Very weak solutions and regularity for the Navier-Stokes equations with Dirichlet boundary conditions or with non standard boundary conditions

Chérif Amrouche (Pau), Joint work with M.A Rodriguez Bellido (Sevilla) and N.E.H. Seloula (Bordeaux 1)

Part I. The concept of very weak solution introduced by Giga [6] for the stationary Stokes equations (and also by Lions-Magenes [7] for the Laplace's equation, in a domain Ω of class $\mathcal{C}^\infty$) has been intensively studied in the last years for the stationary Navier-Stokes equations with Dirichlet boundary conditions. We give here a new and simpler proof of the existence of very weak solutions $(\boldsymbol{u}, \pi) \in \mathbf{L}^p(\Omega) \times W^{-1,p}(\Omega)$, with $1 < p < \infty$, for the stationary Navier-Stokes equations, based on density arguments and an adequate functional framework in order to define more rigourously the traces of non regular vector fields. We also obtain regularity results in fractional Sobolev spaces. All these results are obtained in the case of a bounded open set, connected of class $\mathcal{C}^{1,1}$ of $\mathbb{R}^3$ and can be extended to the Laplace's equation and to other dimensions.

Part II. We consider in this part the Stokes and Navier-Stokes problems in a bounded domain, eventually multiply connected, whose boundary consists of multi-connected components. We investigate the solvability in L^p-theory, with $1 < p < \infty$, under the non standard boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = g, \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \quad \text{or} \quad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g}, \quad \pi = \pi_* \quad \text{on}\, \Gamma. \tag{1}$$

The main ingredients for this solvability are given by the Inf-Sup conditions, some Sobolev's inequalities for vector fields and the theory of vector potentials satisfying

$$\boldsymbol{\psi} \cdot \boldsymbol{n} = 0 \quad \text{or} \quad \boldsymbol{\psi} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on}\, \Gamma. \tag{2}$$

Those inequalities play a fundamental key and are obtained thanks to Calderon-Zygmund inequalities and integral representations. In the study of ellpitical problems, we consider both generalized solutions and strong solutions that very weak solutions. Finally, this work is an extension of [1] where the authors give the hilbertian theory for vector potentials.

References

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