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Graphs of transverse groups in foliated spaces

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Abstract

There are many algebraic and analytic invariants in foliated spaces. We are interested in the representation of the transverse structure of a foliation through some equivalent groupoids, which provides us another C*-algebraic and K-theoretical invariants (see [1]).

In this article, we study the transverse dynamics of some examples of foliations, through transverse groupoids Morita-equivalent to the holonomy groupoid of the foliated space.

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1. Some definitions

Definition 1 An *oriented graph* is a 4-tuple $\Gamma = (\Gamma^0, \Gamma^1, s, r)$, where:

(i) Γ^0 and Γ^1 are the sets of *vertices* and *edges*, respectively;

(ii) the *incidence maps* $s, r : \Gamma^1 \longrightarrow \Gamma^0$ associate to each edge $a \in \Gamma^1$ its *source* $s(a) \in \Gamma^0$ and *end* $r(a) \in \Gamma^0$.

The *orientation* is implicitly contained in the definition of the incidence maps.

Γ is *locally finite*, if the vertices have finite degree (the degree of a vertex v is the number of edges starting or ending at v).

Definition 2 A *morphism* between two oriented graphs: $f : \Gamma \to \widehat{\Gamma}$, is defined by a couple of maps $f_0 : \Gamma^0 \longrightarrow \widehat{\Gamma^0}$ and $f_1 : \Gamma^1 \longrightarrow \widehat{\Gamma^1}$, which preserve the orientation and verify $\widehat{s} \circ f_1 = f_0 \circ s$ and $\widehat{r} \circ f_1 = f_0 \circ r$.

Definition 3 Let Γ be an oriented graph, where Γ^0 and Γ^1 are provided with the discrete topology. On the disjoint sum $T = \Gamma^0 \bigsqcup \left(\Gamma^1 \times [0,1]\right)$, we consider the equivalence relation $s(a) \sim (a,0)$ and $r(a) \sim (a,1)$, if $a \in \Gamma^1$.

The quotient $R(\Gamma) = T/\sim$, is called the *geometric realization* of Γ.

Definition 4 A *locally finite* and oriented graph of abelian groups of finite type is defined by a locally finite and oriented graph Γ and for each $s \in \Gamma^0$ and $a \in \Gamma^1$, there are groups G_s and H_a and homomorphisms between them $S_a : H_a \to G_{s(a)}$ and $R_a : H_a \to G_{r(a)}$.

A *morphism* between two locally finite and oriented graphs of abelian groups of finite type is defined in the obvious manner (see [3]).

The category of locally finite and oriented graphs of abelian groups of finite type and the morphisms between them will be the most suitable in our work.

Definition 5 Let $(M, \mathcal{F})$ a foliated space, where M^m is a compact riemannian manifold, $\mathcal{F}$ is a codimension one foliation of C^r class ($r \geq 1$), transversally oriented by a vector field Y, tangent to $\partial(M)$, when $\partial(M) \neq \emptyset$. Let $\mathcal{N}$ be a transverse foliation, orientable, of dimension one, defined by the flow corresponding to Y. $(M, \mathcal{F}, \mathcal{N})$ is an *almost without holonomy foliation*, when every non closed leaf $L \in \mathcal{F}$, has trivial holonomy.

2. Our previous work

Let $(M, \mathcal{F}, \mathcal{N})$ be as in **Definition 5**, and $\mathcal{F}$ an almost without holonomy foliation of finite type on M (i.e., with only a finite number of closed leaves, see [2]).

In [3] and [4] we associate to $(M, \mathcal{F}, \mathcal{N})$, in a canonical manner, a locally finite (in fact a finite graph, in the sense that the sets of vertices and edges are finite) and oriented graph of abelian groups of finite type, $\Gamma(\mathcal{F})$, that we call *the graph of the foliation*. We prove in this paper that $\Gamma(\mathcal{F})$ is Morita–equivalent to the holonomy groupoid of $\mathcal{F}$.

In the following lines, we describe this construction, based on the special properties of almost without holonomy foliations (see [2]).

If $F = C_1 \cup \cdots \cup C_n$ is the union of the closed leaves of $(M, \mathcal{F})$ and $M - F = U_1 \cup \cdots \cup U_m$, then for C_i ($i \in \{1, \cdots, n\}$), we observe:

1) there is exactly one connected component in $M - F$, $U_{r(i)}$, such that C_i is left-adherent to $U_{r(i)}$ (the transverse field is "attracting" in $U_{r(i)}$);

2) there is exactly one connected component in $M - F$, $U_{s(i)}$, such that C_i is right-adherent to $U_{s(i)}$ (the transverse field is "expanding" in $U_{s(i)}$).

Note that we can have $U_{s(i)} = U_{r(i)}$.

