Optimal Discretization in Banach Spaces

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Abstract

In the setting of Banach spaces, we consider the abstract problem

$$\begin{cases} \text{Find } u \in \mathbb{U} \text{ such that} \\ Bu = f \text{ in } \mathbb{V}^*, \end{cases}$$

where $\mathbb{U}$ and $\mathbb{V}$ are Banach spaces, $B : \mathbb{U} \to \mathbb{V}^*$ is a continuous, bounded-below, linear operator, and the data $f \in \mathbb{V}^*$ is a given element in the dual space of $\mathbb{V}$. For a given discrete subspace $\mathbb{U}_n \subset \mathbb{U}$, of dimension n, the objective of this talk is to present a Galerkin-based discretization technique which is guaranteed to provide a near-best approximation $u_n \in \mathbb{U}_n$ to the solution u, i.e., u_n satisfies the a priori error estimate

$$\|u - u_n\|_{\mathbb{U}} \leq C \inf_{w_n \in \mathbb{U}_n} \|u - w_n\|_{\mathbb{U}},$$

for some constant $C \geq 1$, independent of n. In this spirit, we initially propose a discretization method to achieve

$$u_n = \arg\min_{w_n \in U_n} \|f - Bw_n\|_{\mathbb{U}}.$$

The method relies in the duality map $J_{\mathbb{V}} : \mathbb{V} \to \mathbb{V}^*$, which extend to Banach spaces the concept of the well-known Riesz map of Hilbert spaces. However, in a non-Hilbert setting, the duality map is nonlinear. To make the method feasible, a discretization of the test space is needed. Hence, by considering a finite-dimensional subspace $\mathbb{V}_m \subset \mathbb{V}$, we end up with a discretization method that achieves

$$u_n = \arg\min_{w_n \in \mathbb{U}_n} \|f - Bw_n\|_{\mathbb{V}_m^*}.$$

We show the well-posedness of these methods, together we error estimates, and some basic numerical experiments in 1D.

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