

# Selectivity, pulse fishing and endogenous lifespan in Beverton Holt models

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## ABSTRACT

Optimal management in multi-cohort Beverton-Holt model with any number of age classes is equivalent to find the optimal fish lifespan by chosen fallow cycles. Optimal policy has two main differences respect the optimal lifespan rule with perfect selectivity. First, the value of weight gain, is valued in terms of the whole population structure. Second, the cost for waiting is the interest rate adjusted for the raise of the pulse length. This point is specially relevant for assessing the role of the selectivity. With imperfect selectivity, a reduction of the interest rate (i.e. in increase of the discount factor) reduce the optimal lifespan and the optimal pulse length. We illustrate our theoretical findings with an numerical example. Result obtained using global numerical methods select the optimal pulse length predicted by the optimal lifespan rule.

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**Keywords:** optimization in age-structured models, pulse fishing.

# 1 Introduction

Clark *et al.* (1973) and Hannesson's pioneer article (1975) show that optimal fishing mortality that maximizes net present profits in Beverton-Holt multi-cohort models may be embodied in pulse fishing. That is, the optimal solution of the management problem is a periodic cycle of fishing followed by fallow periods for recovering the stock.

After Hannesson and Clark *et al.*, many other papers have found, by using numerical methods, that pulse fishing is the optimal policy in some fisheries modeled with the Beverton-Holt model (Horwood, 1987; Bjørndal and Brasão, 2006; Bjørndal *et al.* 2004a, 2004b; Stage, 2006). Finally, Tahvonen (2009), analytically proved that in a 2-age structured fisheries model, optimal harvesting consists of pulse fishing under specific conditions such as nonselective gear.

In this article we extend Tahvonen (2009) analysis to the Clark and Hannesson's multi-cohort Beverton-Holt model. We show that with more than two age classes, optimal management is equivalent to finding the optimal fish lifespan by chosen fallow cycles. That is, optimal management in an age structured model with nonselective gear is closely related with the pioneer Clark and Hannesson's analysis of the optimal harvesting of a single year class fishery.

Suppose, like Hannesson (1975), that one fish, with a lifespan equal to  $A$ , is "*left impounded in a bay in order to be raised to an optimal size and age at which it will be fished. No more fish enter the 'pound' until those now present have been fished*". Assume that it is optimal to wait until the fish reaches its maximum age,  $A$ . This is so whenever there is perfect selectivity gear that allows us to discriminate perfectly which fishes are harvested and which ones are not. At each period, we have  $A$  cohorts of different ages and we only fish the individuals of the maximum age, (see Skonhøj *et al.*, 2011).

However, this reasoning is not valid when there is nonselective gear. In this case, the size and age at which the fish should optimally be harvested cannot be ex-ante determined. Harvesting the maximum number of fishes of age  $A$ , the optimal rule under perfect selectivity, implies to close the fishery for  $A - 1$  periods and as a collateral result we also harvest fishes of age  $A - 1$ ,  $A - 2$ , .... Therefore, in this context of imperfect selectivity it may be optimal not to wait until its maximum age  $A$  is reached.

We show that under imperfect selectivity, when optimal age can not be chosen, optimal

management is equivalent to chooses the pulse length. Therefore to calculate the optimal fish lifespan we have to compare the value of the harvesting under different pulses. This introduces two main differences respect the optimal age rule with perfect selectivity. First, with imperfect selectivity, the value of weight gain, is valued in terms of the whole population structure while with perfect selectivity it is valued in terms of the population of the previous age. Second, the cost for waiting is the interest rate adjusted for the raise of the pulse length. This point is specially relevant for assessing the role of the selectivity. With perfect selectivity, a reduction of the interest rate (i.e. in increase of the discount factor) yields to a raise in the optimal live span of the resource because is more profitable to wait more for harvesting. However, when the selectivity is imperfect, variations of the discount factor also affect positively the cost of rotation. We prove that this second effect dominate the first one and this implies that a reduction of the interest rate reduce the optimal lifespan and the optimal pulse length.

We also show that there exists a relationship between non concavity of the objective function and the non continuity of the optimal solution. In particular, by focusing on the second order conditions, we are able to find sufficient conditions that guarantee that pulse fishing is the optimal solution of problem.

Finally, we illustrate our theoretical findings with an numerical example. We apply our theoretical algorithm to Northern Stock of Hake (NSH) (*Merluccius merluccius*) and we compare this result with the result obtained using global numerical methods. Both select the same optimal pulse length.

The paper is organized in the following manner. We start out by solving a very simple 2-age class model in section 2. In section 3 we extend the results to any number of age classes to show than lifespan is endogenously determined. Section 4 shows a numerical application of our findings to the European Northern Hake Stock. Finally we conclude.

## 2 The simplest fishery with two age classes

Consider the Hannesson's 1975 fishery model with two age classes, juveniles and adults. Let  $N_t^1$ , and  $N_t^2$  be the population of juveniles and adults in period  $t$ , respectively. Population dynamic is very simple. Each year,  $t$ , an exogenous number of juvenile fishes born; without loss of generality we consider that  $N_t^1 = 1$ . Only a part of the juveniles become adults next

Table 1: Dynamic population in a 2 ages class model

	Period t-1	Period t	Period t+1
age 1	$N_{t-1}^1 = 1$	$N_t^1 = 1$	$N_{t+1}^1 = 1$
age 2		$N_t^2 = e^{-(p^1 F_{t-1} + m)}$	$N_{t+1}^2 = e^{-(p^1 F_t + m)}$

period; formally

$$N_t^2 = e^{-p^1 F_{t-1} - m},$$

where  $m$  is the natural mortality and  $p^1$  is the selectivity parameter that indicates how the fishing mortality rate  $F$  affects to juveniles. Figure 1 illustrates the dynamic evolution of this fishery population. Yield in value for year  $t$ , is given by Baranov's equation (1918)

$$Y_t^a = y_t^a(F_t)N_t^a,$$

where,

$$y_t^a(F_t) = pr^a \omega^a \frac{p^a F_t}{m + p^a F_t} [1 - e^{-(p^a F_t + m)}],$$

is the yield in value per unit of fish and  $pr^a$  and  $\omega^a$  are the price and weight of the  $a$ -age class, respectively. Therefore, the net present value of the fishery's yield is equal to

$$\sum_{t=0}^{\infty} \beta^t [y_t^1(F_t) + y_t^2(F_t)N_t^2(F_{t-1})],$$

where  $0 < \beta < 1$  is the discount factor.

First, consider as in Skonhoft *et al.* (2011) that there exist perfect selectivity and  $p^1$ ,  $p^2$  and  $F_t$  can be selected by the regulator. If the value of adults,  $pr^2 \omega^2$ , is greater than the value of juveniles  $pr^1 \omega^1$ , the optimal policy consists of letting all juveniles become adults and catch only adults. Formally we set  $p^2 = 1$ ,  $p^1 = 0$  and  $F_t = F_{t+1} = \infty$ . Therefore, each year we harvest  $e^{-m}$  adults, and the net present values is equal to

$$V^{PS} = \frac{1}{1 - \beta} pr^2 \omega^2 e^{-m},$$

where superscript  $PS$  stands for perfect selectivity.

Now, consider that there is imperfect selectivity. That is  $p^1, p^2$  are given by the fishing technology. This means that is not technically possible to fish adults without fishing juveniles. Moreover, like Tahvonen (2009), assume that there exists “growth overfishing”, that is  $pr^1 = 0$ . The optimal policy is the solution of the problem

$$\begin{aligned} \max_{\{F_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t y_t^2(F_t) N_t^2(F_{t-1}), \\ \text{s.t.} \quad & 0 \leq F_t. \end{aligned} \tag{1}$$

The first order condition of this maximization problem is given by

$$\frac{\partial y_t^2(F_t)}{\partial F_t} N_t^2(F_{t-1}) + \beta y_{t+1}^2(F_{t+1}) \frac{\partial N_{t+1}^2(F_t)}{\partial F_t} + \mu_t = 0, \tag{2}$$

where  $\mu_t \geq 0$  is the Lagrange multiplier associated with the inequality restriction  $F_t \geq 0$ .