The transverse foliation $\mathcal{N}$ is defined by a flow φ. If $\varphi : C_i \times \mathbf{R} \to M$, $\mathcal{F}$ can be lifted on $C_i \times \mathbf{R}$, in an almost without holonomy (but not of finite type) foliation $\varphi^*(\mathcal{F})$. The closed leaves of $\varphi^*(\mathcal{F})$ are isolated, their union K is closed in $C_i \times \mathbf{R}$ and $(C_i \times \mathbf{R}) - K = \bigcup_{n \in \mathbf{N}} W_n$. Observe that when C_i (identified with $C_i \times \{0\}$) is left-adherent to $W_{r(i)}$ and right-adherent to $W_{s(i)}$, then we have $W_{r(i)} \neq W_{s(i)}$.

If $W_i^+ = W_{r(i)} \cup C_i$ and $W_i^- = W_{s(i)} \cup C_i$, then $W_i = W_i^+ \cup W_i^-$ is a neighborhood of C_i in $C_i \times \mathbf{R}$, saturated for $\varphi^*(\mathcal{F})$. And $\varphi^*(\mathcal{F})|_{W_i}$ is an almost without holonomy foliation, with exactly one closed leaf C_i; in fact $\varphi^*(\mathcal{F})|_{W_i}$ is defined by the suspension of a group of abelian homeomorphisms of $\mathbf{R}$, which represents the holonomy of C_i: W_i is the *geometric realization* of the holonomy group of C_i.

We have two maps $\varphi : W_i^\pm \to V_i^\pm = \varphi(W_i^\pm)$. Then, $\varphi(W_i) = V_i$ is a neighborhood of C_i in M, and we have two possibilities: $V_i = V_i^+ = V_i^-$ (when $U_{s(i)} = U_{r(i)}$) or $C_i = V_i^+ \cap V_i^-$.

For $j \in \{1, \cdots, m\}$, $\mathcal{F}|_{U_j} = \mathcal{F}_j$ is a without holonomy foliation. If we fix a base point $x_j \in U_j$, there is a unique transversal $N_j \in \mathcal{N}$ which contains this point. For a convenient election of x_j, N_j is homeomorphic to $\mathbf{R}$, and we consider a parametrization $p_j : \mathbf{R} \to N_j$. $\mathcal{F}_j$ is defined by a group of homeomorphisms without fixed points, of finite type (thus abelian and archimedian) of N_j (then of $\mathbf{R}$, by reciprocal image): it is the global holonomy group G_j of $\mathcal{F}_j$. In the same manner, for $i \in \{1, \cdots, n\}$, the induced foliation $\mathcal{F}|_{V_i}$, is defined by an abelian group of homeomorphisms of a transversal N_i^* (thus, of $\mathbf{R}$, under reparametrization), of finite type, with 0 as fixed point: it is the holonomy group H_i of the leaf C_i. And we have an homomorphism $R_i : H_i \to G_{r(i)}$, defined by $R_i = \lambda_i \circ \rho_i \circ \eta_i$, where:

1) $\eta_i : H_i \to H_i^+$, $H_i^+ = \{f|_{[0,\infty)} : f \in H_i\}$ is the right holonomy group of C_i and η_i is the restriction to $[0, \infty)$,

2) $\rho_i : H_i^+ \to H_{r(i)}$, where $H_{r(i)}$ is the holonomy of the foliation on $W_{r(i)}$, which is the reciprocal image by φ of the foliation over $U_{r(i)}$: we identify $H_{r(i)}$ with the holonomy group of $\mathcal{F}|_{V_{r(i)}}$. ρ_i is the restriction to $(0,\infty)$, composed with the exponential map,

3) $\lambda_i : H_{r(i)} \to G_{r(i)}$ is the natural inclusion.

We define $S_i : H_i \to G_{s(i)}$ in a similar manner.

Then, we have proved the announced result:

Theorem 1 *The graph $\Gamma(\mathcal{F})$ of $(M,\mathcal{F},\mathcal{N})$ is defined by:*

1) the finite and oriented graph Γ, defined by the sets $\Gamma^1 = \{V_1,\cdots,V_n\}$ and $\Gamma^0 = \{U_1,\cdots,U_m\}$ and the maps $s(V_i) \subset U_{s(i)}$ and $r(V_i) \subset U_{r(i)}$, where $r(V_i)$ is an open set in $U_{r(i)}$, represented by the inclusion by φ of $W_{r(i)}$ in $U_{r(i)}$ (similarly for $s(V_i)$),

(2) for $j \in \{1,\cdots,m\}$, G_{U_j} is the global holonomy group G_j of $(U_j,\mathcal{F}_j)$, which is abelian, of finite type and without fixed points,

3) for $i \in \{1,\cdots,n\}$, H_{V_i} is the holonomy group H_i of the closed leaf C_i, which is abelian, of finite type, and with 0 as unique fixed point,

4) for $i \in \{1,\cdots,n\}$, $R_{V_i} : H_{V_i} \to G_{r(V_i)}$ is defined by $R_{V_i} = R_i$. $r(V_i)$ is open in $U_{r(i)}$ and $G_{r(V_i)}$ is a subgroup of $G_{U_{r(i)}}$ (similarly for S_{V_i}).

