to \|v_0\|_{L^2(\Omega\times T^2)}.$$

Then , up to extract a subsequence, the solution v^ε two scale converges to a solution v^0 solution of

$$\begin{aligned}
&\partial_t v^0 + v^0 \cdot \nabla_y v^0 + \nabla_x p^0 + \nabla_y p^1 \\
&\mathrm{div}_y(b v^0) = 0, \qquad \mathrm{div}_x(\overline{b v^0}) = 0 \\
&v^0 \cdot n|_{\partial\Omega} = 0, \qquad v^0|_{t=0} = v_0.
\end{aligned}$$

Highly oscillating bathymetry

Meyers estimate: Let Ω a bounded domain of $\mathbb{R}^2$, $a \in L^\infty(\Omega)$ and $f \in L^{q_0}(\Omega)$ for some $q_0 > 2$, $\int_{\partial\Omega} f \cdot n = 0$. Let $\phi \in H^1(\Omega)$ the solution of

$$\mathrm{div}(a\nabla\phi) = \mathrm{div} f \text{ in } \Omega, \qquad \int_\Omega \phi = 0, \qquad \partial_n \phi|_{\partial\Omega} = 0.$$

There exists $2 < q_m = q_m(\|a\|_{L^\infty}, \Omega) \leq q_0$ such that for all $2 \leq q < q_m$, $\phi \in W^{1,q}(\Omega)$ with

$$\|\phi\|_{W^{1,q}} \leq C\|f\|_{L^q}, \qquad C = C(q, \|a\|_{L^\infty}, \Omega).$$

Highly oscillating bathymetry

Decompose

$$v^\varepsilon = Pv^\varepsilon + \nabla\phi^\varepsilon, \qquad \int_\Omega \phi^\varepsilon = 0.$$

with $\mathrm{div} Pv^\varepsilon = 0 \implies \mathrm{div}(b\nabla\phi^\varepsilon) = -\mathrm{div}(bPv^\varepsilon)$.
Use

$$\partial_t v + \varepsilon(\mathrm{curl} v)v^\perp + \nabla\Big(p^\varepsilon + \frac{\varepsilon|v|^2}{2}\Big) = 0$$

gives bounds

$$v^\varepsilon \in L^\infty(R_+; L^q(\Omega))$$

and thus

$$\partial_t v^\varepsilon, \nabla(p^\varepsilon + \varepsilon|v^\varepsilon|^2/2) \in L^\infty(R_+; L^q(\Omega)).$$

Highly oscillating bathymetry

Use now the two formulations

$$\partial_t(bv) + \mathrm{div}(bv \otimes v) + b\nabla p = 0$$

and

$$\partial_t v + \varepsilon(\mathrm{curl} v)v^{\perp} + \nabla\Big(p^{\varepsilon} + \frac{\varepsilon|v|^2}{2}\Big) = 0$$

with strong compactness of $(v^{\varepsilon})_{\varepsilon}$ in $L^{\infty}(W^{-1,q'}(\Omega))$.
Inroducing the two-scale defect measures α and β such that $|v^{\varepsilon}|^2$, $v^{\varepsilon} \otimes v^{\varepsilon}$ two scale convergence to $|v^0|^2 + \alpha$, $v^0 \otimes v^0 + \beta$.
Using that $\mathrm{curl}_y v^0$ is bounded, we have for almost every x

$$\|\nabla_y v^0(x,\cdot)\|_{L^{\infty}_t(L^p_y)} \leq N(x)p, \quad N(x) < +\infty.$$

We get an ode on $\gamma = \overline{b\alpha}$, namely

$$\partial_t\gamma(t,x) < C(x)p\gamma(t,x)^{1-1/p}.$$

We conclude assuming now defect measure initially.

Highly oscillating bathymetry

What about the limit

$$\partial_t h + \mathrm{div}(hu) = 0$$

$$\partial_t u + \varepsilon u \cdot \nabla u + \frac{\nabla(h-b)}{\varepsilon} = 0$$

with $b = b(x, x/\varepsilon)$.

Use of D.B., G. Métivier + paper by S. Schochet in M2AN.