As in the perfect selectivity context, we may think that an optimal solution to (1) will consist of harvesting the maximum number of fishes of highest value. However, the imperfectness of the selectivity does not allow to fish continuously adults without fishing juveniles. Nevertheless this aim can be indirectly achieved if fishing is carried out in a cyclical manner. In this 2-ages class model, the natural periodic solution consists on one fallow cycle. First, the fishery is lied fallow for one year -by closing the fishery- and, second, in the next year the whole population is harvested. Formally we are describing a pulse fishing characterized by  $F_t = F_{t+2} = \dots = 0$  and  $F_{t+1} = F_{t+3} = \dots = \infty$ . The following proposition proves that this periodic pulse is a solution that satisfies the first order condition (2). Superscript <sup>IS</sup> stands for imperfect selectivity.

**Proposition 1.** *In a 2-ages class model, a periodic pulse with length equal to two periods such that  $F_t = F_{t+2} = \dots = 0$  and  $F_{t+1} = F_{t+3} = \dots = \infty$ , is a local optimum for problem (1). Moreover, its net present value is equal to*

$$V_{pulse}^{IS} = \frac{\beta}{1 - \beta^2} pr^2 \omega^2 e^{-m}. \tag{3}$$

**Proof:** See Appendix.

Comparing  $V^{PS}$  and  $V_{pulse}^{IS}$ , it is clear that the net present value of the  $IS$  solution is lower than the one from the  $PS$  solution. The intuition is straightforward, with perfect selectivity we are able to harvest continuously adults without fishing juveniles. However with imperfect selectivity, the impossibility of keeping the population structure of the perfect selectivity makes necessary the closure of the fishery for one year for maximizing the next period stock. This closure implies a lower value for the fishery.

Is the periodic fishing (the one fallow cycle) the optimal harvesting rule? Tahvonen (2009) shows in a similar context that the maximization problem (1) also has a stationary solution. Let  $F_{ss}$  be the stationary solution such that  $F_t = F_{t+1} = F_{ss}$  associated to  $\mu_t = 0$ . For this case, the optimal condition (2) can be expressed as the following Lerner rule

$$\varepsilon_{ss} = \beta p^1 F_{ss}, \quad (4)$$

where  $\varepsilon_{ss}^a = \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{F_{ss}}{y_{ss}^a}$  is the fishing effort elasticity of the yield in the stationary solution and  $\beta p^1$  is the future marginal cost per unit of adult fish

$$-p^1 = \frac{\partial N_{ss}^2}{\partial F_{ss}} \frac{1}{N^2}.$$

The economic interpretation of the Lerner rule (4) is very intuitive. In the optimal stationary solution, an increase in the mortality rate leads to an increase in the current yield per unit of fish (left hand side, in percentage terms) that is compensated with the decrease of adult population per unit of fish in the next period (right hand side). The net present value associated with  $F_{ss}$  is equal to

$$V_{ss}^{IS} = \frac{1}{1 - \beta} y_{ss}^2 e^{-(p^1 F_{ss} + m)}, \quad (5)$$

where  $y_{ss}^2 = pr^2 \omega^2 \frac{p^2 F_{ss}}{p^2 F_{ss} + m} \left[ 1 - e^{-(p^2 F_{ss} + m)} \right]$ .

Notice that the stationary solution implies a distortion of the structure population generated with perfect selectivity. As a result the value of the fishery with the stationary solution is lower than with perfect selectivity,  $V_{ss}^{IS} < V^{PS}$ .<sup>1</sup>

However, comparing (5) and (3) it is not clear if the pulse solution is better or not than the stationary solution,  $V_{ss}^{IS} \leq V_{pulse}^{IS}$ . Nevertheless, it can be proved that the more imperfect

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<sup>1</sup>Notice that  $\frac{p^2 F_{ss}}{m + p^2 F_{ss}} \left[ 1 - e^{-(p^2 F_{ss} + m)} \right] e^{-p^1 F_{ss}} < 1$

is the selectivity technology, the lower the fishing effort applied in the stationary solution  $F_{ss}$  is. This means that, given the adult selectivity, if the juvenile selectivity parameter is high enough, pulse fishing dominates the smooth stationary solution. Formally, we can find a lower threshold for the juvenile selectivity parameter that when it is exceeded the periodic pulse provides a higher net present value than the stationary interior solution. The following proposition states this result.

**Proposition 2.** *The pulse solution (stationary solution) is the optimal solution to the maximization problem (1) if  $p^1$  is higher (lower) than  $\underline{p}^1$ , being  $\underline{p}^1$  the value of the juvenile selectivity parameter that satisfies the following equality*

$$y_{ss}^2 e^{-\underline{p}^1 F_{ss}(\underline{p}^1)} = \frac{\beta}{1 + \beta} p r^2 \omega^2.$$

**Proof:** See Appendix.

Figure 1 shows the optimal solutions for a benchmark 2-ages fishery. For each juvenile selectivity parameter,  $p^1$ , and stationary fishing mortality, the net present value of the fishery,  $V_{ss}^{IS}$ , is calculated. The blue-red bell surface represents this function. The figure also represents the value of the fishery for the pulse solution,  $V_{pulse}^{IS}$ , through the yellow horizontal hyperplane. Comparing the bell function with horizontal hyperplane we can see that for values of  $p^1 \lesssim 0.26$ , the stationary solution dominates the pulse solution. However for values of  $p^1 \gtrsim 0.26$  the hyperplane is higher than the bell function, so the pulse solution is better than the stationary solution.

Figure 2 shows the net present value of the fishery for any combination of  $(F_t, F_{t+1})$  for two values of the juvenile selectivity parameter, low  $p^1$  (left plot) and high  $p^1$  (right plot). A point in the horizontal diagonal represents the value of the fishery for an stationary path  $\{F_{ss}\}$ . It is clear that the stationary solution and the pulse solution are always local optimums. However while the stationary solution is the global maximum when  $p^1$  is low, the pulse fishing is the global maximum when  $p^1$  is high. Moreover when the pulse fishing is the global maximum, the function is not concave and the stationary solution is a saddle path rather than a local maximum.

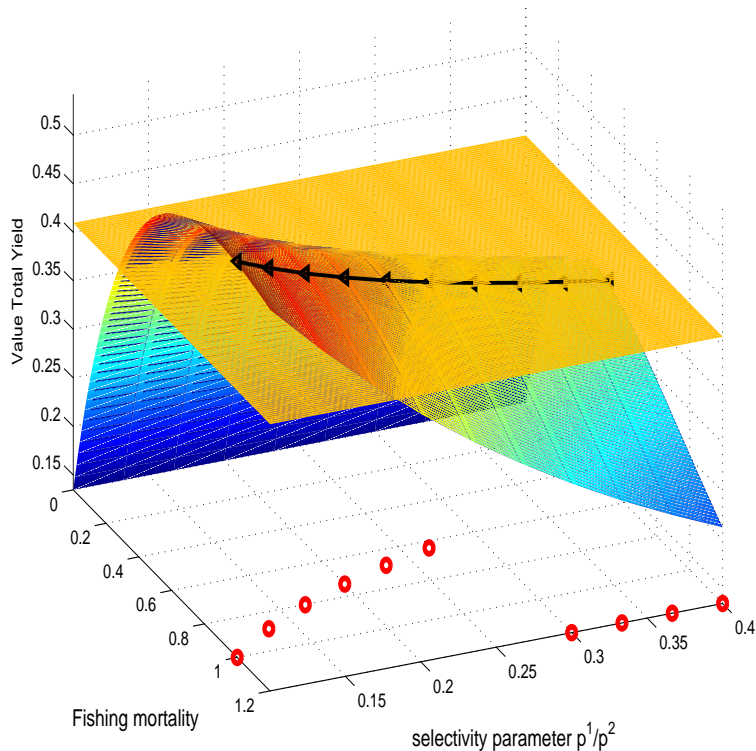


Figure 1: Simulating the net present value of the fishery under the stationary and pulse solutions. Benchmark 2-ages fishery ( $p^2 = 1, m = 0.2, pr^2 = 0$ , and  $pr^2\omega^2 = 1$ ). The blue-red shape bell represents the net present value of the fishery for each pair of juvenile selectivity parameter and stationary fishing rate. The yellow horizontal hyperplane illustrates the value of the fishery for the pulse solution.