$T = (\bigsqcup_{j=1}^{m} N_j) \cup (\bigsqcup_{i=1}^{n} N_i^*)$ is a total transversal to $\mathcal{F}$, naturally embedded in M. In fact, $\Gamma(\mathcal{F})$ is the transverse groupoid of $\mathcal{F}$, relatively to T.

$\Gamma(\mathcal{F})$ is Morita–equivalent to the holonomy groupoid of $(M,\mathcal{F})$, and in particular, the C-algebraic and K-theoretical properties of this two groupoids are the same.*

We explain through two examples the construction described in **Theorem 1**:

Example 1 The Reeb foliation $(\mathbf{S}^3,\mathcal{R})$ has a unique compact leaf C diffeomorphic to $\mathbf{T}^2$ and $\mathbf{S}^3 - C$ has two open connected components U_1 and U_2.

The graph of the foliation has a unique edge (corresponding to the leaf C) and two vertices (defined by the open sets U_1 and U_2). The groups defining the graph are:

(i) for the vertex U_i (for $i = 1,2$), $G_{U_i} \simeq \mathbf{Z}$ is the holonomy group of the induced foliation $(U_i,\mathcal{F}_{U_i})$;

(ii) $H_C \simeq \mathbf{Z}^2$ is the holonomy group of the leaf C.

The homomorphisms of the graph, which explain the contribution of each connected component to the holonomy of the leaf C, are $R_C : H_C \longrightarrow G_{U_1}$ and $S_C : H_C \longrightarrow G_{U_2}$, where $R_C(m,n) = n$ and $S_C(m,n) = m$.

Example 2 Let $\mathcal{F}$ be a foliation on the torus $\mathbf{T}^2$, with a unique compact leaf $C \simeq \mathbf{S}^1$ and the rest of the leaves are lines, obtained by suspension of an homeomorphism of $\mathbf{S}^1$, with an unique fix point. $\mathbf{T}^2 - C$ has a unique open connected component U.

The graph of this foliation is composed by a unique edge (corresponding to the unique leaf C) and a unique vertex (associated with the open set U). The groups defining the graph are described by:

(i) associated to the vertex U, $G_U \simeq \mathbf{Z}$ is the holonomy group of the induced foliation $(U, \mathcal{F}_U)$ (a without holonomy foliation);

(ii) $H_C \simeq \mathbf{Z}$ is the holonomy group of the compact leaf C.

The unique homomorphism of the graph is the identity map $R_C : H_C \longrightarrow G_U$.

3. A work in progress

Our interest is to generalize this type of results to arbitrary foliated spaces, with the following project of work:

(1) firstly, we associate to some foliated spaces $(M, \mathcal{F})$ a (or an inductive limit of) locally finite and oriented graph of groups;

(2) this graph must represent the holonomy groupoid of a complete transversal T to the foliation, and so it must be Morita–equivalent to the holonomy groupoid of $\mathcal{F}$;

(3) the geometric realization M_Γ of this graph (in the sense of **Definition 3**) should furnish us a topological space provided with an equivalence relation $\sim_\Gamma$ induced by the groups defining the graph. We should obtain in this manner a foliation $(M_\Gamma, \mathcal{F}_\Gamma)$, with a simpler structure than $(M, \mathcal{F})$, and by Morita–equivalence, we should retrieve, for example, information about the C*-algebra $C^*(M, \mathcal{F})$ and the K-theory $K^*(M, \mathcal{F})$ of the foliation (see [1]).

We have already started with this project, by studying the cases of almost without holonomy foliations not covered in [3]. We have obtained the same results as in [3], in the case of an almost without holonomy foliation $(M, \mathcal{F})$ without packets of compact leaves: with this hypothesis, we have a countable set $\{C_n\}_{n\in\mathbf{N}}$ of closed leaves in $\mathcal{F}$, and a countable set of open connected components of $M - \left(\bigcup_{n\in\mathbf{N}} C_n\right) = \bigcup_{n\in\mathbf{N}} U_n$.

The general case, in which $(M, \mathcal{F})$ could have packets of compact leaves, has some technical difficulties. We think that the final solution will arrive through an inductive limit of graphs.

After this, our idea is to substitute the abelian groups by amenable one (and more general groups) and to study some particular examples of foliations.

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