## 2.1 The role of the discount factor

Pulse fishing literature claims that higher discounting rates (factor) decrease (increase) the advantage of pulse fishing (Hannesson, 1975). This claim is based on the impact of the discount factor on the pulse solution. Because the value of the fishery is the present value of a constant yield,  $pr^2\omega^2e^{-m}$ , by increasing the discount factor, the net present value of pulse fishing raises.

However, by increasing the discount factor, the net present value of the stationary solution also increases. First, it is well known that as  $\beta \rightarrow 1$ ,  $F_{ss} \rightarrow F_{max}$  (Da Rocha and Gutiérrez, 2011) That is, the stationary solution without discounting is equal to the  $F_{max}$ . Second, it is easy to prove that  $\partial F_{ss}/\partial\beta < 0$ . Therefore, as  $\beta$  increases,  $F_{ss}$  decreases and the stationary

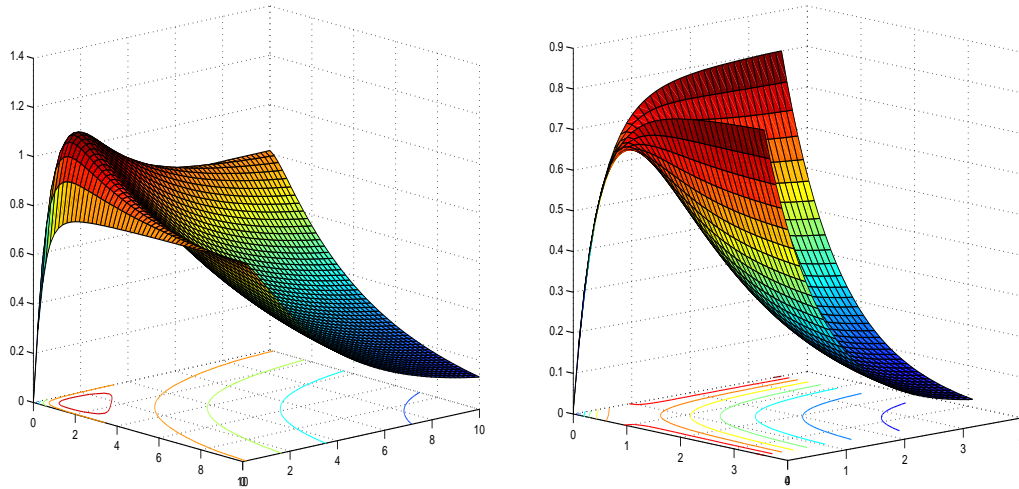


Figure 2: Net present value of the fishery for any any combination of  $(F_t, F_{t+1})$  for two values of the juvenile selectivity parameter,  $p^1$ : left hand side, low  $p^1$ ;right hand side, high  $p^1$ . Benchmark 2-ages fishery ( $p^2 = 1, m = 0.2, pr^2 = 0$ , and  $pr^2\omega^2 = 1$ )

yield raises. At first glance it is not clear whether or not this increase in the value of the stationary fishery is lower than the increase on the value of the pulse fishery. Nevertheless, the following proposition shows that if the discount factor is high enough then any increase in it raises the advantages of the periodic solution.

**Proposition 3.** *In a 2-ages class model, higher discounting rates (factor) decreases (increases) the advantage of periodic fishing over stationary fishing whenever condition  $1 < \beta(1 + \beta)$  holds.*

**Proof:** See Appendix.

### 3 Endogenous Lifespan

In the previous section was not possible to chose the fish lifespan. In both the pulse and the stationary solutions the fish lifespan is equal to two years. Let us extend the 2-ages model of the previous section to a more realistic framework with  $n$ -ages, to show how the fish lifespan is an endogenous variable.

Note, first, that if there were perfect selectivity optimal management would consist of finding the most profitable age to catch the fishes, let them grow until this optimal age is reached and harvesting all of them at this age. Formally, assume that we let any fish grows until age  $L$ . Then the value of the biomass if it is harvested is  $pr^L w^L e^{-(L-1)m}$ . It will be worth waiting one period more without harvesting, if  $\beta pr^{L+1} w^{L+1} e^{-Lm} > pr^L w^L e^{-(L-1)m}$ .

Lets define the value of weight gain for waiting from period  $L$  to period  $L + 1$  as  $\Delta_L^{L+1} = pr^{L+1} w^{L+1} e^{-Lm} - pr^L w^L e^{-(L-1)m}$ . Then we can express the condition under which is worth waiting one more period in term of the returns as

$$\text{Perfect selectivity: } \frac{\Delta_L^{L+1}}{pr^L w^L e^{-(L-1)m}} \geq \frac{1}{\beta} - 1 = r, \quad (6)$$

where  $r$  represents the discount rate. In this perfect selectivity context, the harvesting is carry out in a period  $L^{PS}$  such that

$$\frac{\Delta_{L^{PS}-1}^{L^{PS}}}{pr^{L^{PS}-1} w^{L^{PS}-1} e^{-(L^{PS}-2)m}} \leq r \leq \frac{\Delta_{L^{PS}}^{L^{PS}+1}}{pr^{L^{PS}} w^{L^{PS}} e^{-(L^{PS}-1)m}}.$$

Notice that selecting  $L^{PS}$  is equivalent to choose endogenously the fishes' lifespan.

However, in a context of imperfect selectivity, harvesting fishes of age  $L$  implies that we have to wait  $L - 1$  periods and as a collateral result we also harvest fishes of age  $L - 1$ ,  $L - 2$ , ..... Therefore to calculate the optimal fish lifespan we have to compare the value of the harvesting under different pulses.

Suppose that we harvest with a pulse of length  $L$ . This means that  $F_t = F_{t+1} = \dots = F_{t+L-1} = 0$  and  $F_{t+L} = \infty$ . So the value of the harvest is given by

$$V_L^{IS} = \frac{\beta^{L-1}}{1 - \beta^L} \sum_{a=1}^L pr^a w^a e^{-(a-1)m}. \quad (7)$$

It will be worth harvesting every  $L + 1$  periods if  $V_{L+1}^{IS} > V_L^{IS}$ . In this imperfect selectivity context the condition under which is worth waiting from period  $L$  to period  $L + 1$  can be expressed in terms of the returns as

$$\text{Imperfect selectivity: } \frac{\Delta_L^{L+1}}{\sum_{a=1}^L pr^a w^a e^{-(a-1)m}} \geq r \frac{1}{1 - \beta^L}. \quad (8)$$

Therefore, the harvesting is carry out in a period  $L^{IS}$  such that

$$\frac{\Delta_{L^{IS}-1}^{L^{IS}}}{\sum_{a=1}^{L^{IS}-1} pr^a w^a e^{-(a-1)m}} \leq \left( \frac{1}{\beta} - 1 \right) \frac{1}{1 - \beta^{L^{IS}}} \leq \frac{\Delta_{L^{IS}}^{L^{IS}+1}}{\sum_{a=1}^{L^{IS}} pr^a w^a e^{-(a-1)m}}.$$

Formally  $L^{IS}$  can be selected as the result of

$$V_L^{IS} = \max_L \frac{\beta^{L-1}}{1 - \beta^L} \left[ \sum_{a=1}^L pr^a w^a e^{-(a-1)m} \right].$$

Comparing condition (8) with the one obtained under perfect selectivity, (6), we observe two main differences. First, with imperfect selectivity, the value of weight gain,  $\Delta_L^{L+1}$ , is valued in terms of the whole population structure while with perfect selectivity it is valued in terms of the population of the previous age. Second, the cost for waiting is the interest rate adjusted for the raise of the pulse length. This point is specially relevant for assessing the role of the selectivity. Notice that with perfect selectivity, a reduction of the interest rate (i.e. in increase of the discount factor) yields to a raise in the optimal live span of the resource because is more profitable to wait more for harvesting. However, when the selectivity is imperfect, variations of the discount factor also affect positively the cost of rotation. The following proposition shows that this second effect dominate the first one and this implies that a raise of the discount factor reduce the the optimal lifespan and the optimal pulse length.

**Proposition 4.** *If the discount factor  $\beta$  increases the optimal fish lifespan and the pulse length decreases.*

Proof: See Appendix.

It is worth mentioning that harvesting under a pulse of length  $L$  is a local optimum of the management fishery problem that maximizes the net present value of the the fishery value. In a framework with  $n$ -ages the optimal management problem can be expressed as

$$\begin{aligned} \max_{\{F_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{a=1}^n y_t^a(F_t) \phi_t^a, \\ \text{s.t.} \quad 0 \leq F_t, \end{aligned} \tag{9}$$

where

$$\phi_t^a = \phi(F_{t-1}, F_{t-2}, \dots, F_{t-(a-1)}) = \begin{cases} 1 & \text{for } a = 1, \\ \prod_{i=1}^{a-1} e^{-p^{a-i} F_{t-i-m}} & \text{for } a = 2, \dots, n, \end{cases} \quad (10)$$

is obtained by backward substitution of the population,  $N_t^a = \phi_t^a N_{t-(a-1)}^1 = \phi_t^a$ , and it can be understood as the survival function that shows the probability of a recruit born in period  $t - (a - 1)$  to reach age  $a > 1$  for a given fishing mortality path. The following proposition proves that any periodic pulse of length  $L$  is a solution that satisfies the first order condition of the maximization problem (9).

**Proposition 5.** *In a  $n$ -ages class model, a periodic pulse with length equal to  $L$  such that  $F_t = F_{t+1} = \dots = F_{t+L-1} = 0$  and  $F_{t+L} = \infty$ , is a local optimum for problem (9).*

**Proof:** See Appendix.

There exist also a stationary solution,  $F_t = F_{t+1} = F_{ss}$ , that satisfies the first order condition of maximization problem (9). As in the case of 2-age model, the stationary solution can be expressed in terms of a Lerner rule. In concrete,

$$\sum_{a=1}^n \varepsilon_{ss}^a s^a = F_{ss} \frac{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left( \sum_{j=1}^{a-1} \beta^{a-j} p^j \right)}{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a}, \quad (11)$$

where  $s^a = y_{ss}^a \phi_{ss}^a / \sum_{a=1}^n y_{ss}^a \phi_{ss}^a$  is the share of yield in value of age  $a$  in total age classes yield value and  $\varepsilon_{ss}^a = \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{F_{ss}}{y_{ss}^a}$  is the fishing effort elasticity of the yield of age  $a$ . The following proposition characterizes an interior stationary candidate for global optimum,  $F_{ss}$ .

**Proposition 6.** *In a  $n$ -ages class model, the stationary solution  $F_{ss}$  satisfies the Lerner rule (11). Moreover,  $F_{ss}$  decreases as the selectivity parameter  $p^a$  and/or the discount factor  $\beta$  increases whenever  $y_{ss}^a \phi_{ss}^a$  is an increasing function. Furthermore, the net present value associated with  $F_{ss}$  is equal to*

$$V_{ss}^{IS} = \frac{1}{1 - \beta} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a,$$

where  $\sum_{a=1}^n y_{ss}^a \phi_{ss}^a = \sum_{a=1}^n pr^a \omega^a \frac{p^a F_{ss}}{p^a F_{ss} + m} (1 - e^{-p^a F_{ss} - m}) \prod_{i=1}^{a-1} e^{-(p^i F_{ss} + m)}$ .

**Proof:** See Appendix.

It is well known that there exists a relationship between non concavity of the objective function and the non continuity of the optimal solution (Scarf, 1959, Stokey, Lucas and Prescott, 1989). This link for biomass fishery models was established by Dawid and Kopel (1997, 1999) who show that the optimal pulse length is related with the non concavity of the return function.<sup>2</sup>

We prove that this link exists also for age-structured fishery models. In order to find this link, we analyze the (non-)concavity properties of the objective function of problem (9). In particular, by focusing on the second order conditions, we are able to find sufficient conditions that guarantee that the stationary solution is not the optimal solution of problem (9). The following proposition summarize this result.

**Proposition 7.** *A stationary solution is not the optimal solution of maximization problem (9) whenever the following inequality holds*

$$\sum_{a=1}^n \left[ \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} + y_{ss}^a \left( \sum_{j=1}^{a-1} \beta^{a-j} (-p^j)^2 \right) \right] \phi_{ss}^a \geq 0. \quad (12)$$

**Proof:** See Appendix.

Notice that  $\partial^2 y_{ss}^a / \partial F_{ss}^2 < 0$  because the yield per unit of fish is concave function. However as  $\beta \rightarrow 1$ , it is more likely that inequality (12) be positive.

As in the 2-age model, we study under which conditions a raise in the discount factor can promote pulse fishing comparing with the stationary smooth solution. In particular we find that a sufficient condition to guarantee this result is that the stationary yield,  $y_{ss}^a \phi_{ss}^a$ , be an increasing function with respect to  $F_{ss}$ . We state this result in the following proposition.

**Proposition 8.** *In a n-ages class model, higher discounting rates (factor) decreases (increases) the advantage of periodic fishing over stationary fishing whenever  $y_{ss}^a \phi_{ss}^a$  is an increase function.*

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<sup>2</sup>Maroto and Moran (2008) also show that concavity plays a relevant role in the appearance of resource extinction which can be considered a non continuous solution.

**Proof:** See Appendix.

Finally, we can conclude that in order to find the optimal fishing path with imperfect selectivity is enough to compare the value of fishery for a reduce number of possibilities. In particular, we should calculate the value of the fishery for the pulses  $V_L^{IS}$  for  $L = 2, 3 \dots n$  and for the stationary solution  $V_{ss}^{IS}$ . The optimal solution will be the one that yields to the maximum value of the fishery. In the next section we show a numerical illustration.

## 4 A numerical illustration

In this section we apply the results obtained in above section to Northern Stock of Hake (NSH) (*Merluccius merluccius*) in order to select the optimal fishing path. That is we calculate the value of the fishery for the pulses  $V_L^{IS}$  for  $L = 2, 3 \dots n$  and the stationary solution  $V_{ss}^{IS}$ . Moreover we compare this result with the result obtained using global numerical methods to solve the management problem (9). We will see that we obtain similar results.

The NSH includes all fisheries in International Council for Exploitation for of the Sea (ICES) subareas VII and VIII and also some fisheries in Subareas IV and VI. Landings in 2008 were 47,800 tones, below the regulated TAC of 54,000 tones. Spain accounts for the main part of the landings with 53% of the total captures. France is taking 30% of the total, UK 7%, Denmark 3%, Ireland 3% and other countries (Norway, Belgium, Netherlands, Germany, and Sweden) contributing small amounts (ICES 2009).

A recovery plan was developed in 2004 (EC 811/2004) for this stock. Its aim was to achieve a spawning stock biomass (SSB) of 140,000 tones by limiting fishing mortality to 0.25 and by allowing a maximum change in harvest between consecutive years of 15%. According to ICES, the northern hake SSB for 2009 is estimated to be above the recovery plan target. Article 3 of the recovery plan prescribes that a management plan should be implemented when the target is reached in two consecutive years and ICES considers that SSB has been approximately 140,000 tones in the last two years. Such a plan is under development by the European Commission (ICES 2010).

Table 2: Biological Parameters for the Northern Stock of Hake.

Age	Abundance ( $N$ ) <sup>(1)</sup>	Mortality ( $m$ )	Maturity $\mu$	Weight ( $\omega$ ) <sup>(2)</sup>	Selectivity ( $p$ )	rice ( $pr$ ) <sup>(3)</sup>
Northern Hake						
1	186,213	0.20	0.00	0.06	0.00	2.34
2	152,458	0.20	0.00	0.13	0.01	2.90
3	123,457	0.20	0.00	0.22	0.10	3.39
4	100,213	0.20	0.23	0.34	0.22	3.82
5	67,409	0.20	0.60	0.66	0.20	4.51
6	35,551	0.20	0.90	0.99	0.30	5.18
7	19,674	0.20	1.00	1.44	0.40	5.76
8	10,206	0.20	1.00	1.83	0.47	6.17
9	9,1503	0.20	1.00	2.68	0.47	6.86
10	4,080	0.20	1.00	2.68	0.47	6.86
11	1,821	0.20	1.00	2.68	0.47	6.86

Source: Meeting on Northern Hake Long-Term Management Plans (STECF/SGBRE-07-03). <sup>(1)</sup> Thousand; <sup>(2)</sup> kg; <sup>(3)</sup> euro per kg

In order to calibrate the age structured model for this fishery two data sources have been used. First, the information regarding the biological parameters of the fishery comes from Expert Working Group (STECF, 2008a). Most of the parameters emanate from the summary of XSA results from the 2006 update (ICES, 2007). Second, as the Spanish fleet accounts for most of the northern hake landings (59% of the total in 2006; ICES, 2007), we use 2007 daily sale prices for the Spanish fleet.

Table 2 shows, for each age, the number of fishes at the initial conditions, the parameters of the population dynamics (selection pattern, weight and maturity) and the prices.

Using this data calibration we calculate the value of the fishery,  $V_L^{IS}$ , for  $L = 2, 3, \dots, 11$  defined in (7) for the NHS. Table 3 shows the results obtained. We can observe that the fishery reach the maximum value applying a pulse of 9 periods, that is the fishery is lied fallow for eight years -by closing the fishery- and in the ninth period the whole population is harvested.

Table 3: Fishery value under pulse fishing with imperfect selectivity for the Northern Stock of Hake.

$L$ (periods)	2	3	4	5	6	7	8	9	10	11
$V_L^{IS}$ (€ million)	2.39	2.57	2.65	3.61	4.48	4.98	4.72	5.45	3.90	2.82

Source: Own calculations

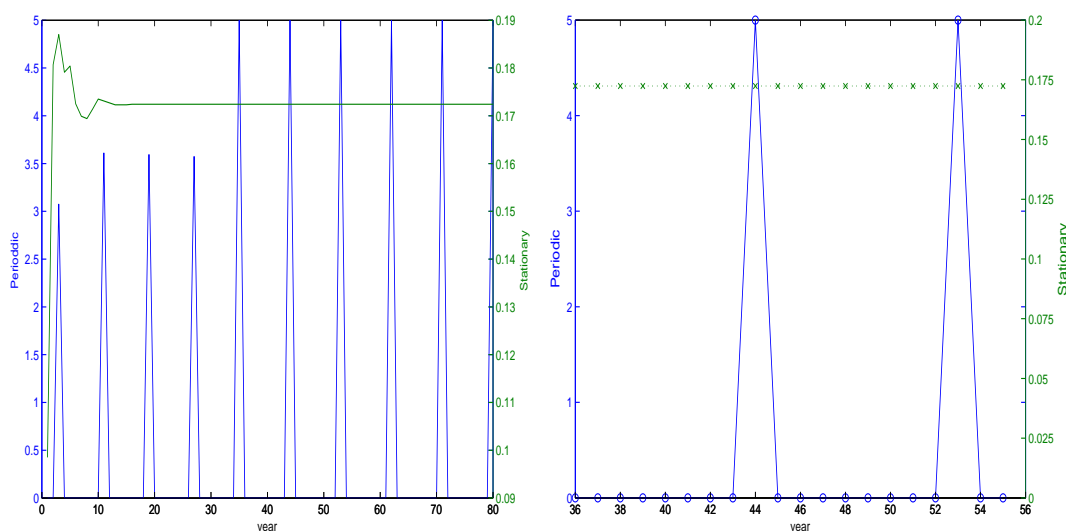


Figure 3: Optimal pulse fishing policy for the NSH obtained by solving (9) with global methods using an hybrid strategy eSS-DHC

This numerical solution has been validated by obtaining the optimal harvesting policy for this NSH stock that solves the management problem (9). Since the Beverton-Holt multi-cohort models used to assess the stock are not globally concave, the stationary solution described may be a local rather than a global optimum (Tavhonen, 2009). Because of this, to find the global solution of (9), we start by transforming the original dynamic optimization problem of infinite dimension, into a low dimension non-linear optimization problem by using the control vector parametrization approach (Vassiliadis, 1993; Vassiliadis *et al.*, 1994).

In order to select the proper algorithm that leads to the best results, both the *robustness* (the capability of the solver to attain consistently good final solutions) as well as the *efficiency*

Table 4: Pulse fishing and stationary solution for the NHS. Using global methods with an hybrid strategy eSS-DHC

	Stationary	Pulse
Yield		
weight ('000 t.)	53.6	567.2 (each 9 years)
value ('000 €)	299.1	3,325.1 (each 9 years)
Average price per kilo (€)	5.5802	5.8623
Profits		
net present value (€ millions)	5,799.1	6,734.2

(the speed to converge to the final solution) of solvers must be evaluated. Based on previous experience on this kind of optimal fishing policy definition, an hybrid strategy is revealed as the most suitable to solve the transformed optimization problem for the NSH. The hybrid methods combine global stochastic and local optimization algorithms. In this article, the hybrid strategy considered is based on the *Enhanced Scatter Search* (eSS-SSm, Egea *et al.*, 2009) as global solver plus *Dynamic Hill Climbing* (DHC, Maza and Yuret, 1994) as local algorithm.

We have use these algorithms to determine the trajectory that drives the NSH fishery for the initial conditions to the global optimal solution. In this framework, the selected hybrid method converges to a global optimal solution that consists on a pulse fishing every 9 years by applying a fishing rate  $F = 5^3$  in the harvesting years and 8 consecutive fallow years. Figure 3 shows the evolution and magnitude of cited pulses. Finally, Table ?? shows the quantitative results associated to the stationary and the pulse solutions using the global methods. Note that average price per kilo is higher in the pulse fishing solution than in the stationary solution. That is due to the fact that age distribution is bias towards older ages in the pulse solution.

We can conclude that numerical global methods support the optimal pulse selected by comparing the valuation of the different pulses  $V_L^{IS}$ .

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<sup>3</sup>We have checked that increasing this value does not change the results.

## 5 Conclusions

We extend Tahvonen (2009) analysis to the Hanneson's multi-cohort Beverton-Holt model with any number of age classes. We show that under imperfect selectivity, when optimal age can not be chosen, optimal management is equivalent to chooses the pulse length that induce the optimal average fish lifespan. Therefore, with imperfect selectivity, optimal lifespan is valued in terms of the whole population structure. Moreover, with imperfect selectivity, a reduction of the interest rate (i.e. in increase of the discount factor) yields to a reduction in the optimal lifespan and the optimal pulse length. Finally, as in the biomass models, we show that pulse fishing is related with the non concavity of the return function.

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## 6 Appendix

### 6.1 Proof of Proposition 1

Valuing the optimal condition (2) in  $t$  and  $t + 1$  and taking into account that  $F_t = F_{t+2} = \dots = 0$  and  $F_{t+1} = F_{t+3} = \dots = \infty$ , it is obtained, respectively

$$\begin{aligned}\mu_t &= - \lim_{\substack{F_{t-1}=F_{t+1} \rightarrow \infty \\ F_t=0}} \left[ \frac{\partial y_t^2(F_t)}{\partial F_t} N_t^2(F_{t-1}) + \beta y_{t+1}^2(F_{t+1}) \frac{\partial N_{t+1}^2(F_t)}{\partial F_t} \right] \\ &= - \lim_{\substack{F_{t-1}=F_{t+1} \rightarrow \infty \\ F_t=0}} \beta p r^2 \omega^2 \frac{p^2 F_{t+1}}{p^2 F_{t+1} + m} (1 - e^{-p^2 F_{t+1} - m}) (-p^1) e^{-p^1 F_t - m} = p^1 \beta p r^2 \omega^2 e^{-m} > 0.\end{aligned}$$

Therefore a stationary pulse with  $F_t = F_{t+2} = \dots = 0$  and  $F_{t+1} = F_{t+3} = \dots = \infty$  satisfies the optimal condition (2). Taking into account that in periods 0, 2, 4, ... the yield is zero and in periods 1, 3, 5, ... the fishing mortality tends to infinity, then the net present value of yield in value for the cycle is given by

$$\begin{aligned}V_{pulse}^{IS} &= \sum_{t=0}^{\infty} \beta^t y_t^2(F_t) N^2(F_t) = \\ &= \sum_{t=0}^{\infty} \beta^{2t+1} p r^2 \omega^2 \lim_{F_{2t+1} \rightarrow \infty} \frac{p^2 F_{2t+1}}{p^2 F_{2t+1} + m} (1 - e^{-p^2 F_{2t+1} - m}) e^{-m^1} \\ &= p r^2 \omega^2 e^{-m^1} [\beta + \beta^3 + \beta^5 + \dots] \\ &= p r^2 \omega^2 e^{-m^1} \frac{\beta}{1 - \beta^2}. \quad \blacksquare\end{aligned}$$

### 6.2 Proof of Proposition 2

Let  $\underline{p}^1$  be the value that satisfies the following equality

$$y_{ss}^2 e^{-\underline{p}^1 F_{ss}(\underline{p}^1)} = \frac{\beta}{1 + \beta} p r^2 \omega^2.$$

First we prove that  $F_{ss}$  decreases as the selectivity parameter  $p^1$  increases. Notice that if  $F_{ss}$  is an interior stationary solution then  $\mu_t = 0$  and the optimal condition (2) valued in  $F_{ss}$  can be expressed as

$$\frac{\partial y_{ss}^2}{\partial F_{ss}} - \beta y_{ss}^2 p^1 = 0.$$

Total differentiation of this optimal condition implies

$$\frac{\partial F_{ss}}{\partial p^1} = \frac{\beta y_{ss}^2}{\frac{\partial^2 y_{ss}^2}{\partial F_{ss}^2} - \beta p^1 \frac{\partial y_{ss}^2}{\partial F_{ss}}} < 0.$$

So if  $p^1 > \underline{p}^1$ , then  $y_{ss}^2 e^{-p^1 F_{ss}(p^1)} < y_{ss}^2 e^{-\underline{p}^1 F_{ss}(\underline{p}^1)}$ . This implies that  $y_{ss}^2 e^{-p^1 F_{ss}(p^1)} < \frac{\beta}{1+\beta} pr^2 \omega^2$ . Therefore  $V_{ss}^{IS} = \frac{1}{1-\beta} y_{ss}^2 (p^1) e^{-p^1 F_{ss}(p^1) - m} < \frac{\beta}{1-\beta^2} pr^2 \omega^2 e^{-m} = V_{pulse}^{IS}$ . ■

### 6.3 Proof of Proposition 3

Taking partial derivative in  $V_{pulse}^{IS}$  defined in Proposition ?? we obtain

$$\frac{\partial V_{pulse}^{IS}}{\partial \beta} = \frac{1 + \beta^2}{(1 - \beta^2)^2} pr^2 \omega^2 e^{-m} > 0. \quad (13)$$

Doing the same over  $V_{ss}^{IS}$  defined in (5)

$$\begin{aligned} \frac{\partial V_{ss}^{IS}}{\partial \beta} &= \frac{1}{(1 - \beta)^2} y_{ss}^2 e^{-(p^1 F_{ss} + m)} + \frac{1}{1 - \beta} \frac{\partial y_{ss}^2}{\partial F_{ss}} \frac{\partial F_{ss}}{\partial \beta} e^{-(p^1 F_{ss} + m)} - p^1 \frac{\partial F_{ss}}{\partial \beta} \frac{1}{1 - \beta} y_{ss}^2 e^{-(p^1 F_{ss} + m)} = \\ &= \frac{1}{(1 - \beta)^2} y_{ss}^2 e^{-(p^1 F_{ss} + m)} + \frac{1}{1 - \beta} \frac{\partial F_{ss}}{\partial \beta} e^{-(p^1 F_{ss} + m)} \left[ \frac{\partial y_{ss}^2}{\partial F} - p^1 y_{ss}^2 \right]. \end{aligned} \quad (14)$$

Given that  $F_{ss}$  satisfies the Lerner rule (4), this expression can be write as

$$\begin{aligned} \frac{\partial V_{ss}^{IS}}{\partial \beta} &= \frac{1}{(1 - \beta)^2} y_{ss}^2 e^{-(p^1 F_{ss} + m)} - \frac{\partial F}{\partial \beta} e^{-(p^1 F_{ss} + m)} p^1 y_{ss}^2 = \\ &= y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[ \frac{1}{(1 - \beta)^2} - \frac{\partial F}{\partial \beta} p^1 \right] \\ &= y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[ \frac{1}{(1 - \beta)^2} - \frac{(p^1)^2 y_{ss}^2}{\frac{\partial^2 y_{ss}^2}{\partial F_{ss}^2} - \beta p^1 \frac{y_{ss}^2}{\partial F_{sst}}} \right] \\ &= y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[ \frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2 - \frac{\partial^2 y_{ss}^2}{\partial F_{ss}^2} \frac{1}{(p^1)^2 y_{ss}^2}} \right]. \end{aligned} \quad (15)$$

Given that  $y^2$  is convex,

$$\frac{\partial V_{ss}^{IS}}{\partial \beta} < y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[ \frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2} \right]. \quad (16)$$

Now, let us assume that for a given  $\beta$ , both solutions are equivalent,  $V_{pulse}^{IS} = V_{ss}^{IS}$ . This implies that

$$y_{ss}^2 e^{-(p^1 F_{ss} + m)} = \frac{\beta}{1 + \beta} pr^2 \omega^2 e^{-m}.$$

Taking this into account and (13), (16) can be expressed as

$$\begin{aligned} \frac{\partial V_{ss}^{IS}}{\partial \beta} &< y_{ss}^2 e^{-(p^1 F_{ss} + m)} \left[ \frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2} \right] = \left[ \frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2} \right] \frac{\beta}{1 + \beta} pr^2 \omega^2 e^{-m} \\ &= \left[ \frac{1}{(1 - \beta)^2} + \frac{1}{\beta^2} \right] \frac{\beta}{1 + \beta} \frac{(1 - \beta^2)^2}{1 + \beta^2} \frac{\partial V_{pulse}^{IS}}{\partial \beta} \\ &= \frac{[1 - 2\beta(1 - \beta)](1 + \beta)}{\beta(1 + \beta^2)} \frac{\partial V_{pulse}^{IS}}{\partial \beta}. \end{aligned}$$

Note that  $[1 - 2\beta(1 - \beta)](1 + \beta) < \beta(1 + \beta^2)$  implies  $1 - 2\beta + \beta^3 < 0$ . And this implies that condition  $1 < \beta(1 + \beta)$  holds. Therefore, whenever  $1 < \beta(1 + \beta)$  is satisfied  $\frac{\partial V_{ss}^{IS}}{\partial \beta} < \frac{\partial V_{pulse}^{IS}}{\partial \beta}$ . ■

## 6.4 Proof of Proposition 4

The optimal lifespan of the resource is given by an age  $L^{IS}$  such that (8) holds. Notice that

$$\frac{\partial}{\partial \beta} \left[ \left( \frac{1}{\beta} - 1 \right) \frac{1}{1 - \beta^{L^{IS}}} \right] = \frac{1}{\beta} \left( \frac{1}{1 - \beta^{L^{IS}}} \right) \left[ \frac{L^{IS} \beta^{L^{IS}-1} (1 - \beta)}{1 - \beta^{L^{IS}}} - \frac{1}{\beta} \right],$$

which is negative for  $L^{IS} > 1$  and  $0 < \beta < 1$ . This means that if  $\beta$  increases, the  $L^{IS}$  that satisfies (8) decreases. ■

## 6.5 Proof of Proposition 5

The Lagrangian associated to the maximization problem (9) is given by

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ \sum_{a=1}^n y_t^a(F_t) N_t^a + \mu_t F_t \right\}.$$

The first order conditions of this maximization problem can be written as

$$\frac{\partial L}{\partial F_t} = 0, \quad \implies \quad \sum_{a=1}^n \frac{\partial y_t^a(F_t)}{\partial F_t} \phi_t^a + \mu_t - \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^j y_{t+j}^{a+j}(F_{t+j}) \phi_{t+j}^{a+j} \right\} = 0, \quad (17)$$

where we have taken into account that

$$\frac{\partial \phi_{t+j}^a}{\partial F_t} = \begin{cases} 0 & \text{for } j = 0, \\ -p^a \phi_{t+j}^a & \text{for } j = 1, \dots, n-1. \end{cases}$$

Valuing the condition (17) for a pulse of length  $L$ , i.e.  $F_t = F_{t+1} = \dots = F_{t+L-1} = 0$  and  $F_{t+L} = F_{t+2L} = \dots = \infty$ , we obtain

$$\begin{aligned} \mu_t &= - \lim_{\substack{F_{t+L}=F_{t+2L} \rightarrow \infty \\ F_t=F_{t+1}=\dots=F_{t+L-1}=0}} \left[ \sum_{a=1}^n \frac{\partial y_t^a(F_t)}{\partial F_t} \phi_t^a - \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^j y_{t+j}^{a+j}(F_{t+j}) \phi_{t+j}^{a+j} \right\} \right] \\ &= \lim_{\substack{F_{t+L}=F_{t+2L} \rightarrow \infty \\ F_t=F_{t+1}=\dots=F_{t+L-1}=0}} \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^{Lj} y_{t+Lj}^{a+Lj}(F_{t+Lj}) \phi_{t+Lj}^{a+Lj} \right\} \\ &= \lim_{\substack{F_{t+L}=F_{t+2L} \rightarrow \infty \\ F_t=F_{t+1}=\dots=F_{t+L-1}=0}} \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^{Lj} \left[ p r^{a+Lj} \omega^{a+Lj} \frac{p^{a+Lj} F_{t+Lj}}{p^{a+Lj} F_{t+Lj} + m} (1 - e^{-p^2 F_{t+Lj} - m}) \right] \phi_{t+Lj}^{a+Lj} \right\} > 0. \end{aligned}$$

Therefore any pulse of length  $L$  with  $F_t = F_{t+1} = \dots = F_{t+L-1} = 0$  and  $F_{t+L} = \infty$  satisfies the optimal condition (17).

## 6.6 Proof of Proposition 6

If  $F_{ss}$  is an interior stationary smooth solution then  $\mu_t = 0$  and the optimal condition (17) valued in  $F_{ss}$  can be expressed as

$$\sum_{a=1}^n \frac{\partial y_{ss}^a}{\partial F_{ss}} \phi_{ss}^a = \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^j y_{ss}^{a+j} \phi_{ss}^{a+j} \right\}. \quad (18)$$

Notice that

$$\begin{aligned} & \sum_{a=1}^{n-1} p^a \left( \sum_{j=1}^{n-a} \beta^j y_{ss}^{a+j} \phi_{ss}^{a,j} \right) = \\ & p^1 [\beta y_{ss}^{2,j} \phi_{ss}^{2,j} + \beta^2 y_{ss}^{3,j} \phi_{ss}^{3,j} + \dots + \beta^{n-1} y_{ss}^n \phi_{ss}^n] + \\ & p^2 [\beta y_{ss}^3 \phi_{ss}^3 + \beta^2 y_{ss}^4 \phi_{ss}^4 + \dots + \beta^{n-2} y_{ss}^n \phi_{ss}^n] + \dots + \\ & p^{n-2} [\beta y_{ss}^{n-1} \phi_{ss}^{n-1} + \beta^2 y_{ss}^n \phi_{ss}^n] + p^{n-1} \beta y_{ss}^n \phi_{ss}^n = \\ & y_{ss}^n \phi_{ss}^n [p^1 \beta^{n-1} + p^2 \beta^{n-2} + \dots + p^{n-1} \beta] + \\ & y_{ss}^{n-1} \phi_{ss}^{n-1} [p^1 \beta^{n-2} + p^2 \beta^{n-3} + \dots + p^{n-2} \beta] + \dots \\ & y_{ss}^3 \phi_{ss}^3 [p^1 \beta^2 + p^2 \beta] + y_{ss}^2 \phi_{ss}^2 p^1 \beta = \\ & \sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left( \sum_{j=1}^{a-1} \beta^{a-j} p^j \right). \end{aligned}$$

Therefore, the first order condition valued in the stationary solution, (18), can be expressed as

$$\sum_{a=1}^n \frac{\partial y_{ss}^a}{\partial F_{ss}} \phi_{ss}^a = \sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left( \sum_{j=1}^{a-1} \beta^{a-j} p^j \right).$$

Defining  $\varepsilon_{ss}^a = \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{F_{ss}}{y_{ss}^a}$  as the fishing effort elasticity of the yield of age  $a$  in the stationary solution, the above expression can be written as

$$\sum_{a=1}^n \varepsilon_{ss}^a s^a = F_{ss} \frac{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left( \sum_{j=1}^{a-1} \beta^{a-j} p^j \right)}{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a},$$

where

$$s^a = \frac{y_{ss}^a \phi_{ss}^a}{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a}.$$

Moreover, total differentiation of (18) implies

$$\begin{aligned} \frac{\partial F_{ss}}{\partial \beta} &= \frac{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left( \sum_{j=1}^{a-1} (a-j) \beta^{a-j-1} p^j \right)}{\sum_{a=1}^n \left[ \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} \phi_{ss}^a + \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{\partial \phi_{ss}^a}{\partial F_{ss}} - \frac{\partial y_{ss}^a \phi_{ss}^a}{\partial F_{ss}} \left( \sum_{j=1}^{a-1} \beta^{a-j} p^j \right) \right]}, \\ \frac{\partial F_{ss}}{\partial p^a} &= \frac{\sum_{a=1}^n y_{ss}^a \phi_{ss}^a \left( \sum_{j=1}^{a-1} \beta^{a-j} \right)}{\sum_{a=1}^n \left[ \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} \phi_{ss}^a + \frac{\partial y_{ss}^a}{\partial F_{ss}} \frac{\partial \phi_{ss}^a}{\partial F_{ss}} - \frac{\partial y_{ss}^a \phi_{ss}^a}{\partial F_{ss}} \left( \sum_{j=1}^{a-1} \beta^{a-j} p^j \right) \right]}. \end{aligned}$$

Since  $y_{ss}^a$  is a concave function and the survival function  $\phi_{ss}^a$  is a decrease function, both expressions are negative whenever  $y_{ss}^a \phi_{ss}^a$  are increasing functions.

Therefore, the net present value of yield in value for  $F_{ss}$  in the imperfect selectivity n-age model is given by

$$\begin{aligned}
V_{ss}^{IS} &= \sum_{t=0}^{\infty} \beta^t \sum_{a=1}^n y_{ss}^a \phi_{ss}^a = \\
&= \left[ \sum_{a=1}^n pr^a \omega^a \frac{p^a F_{ss}}{p^a F_{ss} + m} (1 - e^{-p^a F_{ss} - m}) \prod_{i=1}^{a-1} e^{-p^{a-i} F_{ss} - m} \right] [1 + \beta + \beta^2 + \beta^3 + \dots] \\
&= \frac{1}{1 - \beta} \sum_{a=1}^n y_{ss}^a \prod_{i=1}^{a-1} e^{-p^{a-i} F_{ss} - m}
\end{aligned}$$

## 6.7 Proof of Proposition 7

The first order conditions of maximization problem (9) are given by equation 17). Notice that this condition can be understood as a difference equation on  $F_t$  of order  $2(A - 1) + 1$  because  $\{F_{t-(n-1)}, \dots, F_{t-1}, F_t, F_{t+1}, \dots, F_{t+(n-1)}\}$  appear on it. For interior solution this condition can be written as the following difference equation system,  $\forall t$

$$\Psi_t = \sum_{a=1}^n pr^a \frac{\partial y_t^a(F_t)}{\partial F_t} \phi_t^a(F_{t-1}, F_{t-2}, \dots, F_{t-(a-1)}) - \sum_{a=1}^{n-1} p^a \left\{ \sum_{j=1}^{n-a} \beta^j y_{t+j}^{a+j}(F_{t+j}) \phi_{t+j}^{a+j}(F_{t+j-1}, F_{t+j-2}, \dots, F_{t+j-(a-1)}) \right\} = 0 \quad (19)$$

So, second order conditions associated to maximization problem (9) can be analyzed using the Hessian matrix given by

$$\begin{bmatrix}
\frac{\partial \Psi_t}{\partial F_t} & \frac{\partial \Psi_t}{\partial F_{t+1}} & \cdots & \frac{\partial \Psi_t}{\partial F_{t+(n-1)}} \\
\frac{\partial \Psi_{t+1}}{\partial F_t} & \frac{\partial \Psi_{t+1}}{\partial F_{t+1}} & \cdots & \frac{\partial \Psi_{t+1}}{\partial F_{t+(n-1)}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial \Psi_{t+(n-1)}}{\partial F_t} & \frac{\partial \Psi_{t+(n-1)}}{\partial F_{t+1}} & \cdots & \frac{\partial \Psi_{t+(n-1)}}{\partial F_{t+(n-1)}}
\end{bmatrix}.$$

We know that any solution satisfying first order condition (17) and for which the hessian is not a negative definite cannot be a maximum.

In the stationary solution all the elements in the diagonal are given by

$$\left. \frac{\partial \Psi_{t+k}}{\partial F_{t+k}} \right|_{F_{ss}} = \beta^k \left. \frac{\partial \Psi_t}{\partial F_t} \right|_{F_{ss}} \quad \forall k = -(A - 1), \dots, 0, \dots, (A - 1)$$

where

$$\begin{aligned}
\left. \frac{\partial \varphi_t}{\partial F_t} \right|_{F_{ss}} &= \beta^t \left[ \sum_{a=1}^n \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} \phi_{ss}^a + \sum_{j=1}^{n-1} \beta^j \sum_{a=j+1}^n y_{ss}^a (-p^{a-j})^2 \phi_{ss}^a \right] \\
&= \beta^t \left[ \sum_{a=1}^n \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} \phi_{ss}^a + \sum_{a=1}^n y_{ss}^a \phi_{ss}^a \sum_{j=1}^{a-1} \beta^{a-j} (-p^j)^2 \right] \\
&= \beta^t \sum_{a=1}^n \left[ \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} + y_{ss}^a \sum_{j=1}^{a-1} \beta^{a-j} (-p^j)^2 \right] \phi_{ss}^a
\end{aligned}$$

Notice that if  $\sum_{a=1}^n \left[ \frac{\partial^2 y_{ss}^a}{\partial F_{ss}^2} + y_{ss}^a \sum_{j=1}^{a-1} \beta^{a-j} (-p^j)^2 \right] \phi_{ss}^a \geq 0$ , then the principal minor of order one in the Hessian is not negative and we can state that the stationary solution is not optimal. ■

## 6.8 Proof of Proposition 4

The value of the fishery harvested with a pulse of length  $L$  under imperfect selectivity is given by (7). Taking partial derivative we obtain

$$\frac{\partial V_L^{IS}}{\partial \beta} = \frac{(L-1)\beta^{L-2} + \beta^{2(L-1)}}{(1-\beta^L)^2} \sum_{a=1}^L pr^a w^a e^{-(a-1)m} > 0.$$

Doing the same over  $V_{ss}^{IS}$  defined in Proposition 6

$$\begin{aligned}
\frac{\partial V_{ss}^{IS}}{\partial \beta} &= \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a + \frac{1}{1-\beta} \frac{\partial F_{ss}}{\partial \beta} \sum_{a=1}^n \left[ \frac{\partial y_{ss}^a}{\partial F_{ss}} \phi_{ss}^a + y_{ss}^a \frac{\partial \phi_{ss}^a}{\partial F_{ss}} \right] \\
&= \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a + \frac{1}{1-\beta} \frac{\partial F_{ss}}{\partial \beta} \sum_{a=1}^n \left[ \frac{\partial y_{ss}^a}{\partial F_{ss}} \phi_{ss}^a - p^a y_{ss}^a \phi_{ss}^a \right] \\
&= \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a + \frac{1}{1-\beta} \frac{\partial F_{ss}}{\partial \beta} \sum_{a=1}^n \frac{\partial y_{ss}^a \phi_{ss}^a}{\partial F_{ss}}
\end{aligned}$$

Assuming that  $y_{ss}^a \phi_{ss}^a$  is an increase function, proposition (4) shows that  $\partial F_{ss} / \partial \beta < 0$ . This implies

$$\frac{\partial V_{ss}^{IS}}{\partial \beta} = \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a + \frac{1}{1-\beta} \frac{\partial F_{ss}}{\partial \beta} \sum_{a=1}^n \frac{\partial y_{ss}^a \phi_{ss}^a}{\partial F_{ss}} < \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a \quad (20)$$

Now, let us assume that for a given  $\beta$ , both solutions are equivalent,  $V_{pulse}^{IS} = V_{ss}^{IS}$ . This implies

that

$$\frac{1}{1-\beta} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a = \frac{\beta^{L-1}}{1-\beta^L} \sum_{a=1}^L pr^a w^a e^{-(a-1)m}.$$

Taking this into account (20) can be expressed as

$$\begin{aligned}
\frac{\partial V_{ss}^{IS}}{\partial \beta} &< \frac{1}{(1-\beta)^2} \sum_{a=1}^n y_{ss}^a \phi_{ss}^a = \frac{1}{(1-\beta)^2} \frac{(1-\beta) \beta^{L-1}}{1-\beta^L} \sum_{a=1}^L pr^a w^a e^{-(a-1)m} \\
&= \frac{1}{(1+\beta)} \frac{\beta^{L-1}}{1-\beta^L} \frac{(1-\beta^L)^2}{(L-1)\beta^{L-2} + \beta^{2(L-1)}} \frac{\partial V_L^{IS}}{\partial \beta} \\
&= \frac{\beta^{L-1} (1-\beta^L)}{(1+\beta) [(L-1)\beta^{L-2} + \beta^{2(L-1)}]} \frac{\partial V_L^{IS}}{\partial \beta}
\end{aligned}$$

We can prove that the factor multiplying  $\partial V_L^{IS}/\partial \beta < 1$  by contradiction. Assume  $\partial V_L^{IS}/\partial \beta > 1$  then it should holds that

$$\begin{aligned}
\beta^{L-1} (1-\beta^L) &> (1+\beta) [(L-1)\beta^{L-2} + \beta^{2(L-1)}], \\
\beta^{L-1} - \beta^{2L-1} &> (L-1)\beta^{L-2} + \beta^{2(L-1)} + (L-1)\beta^{L-1} + \beta^{2L-1}, \\
(2-L)\beta^{L-1} - 3\beta^{2L-1} - (L-1)\beta^{L-2} &> 0.
\end{aligned}$$

But this is not true because  $1 < L < n$ .

Therefore, whenever  $y_{ss}^a \phi_{ss}^a$  is an increase function  $\frac{\partial V_{ss}^{IS}}{\partial \beta} < \frac{\partial V_{pulse}^{IS}}{\partial \beta}$ . ■